fixed pt. theorem for nuclear operators order zero (Compact & trace class) in a pos. cone in Banach space (**)

$\Rightarrow$ $\exists!$ eigenfunction of norm one f_s of $\mathcal{L}_s$ in the interior of the cone K

with eigenvalue $\lambda_{0,s}$ positive and simple

All other eigenvalues have strictly smaller ~~...~~ modulus (spectral margin $q(\mathcal{L}_s) < 1$)

$$(*) \qquad \mathcal{L}_s^n f = \text{const.}\ \lambda_{0,s}^n f_s + O(c(\epsilon)(q+\epsilon)^n \lambda_{0,s}^n)$$

for any $f \in \mathcal{B}$ and $\varepsilon > 0$

Note: for $s=1$ have an explicit positive eigenfunction density $\frac{1}{x+1}$ of inv. measure of T on $(0,1)$ with uniform measure in $t \in P$.

(**) K cone in a real Banach space $\mathcal{B}$ with $\overline{K-K} = \mathcal{B}$ and $\mathcal{L}$ compact operator on $\mathcal{B}$ with $\mathcal{L}K \subseteq K$ w/ positive spectral radius $r(L)$

$\Rightarrow$ $r(L)$ is an eigenvalue with eigenfunction in K

Also to see top eigenvalue of an operator w/ an invariant cone is simple:

say $\mathcal{L}$ is u-bounded, for a function $u \in K$ if $\forall f \in K$ $\exists n > 0$ s.t. $au \leq \mathcal{L}^n f \leq bu$ $a, b > 0$

then one sees that $\mathcal{L}_s$ is u-bounded for constant function $u(x,t) \equiv 1$ i.e. $\exists\, a,b > 0$

$a \le \mathcal{L}_s^n f \le b$: difficult part show $a \neq 0$

Suppose for some f lower bound is $= 0$

$\Rightarrow$ $\mathcal{L}_s^n f$ vanishes at some pt. (x_n, t_n) with x_n in $\mathbb{D}$ w/ x_n real

since all summands in

$$(\mathcal{L}_s f)(x,t) = \sum \frac{1}{(x+k)^{2s}} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} t\right)$$

are positive real for x real (s real here)

then f also $= 0$ on all pts of $\bigcup_n \mathrm{Red}_n(x_n, t_n)$

$\Rightarrow$ because $\mathbb{P} = \mathrm{Red}_m(t)$ for some m and all $t \in \mathbb{P}$

$\Rightarrow$ f has zeros on all shells $\mathbb{D} \times \{t\}$ $t \in \mathbb{P}$

$\Rightarrow$ $\exists$ sequence $m_n \to \infty$ and $y_n \in \mathbb{D}$ s.t. f vanishes at all $x \in \bigcup_n \mathrm{Red}_{m_n}(y_n)$ intersects densely $[0,1]$

holomorphic functions $\Rightarrow$ identically vanishing

In u-boundedness condition the lower bound

$\Rightarrow$ spectral radius $r(\mathcal{L}_s) > 0$

Then the fact that eigenvalue λ_0 is simple is like Perron-Frobenius eigenvector argument of linear algebra:

f interior pt of cone; iterates of $\mathcal{L}_s$ map cone to interior of cone $\rightsquigarrow$ unique fixed pt to see spectral gap with other eigenvalues

$\mathcal{L}_s$ u-bounded $\Rightarrow$ $\mathcal{L}_s$ f_s-bounded where

f_s = unique eigenfunction of $\lambda_{0,s}$ eigenvalue

So if f_λ with $\mathcal{L}_s f_\lambda = \lambda f_\lambda$ $\quad \lambda \neq \lambda_{0,s}$

$\Rightarrow$ $\quad -\alpha(\lambda_{0,s}-\varepsilon) f_s \leq \lambda f_\lambda \leq \alpha(\lambda_0-\varepsilon) f_s$ $\quad$ for some $\varepsilon > 0$

where α smallest $\alpha > 0$ s.t.

$$-\alpha f_s \leq f_\lambda \leq \alpha f_s$$

$$\Rightarrow |\lambda| \leq \lambda_0 - \varepsilon$$

Since $\mathcal{L}_s$ compact pt. spectrum : other eigenvalues all have $|\lambda| \leq q\lambda_0$ for some $q<1$

then sequence of iterates $f_{n+1} = \mathcal{L}_s f_n$ converges to eigenfunction f by

set $\quad Uh := f^*(h) f$

f^* = eigenf. of adjoint $\mathcal{L}_s^*$ on dual Banach space $\mathcal{B}'$

$\mathcal{L}^* f^* = \lambda f^*$

if $f^*(f) > 0$ then $\lambda = \lambda_0$ since

$$\lambda_0 f^*(f) = f^*(\lambda_0 f) = f^*(\mathcal{L}_s f) = (\mathcal{L}^* f^*)(f) = \lambda f^*(f)$$

Set also $U^\perp h := h - f^*(h) f$

$$\Rightarrow \lim_n \frac{\| U^\perp f_n \|}{\| U f_n \|} = 0$$

Hence get unique invariant measure on $[0,1] \times \mathbb{P}$

w/ density $\quad \frac{1}{x+1} \cdot \mu_{\mathbb{P}}$

unif. measure on $\mathbb{P}$

Digression: Chaotic cosmologies and modular curves (mixmaster universe models)

Lorentzian spacetimes with

$$ds^2 = dt^2 - a(t)^2 dx^2 - b(t)^2 dy^2 - c(t)^2 dz^2$$

Kasner metrics solutions to Einstein equations w/ non-homogeneous or anisotropic spacetimes

$$a(t) = t^{p_1} \qquad b(t) = t^{p_2} \qquad c(t) = t^{p_3}$$

with $\sum p_i = \sum p_i^2 = 1$

Mixmaster models (with chaotic behavior approaching singularity at $t \to 0$)

locally given by Kasner metrics where p_i vary with a parameter

$$\circledast \qquad p_1 = -\frac{u}{1+u+u^2} \qquad p_2 = \frac{1+u}{1+u+u^2} \qquad p_3 = \frac{u(1+u)}{1+u+u^2}$$

a dynamical system governs the evolution (discrete) of the parameter u

logarithmic time $d\Omega := \frac{dt}{abc}$ $\qquad \Omega \to \infty$ when $t \to 0$

Kasner epochs $[\Omega_n, \Omega_{n+1}]$

in each epoch $\circledast$ exponent with some $u_n > 1$

transition to a new epoch

$$u_{n+1} = \frac{1}{u_n - [u_n]} \qquad (**)$$

Within each Kasner era Kasner cycles (u_n diminishes by 1 until becomes $u_n < 1$ then transition $(**)$) where one of the exponents $p_i(u)$ dominates

* In each Kasner era u decreases

- discretized in steps $u_n, u_n - 1, u_n - 2, \ldots$ until < 1
(Kasner cycles)

* Each Kasner cycle $u \mapsto u-1$ comes with a permutation $(12)(3)$ of space coordinates

replaces a phase of contraction with one of expansion
$(u \mapsto -u)$

* At the end of a Kasner era (when u reaches a value $u<1$ after a series of cycles)

bouncing $\quad u_{n+1} = \dfrac{1}{u_n - [u_n]}$

together with a permutation of spatial coordinates
$(1)(23)$ which restores $p_1 < p_2 < p_3$

Belinski – Khalatnikov – Lifschitz (~ 1970)

Notice: $\quad Tx = \frac{1}{x} - [\frac{1}{x}]$ with $x_{n+1} = T x_n$ and

$u_n = \frac{1}{x_n}$ relates transition of mixmaster universe to continued fraction expansion

Observation: trajectories of mixmaster univ. (time evol. of mixmaster cosmology)
parameterized by geodesics on a modular curve

$$\Gamma_0(p) = \{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2,\mathbb{Z}) \; : \; c \equiv 0 \mod p \} \quad (p \text{ prime})$$

For $p = 2$ $\quad \mathbb{P} = \Gamma / \Gamma_0(2) = \mathbb{P}^1(\mathbb{F}_2) = \{0, 1, \infty\}$ $\quad$ 3-pts

$$X_{\Gamma_0(2)} = \Gamma \backslash \mathbb{H} \times \mathbb{P} \qquad = \Gamma_0(2)\backslash \mathbb{H} \leftarrow \text{upper half plane}$$

So a geodesic in $X_{\Gamma_0(2)}$ is the image of a geodesic in $\mathbb{H}\times\mathbb{P}$ w/ endpoints on $\mathbb{P}^1(\mathbb{R})\times\mathbb{P}$ under the quotient map

$$\pi : \mathbb{H}\times\mathbb{P} \longrightarrow \Gamma\backslash \mathbb{H}\times\mathbb{P} = X_G$$

Identify the 3-pts of $\mathbb{P} = \mathbb{P}^1(\mathbb{F}_2)$ with labelings of the 3 spatial coordinate axes

$$0 = [0:1] \longmapsto z$$
$$\infty = [1:0] \longmapsto y$$
$$1 = [1:1] \longmapsto x$$

So any geodesic with (irrational) endpts coded by

(ω^-, ω^+, s) $\qquad \omega^\pm \in \mathbb{P}^1(\mathbb{R})$ endpts in $\mathbb{P}^1(\mathbb{R})\times\{s\} \subset \mathbb{P}^1(\mathbb{R})\times\mathbb{P}$

can pick $\omega^- \in (-\infty, 1]$ & $\omega^+ \in [0,1]$ up to Γ-action

$\Rightarrow$ Continued fraction expansion

$$\omega^+ = [k_0, k_1, k_2, \dots] = \cfrac{1}{k_0 + \cfrac{1}{k_1 + \cdots}}$$

$$\omega^- = [k_{-1}; k_{-2}, k_{-3}, \dots] = k_{-1} + \cfrac{1}{k_{-2} + \cfrac{1}{k_{-3} + \cdots}}$$

The shift map T acts by

$$T(\omega^+, s) = \left(\frac{1}{\omega^+} - \left[\frac{1}{\omega^+}\right], \begin{pmatrix} -[1/\omega^+] & 1 \\ 1 & 0 \end{pmatrix} s\right)$$

$$T(\omega^-, s) = \left(\frac{1}{\omega^- + [1/\omega^+]}, \begin{pmatrix} -[1/\omega^+] & 1 \\ 1 & 0 \end{pmatrix} s\right)$$

or equivalently

$$T\,[k_0, k_1, k_2, \dots] = [k_1, k_2, k_3, \dots]$$
$$T\,[k_{-1}, k_{-2}, k_{-3} \dots] = [k_0; k_{-1}, k_{-2}, k_{-3}, \dots]$$

Quotient of Γ-action on geodesics = quotient by iterations of T
$\omega = (\omega^+, \omega^-)$ on data (ω^-, ω^+, s)

Given (ω, s) $\rightsquigarrow$ construct a solution of mixmaster universe

$k_n = [u_n] = [\frac{1}{x_n}]$ for k_i sequence coding ω

Kasner epochs with $x_{n+1} = T x_n$

and permutations of the spatial axes $\begin{pmatrix} -k_n & 1 \\ 1 & 0 \end{pmatrix}$

In fact: this is permutation

$0 \mapsto \infty \quad 1 \mapsto 1 \quad \infty \mapsto 0$

when k_n even and permutation

$0 \mapsto \infty \quad 1 \mapsto 0 \quad \infty \mapsto 1$ when k_n odd

this is product of $(12)(3)$ and $(1)(23)$

this is $(1)(23)$

Which is what happens by iterating sequence of Kasner cycles within each epoch & then passing to next Kasner epoch

(ω, s) & $T^m(\omega, s)$ same solution up to a different choice of initial time
(same behavior as $\Omega \to \infty$; i.e. $t \to 0$)

The invariant measure

$$d\mu(x,s) = \frac{\delta(s)\,dx}{3\log(2)\,(1+x)} \quad \text{on } [0,1]\times \mathbb{P}$$

$\Rightarrow$ alternation of space axes in mixmaster dynamics is uniform (each occurs with equal frequency as leading expansion or contraction)

Also special classes of solutions from special kinds of geodesics : example where $k_i \leq N$ for some fixed N : bound unif. on oscillations in each epoch

Hensley Cantor sets $E_N \subset [0,1]$

G fin. index subgrp of $PSL_2(\mathbb{Z})$

$\alpha, \beta \in \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \Rightarrow$ modular symbols (Manin)

real homology class

$\{\alpha, \beta\}_G$ in $H_1(X_G, \mathbb{R})$

Seen as a functional on differential forms

$$\int_{\{\alpha,\beta\}_G} \omega := \int_\alpha^\beta \pi_G^*(\omega)$$

integral along the geodetic in $\mathbb{H}$ w/ endpoints α, β

$$\pi_G : \mathbb{H} \longrightarrow G\backslash\mathbb{H} = X_G$$

Additivity and invariance properties:

$$\{\alpha, \beta\}_G + \{\beta, \gamma\}_G = \{\alpha, \gamma\}_G$$

$$\forall g \in G \qquad \{g\alpha, g\beta\}_G = \{\alpha, \beta\}_G$$

Modular symbols and continued fractions:

$$\int_0^\alpha \pi^*(\omega) \qquad \text{for } \alpha \in \mathbb{Q}$$

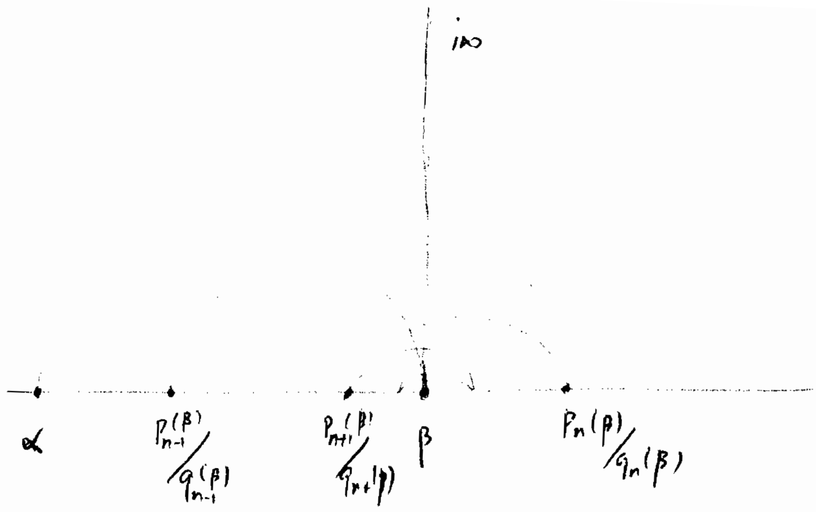
$$\frac{p_{k-1}(\alpha)x + p_k(\alpha)}{q_{k-1}(\alpha)x + q_k(\alpha)}$$

then finite continued fraction expansion $\alpha = [k_1, k_2, \ldots, k_n]$

$$g_k := \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix} \qquad \alpha = \frac{p_n(\alpha)}{q_n(\alpha)} \qquad \frac{p_k(\alpha)}{q_k(\alpha)} = g_k(\alpha)(0)$$

$$\frac{p_{k-1}(\alpha)}{q_{k-1}(\alpha)} = g_k(\alpha)(i\infty)$$

$$\int_0^\alpha \pi^*(\omega) = \sum_{k=1}^n \int_{p_{k-1}(\alpha)/q_{k-1}(\alpha)}^{p_n(\alpha)/q_n(\alpha)} \pi^*(\omega) = -\sum_{k=1}^n \int_{\{g_k(0),\, g_k(i\infty)\}_G} \omega$$



Use of modular symbols: integrals of modular functions along mod symbols

$\rightsquigarrow$ relations between periods

What about the "invisible" part of the boundary $\mathbb{P}^1(\mathbb{R}) \smallsetminus \mathbb{P}^1(\mathbb{Q})$?

Can have a theory of modular symbols there?

asymptotic cycles

say end $\beta \in \mathbb{R} \smallsetminus \mathbb{Q}$ then limiting cycle

$$\{\{*, \beta\}\}_G := \lim_{T\to\infty} \frac{1}{T(x)} \{x_0, x\}_G \in H^1(X_G, \mathbb{R})$$

limit along geodesic from starting pt. x_0 to β

x moves towards β along the geodesic

$T(x) =$ geod. length from x_0 to x

(1) Whenever limit exists it is indep. of starting pt. x_0 and of α other endpt α

Take two different geodesics γ_1, γ_2 w/ same β endpt.

Say γ_1 from $i\infty$ & γ_2 from $\alpha \in \mathbb{Q}$

(*) have $\frac{\alpha+\beta}{2} < \frac{p_{n-1}(\beta)}{q_{n-1}(\beta)} < \beta < \frac{p_n(\beta)}{q_n(\beta)}$

for sufficiently large n or exchanged depending on parity of n

also always have

(**) $$\left| \frac{p_{n-1}}{q_{n-1}} - \beta \right| > \left| \frac{p_n}{q_n} - \beta \right|$$

take intersection pts of geodesic arcs

$\tilde{\gamma}_n = \gamma\left(\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}\right)$ and γ_1 & γ_2 geodesics

$$z_n = \gamma_1 \cap \tilde{\gamma}_n \qquad \xi_n = \gamma_2 \cap \tilde{\gamma}_n$$

have from (*) & (**) estimates

$$\frac{1}{2 q_n q_{n+1}} < \operatorname{Im}(z_n) < \frac{1}{2 q_{n-1} q_n}$$

and for n large $\exists\, \theta$ a constant $0 < \theta < 1$ s.t.

$$\frac{\theta}{2 q_n q_{n+1}} < \theta \operatorname{Im}(z_{n+1}) < \operatorname{Im} \xi_n \leq \frac{1}{2 q_{n-1} q_n}$$

because for large n

Along γ_1 distance from fixed z_0 to z is $-\log \operatorname{Im}(z)$ up to a constant $+O(1)$

Along γ_2 similarly from fixed ξ_0 to ξ $-\log \operatorname{Im}(\xi) + O(1)$

$$\frac{1}{T(z_0, z_n)} \{z_0, z_n\} = \frac{1}{T(\xi_0, \xi_n) + O(1)} \left[\{\xi_0, \xi_n\} + O(1) \right]$$

So if lim exists they are same

For continued fraction expansion also known that

$$\log q_n(\beta) = C\, n\, (1 + o(1))$$

for almost all β (in Lebesgue measure on $[0,1]$)

with $C = \dfrac{\pi^2}{12 \log 2}$

$\Rightarrow$ Limit computing mod symbol $\{\{*, \beta\}\}_G$ written as

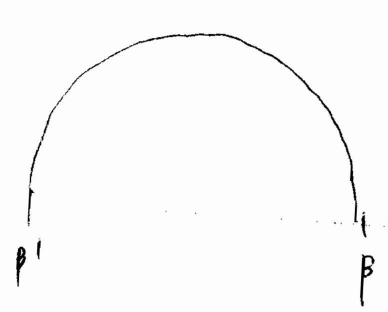
$$\lim_{n \to \infty} \frac{1}{2Cn} \{ i\infty, z_n \} = \lim_{n \to \infty} \frac{1}{2Cn} \sum_{k=1}^{n} \left\{ \frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}, \frac{p_k(\beta)}{q_k(\beta)} \right\}$$

(1) Case of closed geodesics: β is a quadratic irrationality
pt in a real quadratic field in $\mathbb{R}$

$\Rightarrow$ $\exists$ a closed geodesic in X_G whose lift is

β' = Gal conj of β

in a sheet of $\mathbb{H} \times \mathbb{P}$ $\quad \mathbb{P} = \Gamma/G$



$\Rightarrow$ $\{\{0, \beta\}\} = \dfrac{\{0, g(0)\}}{\lambda(g)}$

$\lambda(g)$ = length of geod. $= \log \lambda_g^+$ (eigenvalue of g)

β, β' fixed pts of an hyperbolic element g