Euler-Lagrange equations

$$\frac{\partial L}{\partial q^i}(q,\dot q) = \frac{d}{dt}\frac{\partial L}{\partial \dot q^i}(q,\dot q)$$

$$= \sum_{j=1}^{n} \frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}(q,\dot q)\, \ddot q^j + \frac{\partial^2 L}{\partial \dot q^i \partial q^j}(q,\dot q)\, \dot q^j \Big)$$

If want to make this solvable for highest derivative
need to invert matrix

$$H_L(q,\dot q) = \left(\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}(q,\dot q)\right)_{ij}.$$

This is the Hessian matrix of L w/ respect to the $\dot q$ coordinates

<u>Def</u>: (M,L) Lagrangian system <u>non-degenerate</u>
if in every coord. chart H_L is invertible

(Note: $H_L \longrightarrow J(f)^t H_L J(f)$ under coord. changes
so invertibility well def.)

<u>Equivalent description of the invertibility condition</u>

1-form $\quad \theta_L = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot q^i} dq^i = \frac{\partial L}{\partial \dot q} \cdot dq$

well def. under coord. charts

(M,L) nondegenerate iff

2-form $\quad d\theta_L$ is non-degenerate

$$d\theta_L = \sum_{i,j=1}^{n} \left(\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}\, d\dot q^j \wedge dq^i + \frac{\partial^2 L}{\partial \dot q^i \partial q^j}\, dq^j \wedge dq^i \right)$$

non-degen. take $d\theta_L^n = \underbrace{d\theta_L \wedge \cdots \wedge d\theta_L}_{n\text{-times}}$

$2n$-form : top forms

$= f \cdot \omega_{vol}$ some function times volume form

non-deg. if $f \neq 0$ but if write out what this $d\theta_L^n$ is find $\det\left(\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}\right) d\dot q^1 \wedge \cdots \wedge d\dot q^n \wedge dq^1 \wedge \cdots \wedge dq^n$

$I = i_{X'}(\theta_L)$ the Noether integral

$$I(q,\dot q) = \frac{\partial L}{\partial \dot q} \cdot a$$

where $X = \sum a^i(q) \frac{\partial}{\partial q^i}$ vector field of flow of one-param. symmetries

X' lift to TM of vect. field X on M

$$X' = \sum a^i(q,\dot q) \frac{\partial}{\partial q^i} + \sum b^i(q,\dot q) \frac{\partial}{\partial \dot q^i} \quad \text{w/ } b^i = 0 \quad a^i(q,\dot q) = a^i(q))$$

(because θ_L 1-form in $T^*(TM)$ not in T^*M)

Moreover X being infinitesimal symmetry implies

$$\mathcal{L}_{X'}(\theta_L) = 0$$

TM tangent bundle and T^*M cotangent bundle

$(q, \dot{q})$ coordinates $\qquad$ (q, p) coordinates

$$p_i = \text{dual basis to the } \frac{\partial}{\partial q^i} \quad \text{i.e. } dq^i$$

i.e. $p_i(df) = \dfrac{\partial f}{\partial q^i}$

Def: Liouville canonical 1-form on T^*M

$$\theta = \sum_{i=1}^{n} p_i \, dq^i = p \cdot dq$$

(again well def under coordinate changes)

invariant definition:

$v \in T_{(q,p)}(T^*M)$ $\qquad$ $\theta(v) = p\big(\underbrace{\pi_*(v)}_{\in \, TM}\big)$

$\pi : T^*M \to M$

Legendre transformation

$$\begin{tikzcd} TM \arrow[r, "\tau_L"] \arrow[dr] & T^*M \arrow[d] \\ & M \end{tikzcd}$$

$\tau_L : TM \to T^*M$ (fiberwise map)

s.t. $\tau_L^*(\theta) = \theta_L$

$\tau_L(q, \dot{q}) = (q, p)$ with $p = \dfrac{\partial L}{\partial \dot{q}}(q, \dot{q})$

(4) τ_L is a local diffeomorphism iff H_L invertible
(M,L) nondegenerate

<u>Comment</u>: on any mfld have TM & T^*M paired by a duality

<u>but</u> cannot "identify" TM & T^*M unless have extra structure on M

<u>example</u>: if M has a Riemannian metric $g_{\mu\nu}$ then can use this to identify (non-canonically) TM & T^*M (lowering/raising indices)

$$g_{\mu\nu} v^{\nu} = \omega_\mu$$

This is a special case of the identif. via Legendre transform when L ~~only~~ kinetic term given by $g_{\mu\nu}$

$$p_\mu = \left(\frac{\partial L}{\partial \dot q}\right)_\mu = g_{\mu\nu}(q)\, \dot q^\nu$$

$$L(q,\dot q) = g_{\mu\nu}(q)\, \dot q^\mu \dot q^\nu$$

Hamiltonian function $H : T^*M \longrightarrow \mathbb{R}$

$$H \circ \tau_L = E_L = \dot q \frac{\partial L}{\partial \dot q} - L$$

$$\Rightarrow \quad H(q,p) = \left(p\dot q - L(q,\dot q)\right)\Big|_{p = \frac{\partial L}{\partial \dot q}}$$

Legendre transform of Lagrangian

Legendre transform

$H = \mathcal{L}(L)$ is invertible (equiv. to convexity assumption in simple one-dim case)

if τ_L invertible

i.e. $\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}$ invertible ; then $\mathcal{L}$ involution

Assume invertibility: (which in phys. terms is an assumption on pos. def. of kinetic energy term in L)

then Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i} = 0$$

are equivalent to a system of first order equations

$$\begin{cases} \dot p_i = -\dfrac{\partial H}{\partial q^i} \\ \dot q_i = \dfrac{\partial H}{\partial p_i} \end{cases}$$

Hamiltonian equations

Pf:

$$dH = \frac{\partial H}{\partial q}\cdot dq + \frac{\partial H}{\partial p}\, dp$$

$$= \cancel{} \left(\underline{p \cdot d\dot q} + \dot q\, dp - \frac{\partial L}{\partial q}\cdot dq - \underline{\frac{\partial L}{\partial \dot q}\cdot d\dot q} \right)\Bigg|_{p = \frac{\partial L}{\partial \dot q}}$$

$$= \left(\dot q\, dp - \frac{\partial L}{\partial q}\cdot dq \right)\Bigg|_{p = \frac{\partial L}{\partial \dot q}}$$

$$\Rightarrow \dot q = \frac{\partial H}{\partial p} \quad \& \quad -\frac{\partial H}{\partial q} = +\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q} = \dot p$$

(6) Conservation of energy:

H is constant along trajectories of motion

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0$$

Given a Hamiltonian $H : T^*M \longrightarrow \mathbb{R}$

& corresp. equations

$$\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\ \dot{q} = \dfrac{\partial H}{\partial p} \end{cases} \Rightarrow \text{vector field on } T^*M$$

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \right) = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

<u>Hamiltonian Vector field</u>

if can integrate this vector field (integral curves exist for all t)

$\Rightarrow$ Hamiltonian phase flow

$\{g_t\}_{t \in \mathbb{R}}$ one-parameter group of $\mathrm{Diff}(T^*M)$

$g_t(q,p) = (q(t), p(t))$ solution with $(q(0), p(0)) = (q, p)$

Symplectic form on T^*M

Liouville 1-form θ on T^*M $\qquad$ $\theta = p \cdot dq$ in loc. coord's

$\Rightarrow$ $d\theta = \omega$ 2-form $\qquad$ in coord's

$$\omega = dp \wedge dq = \sum_{i=1}^{n} dp_i \wedge dq^i$$

non-degenerate 2-form

$(d\theta)^n$ = volume form

$dp_1 \wedge \cdots \wedge dp_n \wedge dq^1 \wedge \cdots \wedge dq^n$

ω = canonical symplectic form on T^*M (in Darboux coordinates)

<u>Symplectic manifolds</u> (generalization of T^*M)

M smooth manifold dim $2n$ (even real dimension)

w/ a closed 2-form ω $\quad (d\omega = 0)$

which is non-degenerate: $\omega^n \neq 0$ on all M
(nowhere vanishing)

$\frac{\omega^n}{n!}$ = Liouville volume form on M

Lagrangian submanifold $\mathcal{L} \subset M$ if

$\dim \mathcal{L} = \frac{1}{2} \dim M$ & $\omega|_{\mathcal{L}} = 0$

symplectomorphism $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$

smooth $\varphi : M_1 \to M_2$ with $\omega_1 = \varphi^*(\omega_2)$

Note:

Darboux's theorem ω on M sympl. can always be written locally as $\omega = dp \wedge dq$

made people erroneously think sympl. mflds were easy. In fact very complicated classes of mflds when studied w/ better invariants (GW; Floer, etc.)

(8) non-degenerate 2-form ω on T^*M (or more generally on sympl. mfld $\mathcal{M}$)

defines isomorphism $J : \underset{T^*M}{T^*(\mathcal{M})} \longrightarrow \underset{T^*M}{T(\mathcal{M})}$ by

setting

$$\omega(v_1, v_2) = \underbrace{\langle J^{-1}(v_1), v_2 \rangle}$$

$v_i \in T_{(p,q)}(T^*M)$ pairing by duality of $T^*\mathcal{M}$ & $T\mathcal{M}$

i.e. $J^{-1}(v_1) = \omega(v_1, \cdot)$ (just like pairing done via a Riem. metric also a non-deg. quadr. form)

Hamiltonian vector field is just

$$J(dH) = X_H$$

Note: can define a Hamiltonian & Ham. flow on more general symplectic manifolds than just T^*M

Hamiltonian phase flow preserves the symplectic form : symplectomorphisms not just diffeomorphisms

enough to check

$$\frac{d}{dt}(g_t)^*\omega \Big|_{t=0} = \mathcal{L}_{X_H}(\omega) = 0$$

$\mathcal{L}_X(\alpha \wedge \beta)$ derivation : $\mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$

$$\mathcal{L}_X(df) = d(X(f)) \quad \text{by def. of } \mathcal{L}_X \text{ on diff forms}$$

check: $\mathcal{L}_X(dp_i) = -d\left(\frac{\partial H}{\partial q^i}\right)$ $\quad \mathcal{L}_X(dq^i) = d\left(\frac{\partial H}{\partial p_i}\right)$

then get: $\mathcal{L}_X(\omega) = \sum^n \mathcal{L}_X(dp_i) \wedge dq^i + dp_i \wedge \mathcal{L}_X(dq^i) =$

$$= \sum_{i=1}^{n} \left(-d\left(\frac{\partial H}{\partial q^i}\right) \wedge dq^i + dp_i \wedge d\left(\frac{\partial H}{\partial p_i}\right)\right) = -d(dH) = 0$$

$\Rightarrow$ also $\{g_t\}$ Hamiltonian flow preserves Liouville volume form

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Least action principle in phase space

TM
$(q, \dot q)$
configuration space M

here least action:

$$S(\gamma) = \int L(\gamma'(t), t)\, dt$$

$\gamma \in \mathcal{P}(M)$

$$\frac{d}{d\varepsilon} S(\gamma_\varepsilon)\Big|_{\varepsilon=0} = 0$$

T^*M
(q, p)
phase space

Poincaré-Cartan form

$$\theta - H dt = p \cdot dq - H dt$$

(note: relativistic viewpt. $\langle (p, H), (dq, dt) \rangle$ w Lorentzian metric: Energy = time component of momentum

1-form on $T^*M \times \mathbb{R}$ (extended phase space)

$\gamma : [t_0, t_1] \to T^*M$ path

lift σ of γ to extended phase space

$\sigma(t) = (\gamma(t), t)$ admissible paths on $T^*M \times \mathbb{R}$ = lifts of paths on T^*M

$\tilde{\mathcal{P}}(T^*M)$

σ_ε = variation of an admissible path by admissible paths w/ fixed endpts

$$\delta\sigma = \frac{\partial \sigma_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} \quad \text{infinitesimal variation}$$

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$$S(\sigma) = \int_\sigma p\,dq - H\,dt = \int_{t_0}^{t_1} (p\dot{q} - H)\,dt$$

Poincaré action functional on phase space:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\sigma_\varepsilon) = 0 \quad \text{critical points}$$

Check: iff satisfy Hamiltonian equations

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\sigma) = \sum_{i=1}^{n} \int_{t_0}^{t_1} \left(\dot{q}^i \delta p_i - \dot{p}_i \delta q^i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i \right) dt$$

$$+ \sum_{i=1}^{n} p_i \delta q^i \Big|_{t_0}^{t_1} \quad \text{integration by parts}$$

using $\delta q^i(t_0) = 0 = \delta q^i(t_1)$ get only integral term

$\Rightarrow$ vanishing for arbitrary δp_i & δq^i

$$\Rightarrow \quad \dot{q}^i - \frac{\partial H}{\partial p_i} = 0$$

$$\dot{p}_i + \frac{\partial H}{\partial q^i} = 0$$

<u>Note</u>: if σ lift of path γ and $\tau_L : TM \to T^*M$ invertible

$$S(\sigma) = \int_{t_0}^{t_1} (p\dot{q} - H)\,dt = \int_{t_0}^{t_1} L(\gamma'(t), t)\,dt$$

so same as action functional in configuration space