A CHARACTERIZATION OF SCHOTTKY GROUPS(1)

By

BERNARD MASKIT

in Cambridge, Mass., U.SA.

The purpose of this paper is to prove the following consequence of Ahlfors theorem [1], and the planarity theorem [3]. *A finitely generated Kleinian group G is a Schottky group if and only if G is free, and every element of G other than the identity is loxodromic* (hyperbolic transformations are included among the loxodromic).

This characterization is in some sense the best possible. Using the above result, V. Chuckrow [2] proved that there exist finitely generated groups of Möbius transformations that are free and purely loxodromic, but are not discontinuous. Examples of finitely generated Kleinian groups that are free but not purely loxodromic, or purely loxodromic but not free, are well known.

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1. We recall the definition of a Schottky group. Let D be a region on the Riemann sphere (extended complex plane), bounded by $2n$ disjoint simple closed curves, $C_1, C_1', \cdots, C_n, C_n'$. For $i = 1, \cdots, n$, let A_i be a Möbius transformation with $A_i(C_i) = C_i'$, and $A_i(D) \cap D = \emptyset$. Let G be the group generated by $A_1, \cdots, A_n$. Then G is a Schottky group, and $D' = D \cup C_1 \cup C_2 \cup \cdots \cup C_n$ is a fundamental set for G. It is well known that G is a free group on the n generators, that every element of G is loxodromic, that $R(G) = \bigcup_{g \in G} g(D')$ is the full set of discontinuity of G, and that $R(G)$ is connected and dense in the Riemann sphere. Finally, $R(G)/G$ is a closed surface of genus n.

2. Our aim is to prove

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Theorem 1. *Let G be a finitely generated, purely loxodromic, free Kleinian group. Then G is a Schottky group.*

We first prove

Theorem 1′. *Let G be a finitely generated, purely loxodromic free Kleinian group with an invariant region of discontinuity R_0. Then G is a Schottky group.*

Proof. Since G is finitely generated, by Ahlfors theorem, R_0/G is a finite surface, and, by Ahlfors lemma [1, p. 416] since G is purely loxodromic, $R_0/G = S$ is in fact a closed surface of genus g.

Now by the planarity theorem, the regular planar covering $p; R_0 \to S$ is determined by a set $w_1, \cdots, w_q$, of simple disjoint loops on S, and a set of positive integers $\alpha_1, \cdots, \alpha_q$, where $p: R_0 \to S$ is the highest regular covering of S, for which each of the loops $w_1^{\alpha_1}, \cdots, w_q^{\alpha_q}$ lift to loops.

The w_i and α_i are not uniquely determined by the covering. We observe first that since G has no elements of finite order, we can choose the w_i and α_i so that for $i = 1, \cdots, q, \alpha_i = 1$. Under this restriction, we choose some set of loops $w_1, \cdots, w_q$ so that q, the number of loops, is *minimal*. Since G is free, $q > 0$.

It was shown in [3, p. 353] that given S and $w_1, \cdots, w_q$, one can realize G as the fundamental group of a certain 2-complex K, obtained as follows. If w_1 is dividing, we contract w_1 to a point, so as to get, locally, the wedge product of two surfaces. If w_1 is non-dividing, we contract w_1 to a point, which we consider as two points pulled apart so that locally we still have a closed surface, except that the genus has been reduced by one, and then at one of these two points we take the wedge product with a circle. Having done this, we have a 2-complex K_1. We set $K = K_q$.

G is isomorphic to $\pi_1(K)$ which, as one easily sees, is a free product of infinite cyclic groups and fundamental groups of closed surfaces. Since G is free, each of the closed surfaces has genus 0.

Now let S^* be the union of compact bordered surfaces obtained by "cutting" S along each of the w_1. The genus of each component of S^* is precisely the genus of one of the surfaces of K. Hence each component of S^* has genus zero.

Now assume that S^* has more than one component. Then there must be some loop w_j for which the corresponding boundary contours lie in different components, say S_k^* and S_l^*. Then since S_k^* is of genus 0, on S_k^* there would be a free homotopy relation $w_j \sim \prod_{\alpha=1}^{s} w_{i\alpha}$, where for $\alpha = 1, \cdots, s$, $i_\alpha \neq j$. This free homotopy is equally valid on S, and so the smallest normal subgroup of $\pi_1(S)$ containing $w_1, \cdots, w_q$ is equal to the smallest normal subgroup containing $w_1, \cdots, w_{j-1}, w_{j+1}, \cdots, w_q$. Therefore the assumption that S^* is not connected contradicts the minimality of q. Hence S^* is connected.

We thus have shown that $q = g$, that each w_i is nondividing, and that the w_i are homologously independent.

We now pick a point $x_0 \in \text{int}(S^*)$, and a point $z_0 \in R_0$, with $p(z_0) = x_0$. With this choice $p^{-1}(S^*)$ is well defined, for every loop on S^* corresponds to a loop on S which lifts to a loop. We observe further that $p^{-1}|S^*$ is one-to-one, for on S, no conjugate curve to a w_i can lift to a loop. Let $D = p^{-1}(\text{int}(S^*))$, and for $i = 1, \cdots, q$ let A_i be that element of G which identifies the two pre-images of w_i. We have to show that $A_1, \cdots, A_q$ generate G, and that for $i = 1, \cdots, q$, $A_i(D) \cap D = \emptyset$.

That $A_1, \cdots, A_q$ generate G follows at once from the fact that we can pick disjoint arcs $V_1, \cdots, V_q$ in $\bar{D}$, with the endpoints of V_i identified by A_i. Then, up to a choice of base point, the set of loops $w_1, p(V_1), \cdots, w_q, p(V_q)$ generate $\pi_1(S)$. That $A_i(D) \cap D = \emptyset$ follows at once from the fact that p/D is one-to-one.

3. We now prove Theorem 1. G is a finitely generated, purely loxodromic, free Kleinian group. Let R_1 be some region of discontinuity of G and let H be that subgroup of G which leaves R_1 invariant. Let R_0 be the region of discontinuity of H with $R_0 \supset R_1$. By Ahlfors theorem, H is finitely generated, and so, by Theorem 1′, H is a Schottky group.

Now R_0/H and R_1/H are both closed surfaces, and so $R_0 = R_1$. Since H is a Schottky group R_0 is dense in the Riemann sphere, and since H and G both have the same action on R_1, $H = G$. Therefore G is a Schottky group.

References

1. L. V. Ahlfors, Finitely generated Kleinian groups, *Am. J. of Math.* **86** (1964), 413–429.

2. V. Chuckrow, Doctoral dissertation, New York University, 1966.

3. B. Maskit, A theorem on planar covering surfaces with applications to 3-manifolds, *Ann. of Math.* **81** (1965), 341–355.

Massachusetts Institute of Technology
Cambridge, Mass., U.S.A.

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