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Multifractal of the Appolonian Tiling†

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Abstract—We analyse the multifractal properties of Gibbs measures and Markov type measures defined on the complementary set of the Appolonian tiling of the plane into circles or into spheres in higher dimension. Multifractal refers to a notion of size that involves the variations of the weight of the measures. We associate thermodynamical quantities such as free energy functions with partitions of exponentially decreasing diameters, and prove some regularity. Finally we introduce the correlation dimension which refers to a quantity that is the most accessible in numerical computations. Copyright © 1997 Elsevier Science Ltd

1. INTRODUCTION

Multifractal analysis is a useful technique to investigate the properties of singular measures. It gives some information about how densely the singularities of a measure are distributed. Some models have been studied:

- those described by a tree structure such as the multiplicative chaos [1–7] class of random measures obtained by random multiplications—which corresponds to a rigorous study of the phase transition of a system with random interactions: in physics and chemistry [7–9], in polymers, spin glasses, thermodynamics, turbulence, travelling waves, rainfall distribution, etc., random measures with fixed support [4, 7] or random support [2], Gibbs measures invariant with respect to a C^2 Axiom A diffeomorphism defined on a two-compact manifold [10], etc.;
- an iterated function system, for which the ergodic time-averages along the process converge to a measure μ—the measure we see on screen when computing

$$\mu := \lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i(x)}$$

  describing the occupation of the attractor by orbits of points under iterations of the map T [11].

A multifractal is a fine decomposition of the support of a measure where this support may (or may not) be a fractal [12, 13]. The Hausdorff dimension d_F of the support (or attractor) is then one particular value among other notable ones, such as the information dimension and the Hausdorff dimension of the measure μ ($= \inf\{\mathrm{HD}(A)/A$ a Borel set and $\mu(A) = 1\}$).

†This paper is dedicated to my late sister Martine (10/11/64–29/04/96).

In this paper we study the multifractal properties of coverings of the plane by an infinite set of mutually tangential circles [14, 15] associated with some measures which appear in turbulence, foams and liquid crystal textures. This follows the work of Söderberg [15] where the Appolonian tiling and its complement set are described from a group-theoretical point of view (Kleinian or Lorentz groups). There is a symmetric tree structure that we can see as an alphabet, and this is analysed in different dimensions. The particular problem of calculating the Hausdorff dimension d_F is treated in dimension 2 [15], and in any dimension [14] where a lower bound is found which is strictly greater than 1. Although it may not be a fractal, we can analyse its multifractal properties via some measures. The measures we deal with are very common in physics and are very numerous: these are Gibbs measures and Markov measures. There are many examples that are very typical, and are developed for large families of measures: [2, 4, 10, 12, 16–20].

See El Naschie [21–23] for complete references to the applications.

2. THE FRAMEWORK

Let (X, μ, T) be a dynamical system where X is a metric compact space, T a transformation onto X and μ a T-invariant measure on X. Multifractal analysis is concerned with the decay rates of the measures $\mu(B(x, r))$ of balls of radius r as r goes to zero. To further develop the study of the concentration of the weights of the measure μ, we define for this purpose the singularities of the measure μ at a point x by

$$\alpha^+(X) = \overline{\lim_{r\to 0}} \frac{\log \mu(\mathrm{B}(x, r))}{\log r} \qquad \text{and} \qquad \alpha^-(x) = \underline{\lim_{r\to 0}} \frac{\log \mu(\mathrm{B}(x, r))}{\log r} \tag{1}$$

and $\alpha(x) = \alpha^+(x) = \alpha^-(x)$ when they are equal; hence $\alpha(x)$ represents a local dimension, i.e. the measure μ has pointwise dimension $\alpha(x)$: $\mu(U) \sim |U|^{\alpha(x)}$. One has found relevant information in the singularity sets [12]:

$$C_\alpha^+ = \{x/\alpha^+(x) = \alpha\}, \qquad C_\alpha^- = \{x/\alpha^-(x) = \alpha\} \qquad \text{and} \qquad C_\alpha = C_\alpha^+ \cap C_\alpha^- \tag{2}$$

Using the thermodynamic formalism [10, 11, 18], we introduce partition functions associated with a sequence of partitions $(Q_k)_{k\geq 1}$ whose diameters tend to zero and which are defined for any real β by

$$Z_k(\beta) = \sum_{\substack{U\in Q_k \\ \mu(U)>0}} \mu(U)^\beta \tag{3}$$

and when it exists a free energy function derived from

$$F(\beta) = -\lim_{k\to+\infty} \frac{1}{k} \log_{\mathrm{m}} Z_k(\beta) \tag{4}$$

It is proved in Refs [10, 11, 16, 18, 20] that the dimension spectrum $f(\alpha)$, i.e. the Hausdorff dimensions of the singularity sets

$$\mathrm{f}(\alpha) = \mathrm{HD}(C_\alpha) \qquad \text{and} \qquad f \equiv -\infty \text{ when the sets are empty} \tag{5}$$

is actually, under suitable assumptions, the Legendre–Fenchel transform of the free energy function, i.e.

$$f(\alpha) = \inf_{t\in\mathbb{R}} \{t\alpha - F(t)\} \tag{6}$$

In this scheme we have to prove the existence and regularity of the free energy function F (in fact it is the inverse function of a more intrisic free energy function P, the dynamical free energy function, derived from the dynamical partitions: the so-called Markov partitions). One can see also that in these problems thermodynamic quantities like entropy and Lyapunov exponents are involved [10, 16, 20]. Using large deviation results it is easy to prove the inequality $\mathrm{HD}(C_\alpha^\pm) \leqslant f(\alpha)$. To prove the reverse inequality: $\mathrm{HD}(C_\alpha^\pm) \geqslant f(\alpha)$, we apply a Frostman's lemma to a measure constructed on a set $V_\alpha \subset C_\alpha$ (this construction is recursive and depends on several appropriate sequences).

3. THE MODEL

3.1. *Symbolics*

We define first the symbolic dynamical system $(\tilde{S}, \rho, \sigma)$ where we have

$$\tilde{S} = \{1, 2, \ldots, d+1\}^{\mathbb{N}^*}$$

the transformation σ is the shift

$$\sigma((x_i)_{i\geqslant 1}) = ((y_i)_{i\geqslant 1}) \text{ where for any integer } i \geqslant 1,\ y_i = x_{i+1};$$

the measure ρ is σ-invariant and is defined, for any integer $n \geqslant 1$, on the cylinders included in $\tilde{S}$,

$$C(n, \underline{y}) = \{\underline{x} \in \tilde{S}/x_1 = \underline{y}, \ldots, x_n = y_n\}$$

by the formula

$$\rho(C(n, \underline{y})) = \prod_{j=1}^{n} \theta y_j \tag{7}$$

where

$$\forall j = 1, \ldots, d+1 \qquad \theta_j \geqslant 0 \qquad \text{and} \qquad \sum_{j=1}^{d+1} \theta_j = 1$$

By the Kolmogorov consistency theorem, we take the measure ρ to be the unique probability measure on the Borel subsets of $\tilde{S}$ satisfying eqn (7). We can also define a sequence $(X_i)_{i\geqslant 1}$ of independent identically distributed random variables such that for any integer $k \in [1; d+1]$

$$\rho(X_i = k) = \theta_k \tag{8}$$

The measure ρ is the Gibbs measure of the locally constant function

$$b(\underline{x}) = -\mathrm{Log}\,\Pi(x_0, x_1)$$

where the $(d+1) \times (d+1)$ matrix Π is given by

$$\Pi_{i,j} = \rho(X_2 = j/X_1 = i), \qquad \text{i.e. } \mathbf{\Pi} = \begin{pmatrix} \theta_1 & \ldots & \theta_{d+1} \\ \vdots & & \vdots \\ \theta_1 & \ldots & \theta_{d+1} \end{pmatrix} \tag{9}$$

The log of the Jacobian of the measure ρ at a point $\underline{x}$ of $\tilde{S}$ is locally constant in cylinders of a certain level and is given by

$$-\mathrm{Log}\left[\frac{\rho(X_2 = i_1)}{\rho(X_1 = i_2)}\Pi_{i_1,i_2}\right] = -\mathrm{Log}\,\rho(X_1 = i_1) \tag{10}$$

and it is homologous to $b(\underline{x})$. The measure ρ is then the unique equilibrium state associated to $b(\underline{x})$ (see Appendix 1 of Ref. [24] where a shift-invariant measure is approximated weakly and in entropy by a mixing Markov measure).

In general, we can also take the measure ρ to be a Markov measure, i.e.

$$\rho(\{\underline{x} \in \tilde{S}/x_1 = i_1, \ldots, x_k = i_k\}) = p(i_1)\Pi(i_1, i_2) \ldots \Pi(i_{k-1}, i_k) \tag{11}$$

where $\mathbf{\Pi}$ is an irreducible $(d + 1) \times (d + 1)$ matrix and $\mathbf{p}$ a left eigenvector of $\mathbf{\Pi}$ ($\mathbf{p\Pi} = \mathbf{p}$) which is given by the Perron–Frobenius theorem and whose entries are positive (see eqn (9)). The measure ρ, defined on the cylinders of $\tilde{S}$, can be extended by the Kolmogorov's consistency theorem to a unique shift-invariant Borel probability measure on $\tilde{S}$. The sequence $(X_i)_{i \geq 1}$ of eqn (8) is actually a Markov chain.

3.2. *The dynamical system*

We now define the dynamical system $(\bar{S}, \mu, T)$. The set $\bar{S}$ is the self-similar fractal planar set (the complementary set of the Appollonius tiling [15]). The circles are defined in terms of curvatures, and are provided by a recursive method involving linear relations [14, eqn (2.3), 15, p. 1861] (and $d = 2$). However, we can develop the theory for any integer d. The linear relations between the curvatures of the spheres are represented by the transformations T_j for $j = 1, \ldots, d + 1$ which are conformal maps which send the central sphere to the sphere S_j (see Fig. 1 for $d = 2$), and are therefore contractions:

$$\forall j \in [1; d + 1], \qquad ||T_j(x) - T_j(y)|| = K_j||x - y|| \qquad \text{where } K_j \in (0;1)$$

The starting configuration is S_0 and the $(d + 1)S_j$ are generated from S_0 by the T_j (we take

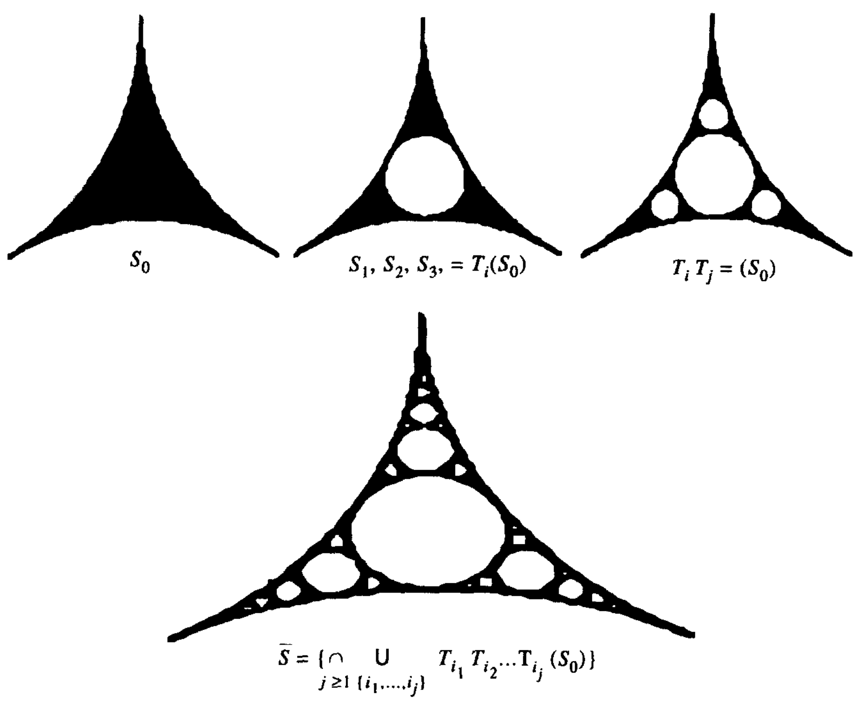

Fig. 1

$T := T_j$ on S_j) [15]. We obviously get the invariance of $\overline{S}$ under the T_j, since we get the self-similarity equation which generates $\overline{S}$ recursively

$$\overline{S} = \bigcup_{k=1}^{d+1} T_k(\overline{S}) = \left\{\bigcap_{j\geq 1} \bigcup_{\{i_1 \ldots, i_j\}} T_{i_1} T_{i_2} \ldots T_{i_j}(S_0)\right\}$$

Figure 1 shows the evolution from S_0 to $\overline{S}$.

Moreover, there exists a surjection φ which codes each point of $\overline{S}$:

$$\varphi: \tilde{S} = \{1, 2, \ldots, d + 1\}^{\mathbb{N}^*} \to \overline{S} \tag{12}$$

$$(i_1, i_2, \ldots) \to \bigcap_{j\geq 1} T_{i_1} T_{i_2} \ldots T_{i_j}(\mathrm{S}_0)$$

The map φ is bounded-to-one and one-to-one on a set of μ measure 1, where the measure μ is the image under the map φ of the measure ρ, i.e.

$$\mu[\varphi(i_1, i_2, \ldots, i_k)] = \rho(i_1, i_2, \ldots, i_k) = p(i_1)\Pi(i_1, i_2) \ldots \Pi(i_{k-1}, i_k) \tag{13}$$

The measure μ is in fact the unique Borel probability measure on $\overline{S}$ satisfying eqn (13), it is a Gibbs measure or a Markov type measure following eqns (9) and (11), and is invariant with respect to the T_j and T. The map φ is then an isomorphism of dynamical systems between $(\tilde{S}, \rho, \sigma)$ and $(\overline{S}, \mu, T)$ [25].

4. THERMODYNAMICS

We compute the dynamical free energy function F eqn (4) associated with the dynamical partition $(\mathcal{P}_n)_{\geq 1}$ which covers $\overline{S}$ (called the Markov partition image under φ of $(\mathcal{C}_n)_{n\geq 1}$, the set of cylinders of length n). Hence we obtain the following proposition:

Proposition: *For any real β the limit of the sequence*

$$-\frac{1}{n} \mathrm{Log}_{d+1} \sum_{A \in \mathcal{P}_n} \mu(A)^{\beta}$$

exists and defines a concave function F.

Let $\beta \in \mathbb{R}$. Actually we prove an inequality (see Refs [11, 26]) of the type

$$\frac{1}{c(\beta)} Z_n(\beta) \frac{1}{c(\beta)} Z_m(\beta) \geq \frac{1}{c(\beta)} Z_{n+m}(\beta) \tag{14}$$

and we know by Refs [11, 18, 26] that the sequence $(\log_{d+1} Z_n(\beta))_{n\geq 1}$, which is subadditive, converges to a concave function, which ends the proposition.

It can be proved that this function F is real analytic, is strictly increasing and is either linear (degenerate case) or strictly concave. To this end, let the matrix $A(\beta, \gamma)$ be defined for any pair of reals (β, γ) by

$$\forall (i, j) \in [1; d + 1]^2, \qquad A_{i,j}(\beta, \gamma) = p(i)^{\beta} \Pi_{i,j}^{\beta} K_j^{\gamma} \tag{15}$$

Each entry of the matrix $A(\beta, \gamma)$ is analytic in both variables, and the matrix $A(\beta, \gamma)$ is irreducible (we can reach any state j from the state i in a finite number of iterates, i.e. $\forall (i, j) \in [1; d + 1]^2$, $\exists n > 0$ $[\{A(\beta, \gamma)\}^n]_{i,j} > 0$—this represents the number of cylinders of $\mathcal{C}_n$ starting at i and finishing at j). The largest zero of a polynomial is an analytic function of the coefficients in the region where it is a simple zero [19, Prop. 3:1], and this assures (by the Perron–Frobenius theory of nonnegative matrices) the existence and the uniqueness of the real γ_1 for which $A(\beta, \gamma)$ has spectral radius equal to 1: we then get $\gamma_1 = -F(\beta)$; for example, when $\beta = 0$, we have $\gamma_1 = -F(0) = d_F$.

The Legendre–Fenchel transform f of the free energy function F (eqn (6)) is defined on an interval $]\alpha_{\inf}; \alpha_{\sup}[\subset \mathbb{R}^{+*}$ (with possibilities of limits at the boundaries) in the general case, and is defined only in one point in the degenerate case (the fractal dimension $d_F = -F(0)$, and we get for any real β: $F(\beta) = d_F(\beta - 1)$). We may note that the function F' realizes a bijection, analytic and strictly decreasing ($F'' < 0$), between $\mathbb{R}$ and $]\alpha_{\inf}; \alpha_{\sup}[$, [27], i.e. $\forall \alpha \in]\alpha_{\inf}; \alpha_{\sup}[$, $\exists!\ \beta \in \mathbb{R}$ such that $\alpha = F'(\beta)$ (conversely, $\forall \beta \in \mathbb{R}, \exists!\ \alpha \in]\alpha_{\inf}; \alpha_{\sup}[$ such that $\beta = f'(\alpha)$). A relation is satisfied between the two conjugate functions which assures that f is really analytic on the interval $]\alpha_{\inf}; \alpha_{\sup}[$:

$$\forall \beta \in \mathbb{R}, \quad f(F'(\beta)) = \beta F'(\beta) - F(\beta) \tag{16}$$

Moreover, the function f is strictly concave on $]\alpha_{\inf}; \alpha_{\sup}[$;

- in the degenerate case, we have

$$\forall \beta \in \mathbb{R}, \quad F(\beta) = d_F(\beta - 1) \quad \text{and} \quad f(d_F) = d_F \quad \text{and} \quad f \equiv -\infty \text{ otherwise};$$

- in the general case, we get from eqn (16)

$$\forall \beta \in \mathbb{R}, \quad f''(\alpha(\beta)) = \frac{1}{F''(\beta)} < 0 \quad \text{and} \quad f \text{ is analytic on }]\alpha_{\inf}; \alpha_{\sup}[$$

and $f \equiv -\infty$ on $\mathbb{R}\backslash[\alpha_{\inf}; \alpha_{\sup}]$.

For any $\underline{x} \in \tilde{S}$, let

$$\eta(\underline{x}/n) = \frac{\operatorname{Log} p(x_{i_1})\Pi(x_{i_1}, x_{i_2}) \ldots \Pi(x_{i_{n-1}}, x_{i_n})}{\prod_{i=1}^{n} K_{X_i}}$$

and let

$$\alpha_{+\infty} = \inf \eta(\underline{x}/n) \quad \text{and} \quad \alpha_{+\infty} = \inf \eta(\underline{x}/n) \tag{17}$$

where the infimum and the supremum are taken over n and all cycle paths, i.e. starting and finishing at the same state with no repetition: $(i, j, k, \ldots, v, i)$ with possibilities of loops: (i, i). Following Refs [10, 11, 19], we prove that

$$\alpha_{+\infty} = \lim_{\beta \to +\infty} F'(\beta) = \alpha_{\inf} \quad \text{and} \quad \alpha_{-\infty} = \lim_{\beta \to -\infty} F'(\beta) = \alpha_{\sup} \tag{18}$$

In the next section we relate the function f to the Hausdorff dimensions of the singularity sets.

5. DIMENSION SPECTRUM

We study the general case when the sets C_α are not empty for some $\alpha \neq d_F$. Using a large deviations theorem [27, Th. II 6.1, p. 47], we easily obtain by a counting argument [10, eqn (2.3.10)]

$$\mathrm{HD}(C_\alpha^{\pm}) \leqslant f(\alpha) \quad \text{for} \quad \alpha \in]\alpha_{+\infty}; \alpha_{-\infty}[\tag{19}$$

The 'reverse' inequality

$$\mathrm{HD}(C_\alpha) \geqslant f(\alpha) \quad \text{for} \quad \alpha \in]\alpha_{+\infty}; \alpha_{-\infty}[\tag{20}$$

which is much harder to prove, is obtained generally by two methods:

• the first [4, 10, eqn (2.3.10)] uses a Frostman's lemma applied to a measure μ_α concentrated on a set $V_\alpha \subset C_\alpha$, i.e. for balls B_ε where $\varepsilon > 0$ and small enough,

$$\mu_\alpha(B_\varepsilon) \leqslant C\varepsilon^{f(\alpha)} \tag{21}$$

This expression gives eqn (20) and provides an estimate of the Hausdorff measure of the sets C_α. The construction of the measure μ_α and of the set V_α is recursive (we see these elements 'appear') and depends on appropriate sequences. This is well developed in Ref. [4];

• the second [2, 10, Th. 3.1, 19, Lemma 4.1] uses the properties of the measure μ_α where

$$\mu_\alpha(C_\alpha) = 1 \quad \text{and} \quad \mathrm{HD}(\mu_\alpha) = \inf\{\mathrm{HD}(A)/A \text{ a Borel set } \mu_\alpha(A) = 1\} = f(\alpha).$$

The measure μ_α is actually a Gibbs measure or a Markov type measure and is obtained by a variational principle. Hence, we do not clearly see the evolution of this measure (this is an existency theorem—and unique) rather than the first method, but it is well fitted to the singularity set C_α since we get

$$\mathrm{HD}(\mu_\alpha) = F(\alpha) = \mathrm{HD}(C_\alpha) \quad \text{and} \quad \forall x \in C_\alpha, \quad \lim_{r\to 0} \frac{\log \mu_\alpha(B(x, r))}{\log r} = f(\alpha)$$

This is done, for example, in Refs [10, 20].

We can see that the second method may also work for the boundaries $\alpha_{+\infty}$ and $\alpha_{-\infty}$ by taking limits in some variational principles [10: eqns (2.3.11), and (2.3.15)] as is the case here.

For numerical computations of the Hausdorff dimension d_{F} of the attractor, for example, one has to choose a length measure, i.e.

$$Z_n(\beta) = O(1) \sum_{C\in\mathscr{C}_n} m(C)^\beta$$

where $m(C)$ describes 'some measure' of C. Since the curvatures of the circles, which satisfy some identities by the T_j [15], the length measure m can be given in terms of $1/q$ where q is the curvature, and gives a relation of the type

$$Z_n(\beta) = O(1) \sum_{C\in\mathscr{C}_n} \frac{1}{q^\beta}$$

and the value d_{F} satisfies $Z_n(d_{\mathrm{F}}) \approx 1$ when n goes to $+\infty$ ($\approx$ means a finite level and we have in Ref. [15] $d_{\mathrm{F}} \sim 1.305$).

6. CORRELATION DIMENSION

We define a natural quantity

$$C(r, \beta) = \frac{\log \int \mu(B(x, r))^\beta \mu(\mathrm{d}x)}{\log r}$$

which is most accessible in numerical computations based on the time-series of a dynamical system [28, 29] (a simple numerical procedure is given in Ref. [28]).

In Ref. [24] we have proved the existence of the limit

$$\forall \beta \in \mathbb{R}, \qquad C(\beta) = \lim_{r\to 0} C(r, \beta) \tag{22}$$

called the correlation dimension [29]. Actually it is pointed out in Ref. [21] that this function plays an important role in the numerical investigation of some models, and differs in general from other characteristic dimensions such as the Hausdorff dimension, the capacity or the information dimension.

We know that in many examples, including the ones we are studying, we get

$$\lim_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} = F'(1)\ \mu\text{a.s.};$$

by a simple argument we obtain

$$\exists\ \theta \in)0;1(,\ F(2) = F(1) + F'(1 + \theta) = F'(1 + \theta)$$

and in Ref. [26] we have proved that for general situations $F(2) = C(1)$. More precisely, we extend this result to the following variational principle:

$$\forall \beta \in \mathbb{R}, \qquad C(\beta) = F(\beta + 1) \tag{23}$$

We only use the existence of a free energy function F associated with a sequence of uniform partitions $(P_n)_{n\geq 1}$. Then it is quite easy to prove that the relation (23) holds.

In our case, using relation (11), derived from the properties of the Markov chain $((X_i)_{i\geq 1}, \rho)$, we prove the existence of a free energy function associated with a sequence of uniform partitions $(P_n)_{n\geq 1}$ since any element A of the dynamical partition $(C_n)_{n\geq 1}$ at the rank n can be covered by a finite number (uniformly bounded in A and n) of elements of P_n [26]. Hence, the dynamical system $(\overline{S}, \mu, T)$ satisfies eqn (23).

Just for illustration we give the shape of the dimension spectrum $f(\alpha)$ in Fig. 2.

7. CONCLUSIONS

We have thus shown that these dynamical systems (associated with measures that appear in physics and chemistry: polymers, spin glasses, turbulence, etc.) satisfy a typical multifractal decomposition. In Ref. [23] El Naschie suggested a connection between the Kolmogorov theory of turbulence, multidimensional Cantor space, the Cantorian model of

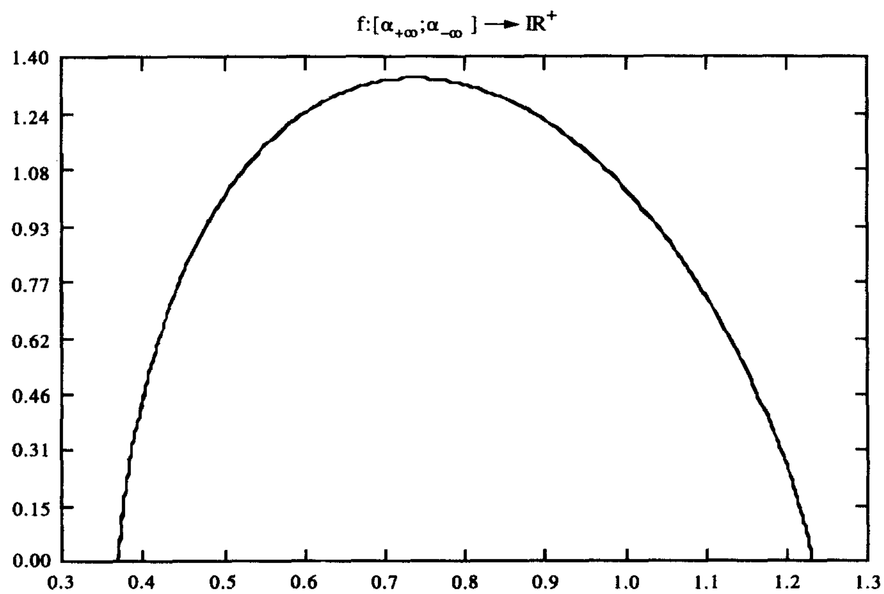

Fig. 2

quantum space–time and the theory of fractal space–time, as, for example, in the continuous trajectories formed by the contact points of the spheres. See Refs [21–23] for more details on the Appolonian tiling and fractal space–time, and the possible connection between classical and quantum mechanics by the chaotic fractals.

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