

Tuesday March 30 (1)

Bost-Connes algebra

First description . generators and relations

$\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ group ring

\mathbb{Q}/\mathbb{Z} additive group $\mathbb{Q} \bmod \mathbb{Z}$
 $r \in \mathbb{Q}/\mathbb{Z}$

basis elements δ_r of $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$
 $\sum a_r \delta_r$ finite sums

where $\delta_r \delta_s = \delta_{r+s}$

use notation $e(r)$ for these generators δ_r

reason: what to "suggest" ... exponential map

$$e(r+s) = e(r) e(s)$$

$$e(0) = 1$$

~~Additional~~ Additional generators μ_n $n \in \mathbb{N}$
 and "adjoints" μ_n^* $n=1, 2, 3, \dots$

with relations

$$\left\{ \begin{array}{l} \mu_n^* \mu_n = 1 \\ \mu_n \mu_n^* = e_n \text{ an idempotent in } \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \\ \mu_{nm} = \mu_n \mu_m \qquad e_n = \frac{1}{n} \sum_{s: ns=0} e(s) \\ \mu_n \mu_m^* = \mu_m^* \mu_n \text{ if } (m,n)=1 \end{array} \right.$$

[need to verify: $e_n^2 = e_n (= e_n^*)$]

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

Algebra defined over \mathbb{Q} $A_{BC, \mathbb{Q}}$

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(over \mathbb{C} : $A_{BC, \mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = A_{BC, \mathbb{C}} = \mathbb{C}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$

\mathbb{C}^* -completion $\mathbb{C}^*(\mathbb{Q}/\mathbb{Z}) \rtimes_p \mathbb{N}$

Note : $\mathbb{C}^*(\mathbb{Q}/\mathbb{Z})$ group \mathbb{C}^* -alg. of abelian group

$\Rightarrow \mathbb{C}^*(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{C}(\hat{\mathbb{Q}/\mathbb{Z}})$ isom. by Fourier transform to alg of cont. funct. on Pontrjagin dual group

$\left. \begin{matrix} \text{Hom}(\mathbb{Q}/\mathbb{Z}, U(1)) \\ \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{C}^*) \end{matrix} \right\}$ a homom. is going to map \mathbb{Q}/\mathbb{Z} to roots of 1 in \mathbb{C}^* because all torsion pts (in $U(1)$)

= All ways of mapping by group homomorphism \mathbb{Q}/\mathbb{Z} to roots of 1

$\rho \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{C}^*) = \lim_{\leftarrow n} \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$ or (the mapping profin completion)
m/n ordered by

pairing $\langle x, \rho \rangle = \exp(2\pi i \rho(x))$
 $x \in \mathbb{Q}/\mathbb{Z}$

$\Rightarrow \mathbb{C}^*(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{C}(\hat{\mathbb{Z}})$

Bord. Connes algebra also described as

$\mathbb{C}(\hat{\mathbb{Z}}) \rtimes_p \mathbb{N}$

Semigroup crossed product:

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The μ_n 's alone generate Semigroup ring $C[N]$

$N =$ multiplicative abelian Semigroup generated by primes

$\{ \mu_p \}_{p=\text{primes}}$ generators

$$\mu_p^* \mu_p = 1 \quad \mu_p \mu_p^* = e_p \text{ projector}$$

\Rightarrow Toeplitz algebra on generators μ_p

$\bigoplus_p \tau_p$
Toeplitz alg on single μ_p generator

Semigroup crossed product:

$A =$ abelian alg. $A = C[\mathbb{Q}/\mathbb{Z}]$ or $\mathcal{Q}[\mathbb{Q}/\mathbb{Z}]$ here

$\forall n \in \mathbb{N} \rightsquigarrow$ endomorphisms

$$\rho_n: \mathcal{Q}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathcal{Q}[\mathbb{Q}/\mathbb{Z}] \text{ endom.}$$

$$\rho_n(e(r)) = \frac{1}{n} \sum_{ns=r} e(s)$$

(note: sends 1 to e_n not to 1)

isom: from $A \otimes \mathcal{Q}[\mathbb{Q}/\mathbb{Z}]$ to reduced alg. by $e_n: e_n A e_n = e_n A$

$$\rho_n(e(r)) = e_n e(s) \quad \forall s \text{ s.t. } ns=r$$

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Also $\rho_n(e(r)) = \mu_n e(r) \mu_n^*$

Have also other family of endom. of $A = \mathcal{O}[\mathbb{Q}/\mathbb{Z}]$ from semigroup \mathbb{N}

$$\sigma_n : \mathcal{O}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathcal{O}[\mathbb{Q}/\mathbb{Z}]$$

$$\sigma(e(r)) = e(nr) \quad \forall n \in \mathbb{N} \quad \forall r \in \mathbb{Q}/\mathbb{Z}$$

These satisfy:

$$\begin{cases} \sigma_n \rho_n(x) = x \\ \rho_n \sigma_n(x) = e_n \cdot x \end{cases}$$

If had a group instead of semigroup action would be implemented by unitaries, not isometries μ_s and would have isomorphisms $U_Y \circ U_Y^*$ and $U_Y^* \circ U_Y$ instead of endom.

Note: multiplication by n

$$n : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \quad \text{is surjective} \\ \text{(divisibility of } \mathbb{Q}/\mathbb{Z} \text{)}$$

Note: ρ_n 's surjective onto $e_n A$ (not onto A !)
 σ_n 's injective $\{ \text{on } e_n A \}$

$$\sigma_n(e(r)) = \mu_n^* e(r) \mu_n$$

Look at representations:

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$$H = \ell^2(\mathbb{N}) \text{ on basis } \{\varepsilon_m\}$$

$$\mu_n \varepsilon_m = \varepsilon_{nm}$$

$$\mu_n^* \varepsilon_m = \begin{cases} 0 & n \nmid m \\ \varepsilon_{\frac{m}{n}} & n \mid m \end{cases} \quad \mu_n^* \mu_n = 1$$

$$\mu_n \mu_n^* = \text{Projection on } \{\varepsilon_m : n \mid m\}$$

$e(r)$: choose an embedding of roots of unity into \mathbb{C} by group homom.

$$\rho \in \hat{\mathbb{Z}}^* = GL_1(\hat{\mathbb{Z}})$$

$$\Rightarrow \pi_\rho(e(r)) \varepsilon_n = \sum_r^n \varepsilon_n$$

$$\sum_r = \rho(e(r)) \text{ root of } 1 : \sum_r = \left(\sum_b\right)^a$$

$r = \frac{a}{b}$ (primitive root of 1)

then all relations satisfied

need to check crossed prod. relation

$$\mu_n \pi(e(r)) \mu_n^* \varepsilon_m = \begin{cases} 0 & n \nmid m \\ \sum_r^{\frac{m}{n}} \varepsilon_m & n \mid m \end{cases}$$

but notice that

$$e_n = \frac{1}{n} \sum_{sn=0} e(s) \text{ acts as}$$

$$\frac{1}{n} \sum_{ns=r} e(s) \quad \epsilon_m = \frac{1}{n} \sum_{ns=r} \sum_s^m \epsilon_m$$

$$s = \frac{a}{b} \quad ns = 0 \quad b/n \quad r = \frac{c}{d}$$

$$\xi_s = \xi_b^a \quad \frac{na}{b} = \frac{c}{d} \quad \frac{n}{b} = \frac{1}{d}$$

- if n/m summing over all roots of 1 of order $\frac{b}{n} = d \Rightarrow$ zero
 - if n/m summing 1 n times with $\frac{1}{n} \Rightarrow$ one
- result

0 n/m } same as projection above
 1 n/m } $e_n = e_n^2 = e_n^*$

Time evolution (Bohr-Connes system)

$$\begin{cases} \sigma_t(\mu_n) = n^{it} \mu_n \\ \sigma_t(e(r)) = e(r) \end{cases}$$

$$(c^*(\mathbb{Z}/2) \rtimes_p \mathbb{N}, \sigma_t)$$

quantum statistical mechanical system

In all representation π_p $f \in \hat{\mathbb{Z}}^*$
on $\mathcal{H} = l^2(\mathbb{N})$

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Hamiltonian of time evolution:

$$\pi_p(\sigma_t(a)) = e^{itH} \pi_p(a) e^{-itH}$$

check for μ_n

$$\pi_p(\mu_n) \varepsilon_m = \varepsilon_{nm}$$

↑
indep of p

$$\begin{aligned} \pi_p(\sigma_t(\mu_n)) \varepsilon_m &= n^{it} \varepsilon_{nm} \\ &= (nm)^{it} m^{-it} \pi_p(\mu_n) \varepsilon_m \end{aligned}$$

get: $e^{itH} \varepsilon_m = m^{it} \varepsilon_m$

$$H \varepsilon_m = \log m \varepsilon_m$$

Hamiltonian (up to constant)

$$\Rightarrow Z(\beta) = \text{Tr}(e^{-\beta H}) = \sum_{n \geq 1} n^{-\beta} = \zeta(\beta)$$

Partition function is the Riemann zeta function

Symmetries of the system:

$\hat{\mathbb{Z}}^*$ acts by automorphisms of the algebra

$$\alpha \in \hat{\mathbb{Z}}^* \quad \alpha(\mu_n) = \mu_n$$

$$\alpha(e(r)) = e(\alpha(r))$$

since $\alpha \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ invertible

Before describing classification of KMS states for $(C^*(\mathbb{Q}/\mathbb{Z})_{\sigma_p, N}, \sigma_T)$

A different more geometric description of the algebra:
 \mathbb{Q} -lattices and commensurability

\mathbb{Q} -lattice in n -dimensions

$\Lambda \subset \mathbb{R}^n$ lattice (abelian group, abstractly isomorphic to \mathbb{Z}^n , embedded in \mathbb{R}^n with compact quotient)

together with

$$\phi: \mathbb{Q}^n / \mathbb{Z}^n \xrightarrow{\phi} \mathbb{Q}\Lambda / \Lambda$$

a group homomorphism (not required to be an isomorphism)

$$\mathbb{R}^n / \Lambda \cong T^n \cong (S^1)^n$$

(torus)

A \mathbb{Q} -lattice (Λ, ϕ) is "invertible" if ϕ is an isomorphism

Equivalence relation on \mathbb{Q} -lattices

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Commensurability $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$

iff $\Lambda_1 \mathbb{Q} = \Lambda_2 \mathbb{Q}$ i.e. the two lattices are commensurable

and $\phi_1 = \phi_2 \pmod{\Lambda_1 + \Lambda_2}$

Note $\Lambda_1 + \Lambda_2$ is a lattice because Λ_1, Λ_2 commensurable (in general not)

It is an equivalence relation:

$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$ and $(\Lambda_2, \phi_2) \sim (\Lambda_3, \phi_3)$

then also $(\Lambda_1, \phi_1) \sim (\Lambda_3, \phi_3)$

• the lattices Λ_i of finite index in $\Lambda_1 + \Lambda_2 + \Lambda_3$

$$\#(\Lambda_1 + \Lambda_2 + \Lambda_3) / \Lambda_i < \infty$$

$$\phi_1 - \phi_2 = 0 \pmod{\Lambda_1 + \Lambda_2} \Rightarrow \pmod{\Lambda_1 + \Lambda_2 + \Lambda_3}$$

$$\phi_2 - \phi_3 = 0 \pmod{\Lambda_2 + \Lambda_3} \Rightarrow \pmod{\Lambda_1 + \Lambda_2 + \Lambda_3}$$

$$\Rightarrow \phi_1 - \phi_3 = 0 \pmod{\Lambda_1 + \Lambda_2 + \Lambda_3}$$

want $\pmod{\Lambda_1 + \Lambda_3}$

induced homom. $\phi_1 - \phi_3 : \mathbb{Q}^n / \mathbb{Z}^n \rightarrow (\Lambda_1 + \Lambda_2 + \Lambda_3) / (\Lambda_1 + \Lambda_3)$

infinitely divisible

finite

\Rightarrow group homom. trivial $\Rightarrow \phi_1 - \phi_3 = 0 \pmod{\Lambda_1 + \Lambda_3}$

Case of 1-dimensional \mathbb{Q} -lattices :

$\Lambda \subset \mathbb{R}$ 1-dimensional
is a scaled copy of \mathbb{Z}

$$\Lambda = \lambda \mathbb{Z} \quad \text{for some } \lambda \in \mathbb{R}_+^*$$

$\phi : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ group homom.

after identifying $\Lambda = \lambda \mathbb{Z}$ ϕ completely determined
by an element

$$\rho \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$

$$\phi = \lambda \rho$$

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Note: in general
commens relation
on \mathbb{Q} -lattices
"bad" equiv. rel.
quotient NC space

1-dimensional \mathbb{Q} -lattices up to scaling

forget the factor $\lambda \in \mathbb{R}_+^*$

\Rightarrow just \mathbb{Z} with an element $\rho \in \hat{\mathbb{Z}}$

algebra of functions: $C(\hat{\mathbb{Z}}) \cong C^*(\mathbb{Q}/\mathbb{Z})$

Commensurability relation:

rescaling lattice by $\frac{n}{m} \lambda$ ~~λ~~ $\frac{n}{m} \mathbb{Z}$ ~~\mathbb{Z}~~

$$\phi_1 = \phi_2 \text{ mod } \lambda_1 + \lambda_2 \quad r = \frac{n}{m} \in \mathbb{Q}_+^*$$

$$\uparrow \quad m\rho \in \hat{\mathbb{Z}}$$

pairs of commens. lattices
 $(\rho_1, \lambda_1) (\rho_2, \lambda_2) \exists r \in \mathbb{Q}_+^* \lambda_2 = r\lambda_1$

up to scaling
 $\rho_2 = r\rho_1$



Groupoid of equiv. rel.

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$$G = \{ (r, p, \lambda) \in \mathbb{Q}_+^* \times \hat{\mathbb{Z}} \times \mathbb{R}_+^* : rp \in \hat{\mathbb{Z}} \}$$

source, range maps

$$(r, p, \lambda) \mapsto (p, \lambda)$$

$$(r, p, \lambda) \mapsto (rp, r\lambda) \quad \text{composition}$$

$$(r_1, p_1, \lambda_1) \circ (r_2, p_2, \lambda_2) = (r_1 r_2, p_2, \lambda_2)$$

$$\text{if } r_2 p_2 = p_1 \quad r_2 \lambda_2 = \lambda_1$$

Groupoid algebra $C^*(G/\mathbb{R}_+^*)$

quotient of groupoid G by
Scaling action
(still groupoid)

$$(r_1, p_1) \circ (r_2, p_2) = (r_1 r_2, p_2) \quad r_2 p_2 = p_1$$

$$\left\{ \begin{aligned} (f_1 * f_2)(r) &= \sum_{r=r_1 r_2} f_1(r_1) f_2(r_2) \\ f^* &= \overline{f} \end{aligned} \right.$$

This algebra is the same as Bost-Connes algebra

$$C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$$

Time evol. by volume of commensurable lattices

$$\sigma_t(f)(r, p) = r^{it} f(r, p) = \left(\frac{\text{Covol}(\Lambda_1)}{\text{Covol}(\Lambda_2)} \right)^{it} f((r_1, p_1), (r_2, p_2))$$

$n\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}$ range of multiplication
(open and closed)

$e_n = \chi_{n\hat{\mathbb{Z}}}$ characteristic function $\in C(\hat{\mathbb{Z}})$

$$e_n e_m = e_{\text{lcm}(n,m)}$$

$e_n = \chi_{\{\mathbb{Q}\text{-lattices divisible by } n\}}$

\uparrow
(Λ, ϕ) divisible by n
if $\phi_n: (\Lambda/\mathbb{Z})^n \rightarrow \Lambda/\Lambda$
is $\phi_n = 0$.

Semigroup action

$$\alpha_n(f)(\Lambda, \phi) = f(n\Lambda, \phi)$$

if $(\Lambda, \phi) \in \text{supp}(e_n)$
commensurability relation
zero otherwise

same as

$$\alpha_n(f)(p) = \begin{cases} f(n^{-1}p) & p \in n\hat{\mathbb{Z}} \\ 0 & p \notin n\hat{\mathbb{Z}} \end{cases}$$

$$\mu_n(r, p) = \begin{cases} 0 & r \neq n \\ 1 & r = n \end{cases}$$

$(r, p) \leftrightarrow (r^{-1}\mathbb{Z}, p), (\mathbb{Z}, p)$
pair of commens.
 \mathbb{Q} -lattices

$$\mu_n * f * \mu_n^* = \alpha_n(f)$$

convol prod
in $C^*(G_1/R_1^*)$