257 1:00 QM

Classical Mechanics

Tuesday Mar 29 ①

M smooth n-dimensional mfld — paracompact Hausdorff top. space

i.e. $\forall x \in M \quad \exists\, U_x$ open set $x \in U_x \subset M$

and diffeomorphism $\varphi : U_x \xrightarrow{\sim} \varphi(U_x) \subset \mathbb{R}^n$ — unique

(coordinate chart)

homeomorphisms

Covering $U_\alpha \subseteq M \quad \varphi_\alpha : U_\alpha \xrightarrow{\sim} \varphi(U_\alpha) \subset \mathbb{R}^n$ Atlas

$x \in U_\alpha \cap U_\beta \qquad \phi_{\alpha\beta}$ = change of coordinates
$\phi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ by diffeom. on $\varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \xrightarrow[g_{\alpha\beta}]{\sim} \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

all these are diffeomorphisms

local coordinates $q = (q_1, \dots, q_n)$

$f(q) = f(q_1, \dots, q_n) \qquad f : \mathcal{U} \longrightarrow \mathbb{R}^n$

$$\frac{\partial f}{\partial q} = \left(\frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_n}\right)$$

Note: indices notation $q = (q^1, \dots, q^n)$ preferred to $(q_1, \dots, q_n)$ (contravariance)

$$\Omega^{\cdot}(M) = \bigoplus_{j=0}^{n} \Omega^j(M) \qquad \text{differential forms on } M$$

$\Omega^0(M) = C^\infty(M)$ (smooth in each chart) $f : M \to \mathbb{R}$ (or $f : M \to \mathbb{C}$)

d = exterior differential

$\langle df, V \rangle = V(f) = V^\mu \frac{\partial f}{\partial x^\mu}$ pairing of 1-forms & vector fields

Cotangent bundle $T^*M \qquad \langle dx^\nu, \frac{\partial}{\partial x^\mu} \rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu$

$\omega = \omega_\mu dx^\mu$ 1-form

$\Lambda^{\cdot} T^*M$ exterior powers: sections (sum over repeated indices)

$\Omega^k(M) \quad dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, dx^{\mu_1} \otimes \dots \otimes dx^{\mu_k}$ totally antisymm. tensor prod

②

Vect(M) vector fields on M (sections of tg. bdl. of X)

$X = X^\mu \frac{\partial}{\partial x^\mu}$

$\left(X^\mu = \frac{dx^\mu(\gamma(t))}{dt}\Big|_{t=0} \right)$ tg vectors to curves in M

Lie algebra :

w/ Lie bracket = commutator of vector fields

$[X, Y] = XY - YX =$

deg 0 derivation on $\Omega(M)$ commutes w/ d & $\mathcal{L}_X(f) = X(f)$ on $\Omega^0(M)$

$\mathcal{L}_X$ = Lie derivative along X $\quad \mathcal{L}_X Y = \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} (\sigma_{-\varepsilon})_* Y\big|_{\sigma_\varepsilon(x)} - Y\big|_x \right]$

$\sigma_{t,X}$ = flow of X

i_X = contraction (inner prod.) w/ X

deg -1 derivation on $\Omega(M)$

$i_X(f) = 0 \quad i_X(df) = X(f)$

$\mathcal{L}_X = i_X \circ d + d \circ i_X = (d + i_X)^2$

$i_{[X,Y]} = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X$

$Y\big|_{\sigma_\varepsilon(x)} = Y^\mu(x^\nu + \varepsilon X^\nu(x))\, e_\mu \big|_{x+\varepsilon X}$ $\quad \frac{\partial}{\partial y^\mu}$

$\Rightarrow \mathcal{L}_X Y = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) e_\nu$

$= [X, Y]$

Covariance vs. Contravariance :

$f : M \longrightarrow N$

can push forward vector ~~[illegible]~~ fields

$f_* : TM \longrightarrow TN \qquad (f_* V)(g) = V(g \circ f)$

$V = V^\mu \frac{\partial}{\partial x^\mu}$ locally $\Rightarrow f_* V = W^\alpha \frac{\partial}{\partial y^\alpha}$

$W^\alpha = V^\mu \frac{\partial}{\partial x^\mu} y^\alpha(x)$

$\frac{\partial y^\alpha}{\partial x^\mu} = J(f)$ Jacobian of f

$W = JV$

can pull back differential forms

$f^* : T^*N \longrightarrow T^*M$

$\omega = \omega_\alpha dy^\alpha \qquad f^*\omega = \xi_\mu dx^\mu \qquad \xi_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}$

$J\xi = \omega$

~~Explicit~~ :

Classical physical system

M as "constraints":
possible "positions" of the system (coordinates)
think M (dim n) inside some $\mathbb{R}^N$ (embeddability)
then system is N-coord's with constraints that force to move only on M

- Smooth manifold M

① • Tangent bundle of M
TM

$(q,\dot{q})$ or (q,v) coord's on TM
are simultaneous specif. of "position & velocity" of system at a given time

- Deterministic: $\underset{TM}{\overset{\cap}{(q,v)}}$ at time t_0 determines $\underset{TM}{\overset{\cap}{(q(t), v(t))}}$ at all $t \geq t_0$

- trajectories

$$\gamma(t) = (q^1(t), \dots, q^n(t)) \quad \text{in } M$$

with $(\dot{q}^1(t), \dots, \dot{q}^n(t)) \in T_q M$ velocity

- What are trajectories of motion?

② Principle of least action

$$\mathcal{P}_{q_0,t_0}^{q_1,t_1}(M) = \text{all smooth parameterized paths}$$

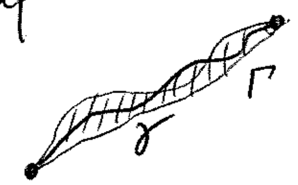
$$\gamma : [t_0, t_1] \to M$$

with $\gamma(t_0) = q_0 \quad \gamma(t_1) = q_1$

∞-dimensional Frechet manifold

(4) a *variation* of $\gamma : [t_0, t_1] \to M$

is a one param. family : smooth map



$$\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] \longrightarrow M$$

$$\Gamma(t, 0) = \gamma(t)$$

$$\Gamma(0, \varepsilon) = q_0 \qquad \Gamma(1, \varepsilon) = q_1$$

∞-dim tg space

$$\delta\gamma = \left.\frac{\partial \Gamma}{\partial \varepsilon}\right|_{\varepsilon = 0} \in T_\gamma \mathcal{P}(M)$$

infinitesimal variation :

$$\delta\gamma : [t_0, t_1] \to \bigcup_{t \in [t_0, t_1]} T_{\gamma(t)}(M)$$

$$\delta\gamma(t) = \Gamma_*\left(\frac{\partial}{\partial \varepsilon}\right)(t, 0) \in T_{\gamma(t)} M$$

$\frac{\partial}{\partial \varepsilon}$ = tg vector at 0 of $[-\varepsilon_0, \varepsilon_0]$

Velocity vector $\gamma'(t)$ of a path $\gamma(t)$ in M

$$\gamma : [t_0, t_1] \to M$$

$$\gamma'(t) = \gamma_*\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)} M \qquad \frac{\partial}{\partial t} \text{ tg vector of } [t_0, t_1] \text{ at } t$$

(write just $\gamma'(t)$ for $(\gamma(t), \gamma'(t)) \in TM$)

<u>Action functional</u> : $S : \mathcal{P}(M) \longrightarrow \mathbb{R}$

- Lagrangian density : smooth function

$$L : TM \times \mathbb{R} \longrightarrow \mathbb{R} \qquad L((q, \dot q), t)$$

evaluate at $L(\gamma'(t), t)$ over paths γ

- Action = integral of Lagr density along a path

$$S(\gamma) = \int_{t_0}^{t_1} L(\gamma'(t), t)\, dt$$

- Motion: (Hamilton's principle)

  A path $\gamma \in \mathcal{P}(M)$ describes the motion of Lagrangian system (M, L) between (t_0, q_0) & (t_1, q_1)

  iff critical point of S

i.e. $\frac{d}{d\varepsilon} S(\gamma_\varepsilon)\Big|_{\varepsilon=0} = 0$ for all variations γ_ε of γ w/ fixed endpts

(*) Note: critical pts not nec. minima; not unique necessarily; also don't know always exists

Expression of equation $\frac{d}{d\varepsilon} S(\gamma_\varepsilon)\Big|_{\varepsilon=0} = 0$ in local coordinates

$(q, v) \in TM$ $\qquad \gamma'(t) = (q(t), \dot{q}(t))$

commonly used notation $(q, \dot{q})$

here not meaning

⑥ $0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon)$ (Gateaux derivative first variation of the action with fixed ends)

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{t_0}^{t_1} L(q(t,\varepsilon), \dot q(t,\varepsilon), t)\, dt$$

$$= \sum_{i=1}^{n} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i}\, \delta q^i + \frac{\partial L}{\partial \dot q^i}\, \delta \dot q^i \right) dt \qquad (*)$$

where $\delta q^i(t) = \frac{\partial q^i(t,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}$ $\quad \delta \dot q^i(t) = \frac{\partial \dot q^i(t,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}$

then use $\delta \dot q^i(t) = \frac{d}{dt}\, \delta q^i(t)$
and integrate by parts second term

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot q^i}\, \delta \dot q^i\, dt = \frac{\partial L}{\partial \dot q^i}\, \delta q^i \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot q^i}\right) \delta q^i$$

but all $q^i(t,\varepsilon)$ have same $q^i(t_0,\varepsilon) = q^i_0$ $\quad q^i(t_1,\varepsilon) = q^i_1$
so $\delta q^i(t_0) = 0 = \delta q^i(t_1)$ so $\frac{\partial L}{\partial \dot q^i}\, \delta q^i \Big|_{t_0}^{t_1} = 0$

then get $(*)$ =

$$\sum_{i=1}^{n} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^i} \right) \delta q^i\, dt$$

this should be $= 0$ for an arbitrary choice of the smooth functions δq^i ($=0$ at endpts)

$$\Rightarrow \quad \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^i} = 0$$

Euler-Lagrange equations

Second order equations

Examples : 1 Small oscillations

$$L = \tfrac{1}{2} m \dot{q}^2 - V(q)$$

with

$$V(q) = \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij} q^i q^j$$

positive-definite quadratic form on $\mathbb{R}^n$

diagonalize by orthogonal transform.

(symmetric matrix $A = (a_{ij})$)

$A = B^* B$ $\quad V(q) = \tfrac{1}{2} \langle Bq, Bq \rangle$

$$L = \tfrac{1}{2} m \left(\dot{q}^2 - \sum_{i=1}^{n} \omega_i^2 (q^i)^2 \right)$$

eigenvectors $\quad (\omega_i > 0)$

normal coordinates

$\Rightarrow$ Euler Lagrange equations $\quad \dfrac{\partial L}{\partial q^i} - \dfrac{d}{dt} \dfrac{\partial L}{\partial \dot{q}^i} = 0$

gives $\quad \ddot{q}^i + \omega_i^2 q^i = 0$

n decoupled (non-interacting) harmonic oscillators ω_i freq.

More generally $V(q)$ w/ minimum at $q = 0$ and with Hessian $\dfrac{\partial^2 V(q)}{\partial q^i \partial q^j}(0) = a_{ij}$ pos-def. quadr. form

$\Rightarrow$ for q near $q = 0$ behaves like n-decoupled harmonic oscillators

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2 Geodesics on Riemannian manifolds

$M \quad ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$ (sum over repeated indices)

metric tensor

$(= \tfrac{1}{2} g_{\mu\nu} v^\mu v^\nu)$

$$L(v) = \tfrac{1}{2} \langle v, v \rangle = \tfrac{1}{2} \|v\|^2 \quad \text{in } TM$$

i.e. for $\gamma' = (x(t), \dot{x}(t))$

$$L(\gamma') = \tfrac{1}{2} g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu$$

(8) $$S(\gamma) = \frac{1}{2}\int_{t_0}^{t_1} \|\gamma'(t)\|^2 \, dt = \frac{1}{2}\int_{t_0}^{t_1} g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu \, dt$$

Euler – Lagrange equations

$$g_{\mu\nu}\, \ddot{x}^\mu + \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\, \dot{x}^\mu \dot{x}^\lambda - \frac{1}{2}\, \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\, \dot{x}^\mu \dot{x}^\lambda$$

(inverse $g^{\sigma\lambda} = (g^{-1})^{\sigma\lambda}$) and sum over λ

$$\ddot{x}^\sigma + \Gamma^\sigma_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = 0$$ where

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\, g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right)$$

Christoffel symbols

$\Rightarrow$ geodesic equation

Note: free moving particle (L = only Kinetic energy, no potential) on a curved mfld M follows geodesics

The role of symmetries

<u>Note</u>: Euler-Lagrange equations are second order diff eq. usually difficult to solve explicitly $\rightsquigarrow$ ways to simplify problem

- Integral of motion (or a conservation law)

$I : TM \to \mathbb{R}$

s.t. $\frac{d}{dt} I(\gamma'(t)) = 0$

on all $\gamma(t)$ extremals (crit. pts of the action functional

- Energy

$$E(q,\dot q,t) = \sum_{i=1}^{n} \dot q^i \cdot \frac{\partial L}{\partial \dot q^i}(q,\dot q,t) - L(q,\dot q,t)$$

$E = \langle \dot q, \frac{\partial L}{\partial \dot q} \rangle - L$ is well def. indep. of local coord's as $E : TM \times \mathbb{R} \to \mathbb{R}$

J = matrix of change of coord's (Jacobian) $q' = F(q)$ $J(q) = F_*(q) \Rightarrow v' = Jv$ pushforw of vectors

$\Rightarrow \dot q' = J(q)\dot q$ $dq' = J\,dq$

$d\dot q' = G(q,q')\,dq + J(q)\,d\dot q$

$$dL = \frac{\partial L}{\partial q'} dq' + \frac{\partial L}{\partial \dot q'} d\dot q' + \frac{\partial L}{\partial t} dt$$

$$= \left(\frac{\partial L}{\partial q'} J + \frac{\partial L}{\partial \dot q'} G\right) dq + \frac{\partial L}{\partial \dot q'} J\,d\dot q + \frac{\partial L}{\partial t} dt$$

$$= \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot q} d\dot q + \frac{\partial L}{\partial t} dt$$

$\Rightarrow \frac{\partial L}{\partial \dot q'} J = \frac{\partial L}{\partial \dot q}$

$\Rightarrow \dot q' \frac{\partial L}{\partial \dot q'} = \dot q \frac{\partial L}{\partial \dot q}$

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q} - \frac{\partial L}{\partial t}$$

$$= \underbrace{\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right)}_{\overset{\shortparallel}{0} \text{ Euler-Lagrange}}\dot{q} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$$

So conservation of energy:

if $\frac{\partial L}{\partial t} = 0$ then also $\frac{dE}{dt} = 0$ along solutions of eq. of motion

<u>Closed system</u> $\frac{\partial L}{\partial t} = 0$

Symmetries of the Lagrangian (assuming $\frac{\partial L}{\partial t} = 0$)

$$L : TM \longrightarrow \mathbb{R}$$

$g : M \longrightarrow M$ $\quad g \in G$ group $G \subset \mathrm{Diff}(M)$ of diffeomorphisms of M

symmetries if

$$L(g_*(v)) = L(v) \quad \forall v \in TM$$

<u>Noether's theorem</u>: symmetries give rise to conservation laws

Noether's theorem

$L : TM \to \mathbb{R}$ Lagrangian

if $\exists \{g_s\}_{s\in\mathbb{R}}$ one parameter family of symmetries

i.e. $\alpha : \mathbb{R} \to \mathrm{Diff}(M)$ $\qquad \alpha(s) = g_s$

s.t. $L((g_s)_*(v)) = L(v) \quad \forall v \in TM$

Then $\exists$ integral of motion

$$I(q,\dot q) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot q^i}(q,\dot q) \left(\frac{d g_s^i(q)}{ds}\Big|_{s=0} \right)$$

$$= \frac{\partial L}{\partial \dot q} \cdot a$$

where $X = \sum_{i=1}^{n} a^i(q) \frac{\partial}{\partial q^i}$ vector field on M associated to the flow g_s

Pf: differentiate the identity

$$L((g_s)_*(\gamma')) = L(\gamma')$$

with resp to s at $s=0$

$$0 = \frac{\partial L}{\partial q} a + \frac{\partial L}{\partial \dot q} \dot a \underset{\text{Euler-Lagr}}{=} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) a + \frac{\partial L}{\partial \dot q} \frac{da}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} a\right)$$

Examples of Noether's theorem

$M = V$ vector space

$L : TM \to \mathbb{R}$ Lagr. invariant under a ~~grp~~ one-param. grp. of translations

$$g_s(q) = q + sv \quad \text{for a given } v \in V$$

then $I = \sum_{i=1}^{n} v^i \frac{\partial L}{\partial \dot{q}^i}$ is the integral of motion coming from this symmetry

e.g. N interacting particles so $V = \mathbb{R}^{3N}$ position vectors $r_a = (r_a^1, r_a^2, r_a^3)$

L = Lagr. invariant translations by $c \in \mathbb{R}^3$ $\quad (c, \dots, c) \in \mathbb{R}^{3N}$

$$I = \sum_{a=1}^{N} \left(c^1 \frac{\partial L}{\partial \dot{r}_a^1} + c^2 \frac{\partial L}{\partial \dot{r}_a^2} + c^3 \frac{\partial L}{\partial \dot{r}_a^3} \right) = c^1 P_1 + c^2 P_2 + c^3 P_3$$

$$\mathbf{P} = \sum_{a=1}^{N} \frac{\partial L}{\partial \dot{\mathbf{r}}_a} \in \mathbb{R}^3 \qquad \left(\mathbf{P} = \sum_{a=1}^{N} m_a \dot{\mathbf{r}}_a \text{ if } L = \sum \tfrac{1}{2} m_a \dot{\mathbf{r}}_a^2 \right)$$

momentum

Generally $\boxed{P_i = \frac{\partial L}{\partial \dot{q}^i}}$ are called momenta of the Lagrangian system

if transl. invariant L w/ resp. to all $v \in V$ then these P_i are conserved quantities

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<u>Question</u>: transform the system of second order equations Euler-Lagrange into a more manageable system of first order equations

$\Rightarrow$ from Lagrangian to Hamiltonian mechanics via Legendre transform one variable example Legendre transform

$F(s)$ smooth convex function

$$\frac{d^2F}{ds^2}(s) > 0$$

or equiv $s(x) = \frac{dF}{ds}(s)$ strictly monotonically increasing

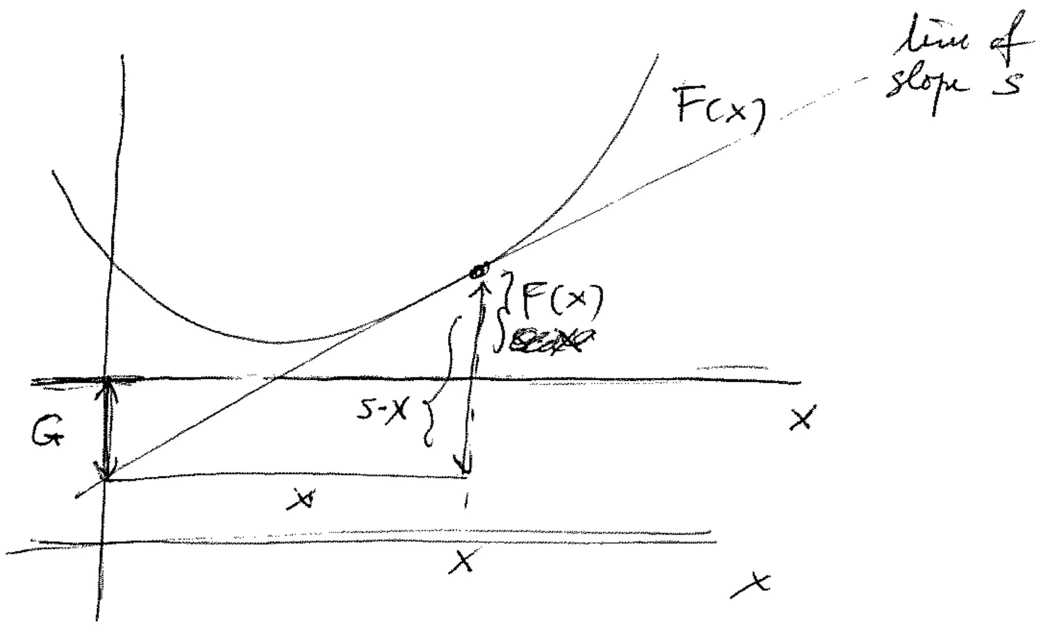
$\Rightarrow$ $x \mapsto s(x)$ can be inverted

$s \mapsto x(s)$

$\Rightarrow$ can make $F(x(s))$ change of variables

Legendre transform of F then is df as

$$G(s) = s\,x(s) - F(x(s)) \qquad G = \mathcal{L}(F)$$



$$G = sx - F(x)$$ express either as fuct of s or x (invertible)

<u>Note:</u> Legendre transform is an involution

(Legendre transf of G is F : $\mathcal{L}(G) = F$)

$$y(s) = \frac{\partial G}{\partial s} = x(s) + s\frac{dx}{ds} - \overbrace{\frac{dF}{ds}}^{s}(x(s))\,\frac{dx}{ds} = x(s)$$

So $\quad s\,y(s) - G(s) = s\,X(s) - s\,X(s) + F(x(s)) = F(x(s))$

<u>Extrema:</u> $\quad F_{min} = \min_x F(x) \quad$ for convex F

$$s(x_{min}) = \frac{dF}{dx}(x_{min}) = 0$$

$$\Rightarrow \quad x_{min}\, s(x_{min}) - F(x_{min}) = G(s(x_{min}))$$

$$\Rightarrow \quad F_{min} = -G(0)$$

similarly $\quad G_{min} = -F(0)$

Higher derivatives relations

$$\frac{dG}{ds} = x \qquad \frac{dF}{dx} = s$$

$$\frac{d^2G}{ds^2} = \frac{dx}{ds} \qquad \frac{d^2F}{dx^2} = \frac{ds}{dx}$$

$$\Rightarrow \quad \left(\frac{d^2G}{ds^2}\right)\left(\frac{d^2F}{dx^2}\right) = 1$$

similarly get relations between higher others

e.g. $$\frac{d^3G}{ds^3}\left(\frac{d^2G}{ds^2}\right)^{-3/2} + \frac{d^3F}{dx^3}\left(\frac{d^2F}{dx^2}\right)^{-3/2} = 0$$

etc.