Differential Forms

M smooth n-dim manifold

T^*M cotangent bundle

sections = 1-forms

$\omega(x) = \sum_{\mu=1}^{n} \omega_\mu(x)\, dx^\mu$ $\qquad \omega_\mu : U \to \mathbb{R}$ smooth functs

Changes of coordinates

$dy^\nu = \frac{\partial y^\nu}{\partial x^\mu} dx^\mu$ $\qquad \frac{\partial y^\nu}{\partial x^\mu}$ = change of base

$\Rightarrow \quad \omega(y) = \sum \tilde{\omega}_\nu(y)\, dy^\nu$

$\tilde{\omega}_\nu = \frac{\partial x^\mu}{\partial y^\nu} \omega_\mu$

$f : M \to \mathbb{R}$ smooth function

$df = \sum \frac{\partial f}{\partial x^\mu} dx^\mu$ $\qquad$ 1-form (exact)

$\Lambda^\cdot(T^*M)$ exterior algebra bundle on T^*M

Note write $\Lambda^\cdot(TM)$ or $\Lambda^\cdot(T^*M)$ different notation but means

i.e. vector bundle of M s.t. fiber over $x \in M$ is exterior algebra of vector space T_x^*M

$\Lambda^r(T_x^*M)$

alternating r-linear forms on $V = T_x^*M$

$L(\alpha_1, \dots \overset{\alpha_i}{|} \dots \overset{\alpha_j}{|} \dots \alpha_r) = -L(\alpha_1, \dots \overset{\alpha_j}{|} \dots \overset{\alpha_i}{|} \dots \alpha_r)$

$$\Rightarrow \quad \Lambda^0(V) = \mathbb{R} \quad , \quad \Lambda^n(V) = \mathbb{R}$$

$$\Lambda^1(V) = V^* \qquad \Lambda^k(V) = V^* \wedge \cdots \wedge V^*$$

basis for $\Lambda^k(TM)$

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$$

with
$$dx^{\mu_i} \wedge dx^{\mu_j} = - dx^{\mu_j} \wedge dx^{\mu_i} \qquad i \neq j$$
$$dx^{\mu_i} \wedge dx^{\mu_i} = 0$$

$$dy^{\nu_1} \wedge \cdots \wedge dy^{\nu_k} = \left(\sum \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} dx^{\mu_1} \right) \wedge \cdots \wedge \left(\sum \frac{\partial y^{\nu_k}}{\partial x^{\mu_k}} dx^{\mu_k} \right)$$

$$= \sum_{\mu_1 \cdots \mu_k} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\nu_k}}{\partial x^{\mu_k}} \underbrace{dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}}$$

note here all terms with same indices = 0 and other can reorder (up to signs) by $\mu_1 < \cdots < \mu_k$

get "det" like expression here

= det for $k = n$

ω_1 & ω_2 forms $\omega_1 \in \Lambda^k$ $\omega_2 \in \Lambda^l$

$\omega_1 \wedge \omega_2$ in Λ^{k+l}

$$\omega_2 \wedge \omega_1 = (-1)^{kl} \, \omega_1 \wedge \omega_2$$

use factorials to eliminate redundancy

$$\omega = \frac{1}{k!}\sum_{\mu_1 \dots \mu_k} \omega_{\mu_1 \dots \mu_k} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$$

$$= \sum_{\mu_1 < \dots < \mu_k} \omega_{\mu_1 \dots \mu_k} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$$

$\exists \; d : \Lambda^k \longrightarrow \Lambda^{k+1}$ exterior differential
$\mathbb{R}$-linear map

1) $f \in \Lambda^0$ $\quad df$ differential

$\omega_1 \in \Lambda^{k_1} \qquad \omega_2 \in \Lambda^{k_2}$

2) $$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2$$

3) $$d^2 = d \circ d = 0$$

For $\omega = \frac{1}{k!} \sum_{\mu_1 \dots \mu_k} \omega_{\mu_1 \dots \mu_k} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$

$$d\omega = \frac{1}{k!} \sum_{\mu_1 \dots \mu_k} d\omega_{\mu_1 \dots \mu_k} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$$

$$d\omega_{\mu_1 \dots \mu_k} = \frac{\partial \omega_{\mu_1 \dots \mu_k}}{\partial x^\nu} dx^\nu$$

$$d^2\omega = d\left(\frac{1}{k!}\sum d\omega_{\mu_1\cdots\mu_k} \wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\right)$$

$$= d\left(\frac{1}{k!}\sum_{\nu,\mu_1,\cdots,\mu_k} \frac{\partial\omega_{\mu_1\cdots\mu_k}}{\partial x^\nu}\, dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\right)$$

$$= \frac{1}{k}\sum_{\nu,\rho,\mu_1,\cdots,\mu_k} \frac{\partial^2\omega_{\mu_1\cdots\mu_k}}{\partial x^\nu\partial x^\rho}\, dx^\rho\wedge dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}$$

Note $\dfrac{\partial^2\omega}{\partial x^\nu\partial x^\rho} = \dfrac{\partial^2\omega}{\partial x^\rho\partial x^\nu}$ while $dx^\rho\wedge dx^\nu = -dx^\nu\wedge dx^\rho$

(and zero when $\rho=\nu$)

so terms cancel in pairs $d^2=0$

i_X contraction of a diff form w/ a vector field

$$X = \sum X^\mu \frac{\partial}{\partial x^\mu}$$

deg -1 derivation on $\Lambda^\cdot(T^*M) = \Omega^\cdot(M)$

$$\omega = \sum \omega_{\mu_1\cdots\mu_k}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}$$

$$X^\mu\,\omega_{\mu\nu_1,\cdots,\nu_{k-1}}$$

$\omega(V_1,\dots,V_k)$ k-linear forms on k-tuple of vector fields

$$i_X\omega(V_1,\dots,V_{k-1}) = \omega(X, V_1,\dots,V_{k-1})$$

Lie derivative

$$\mathcal{L}_X = i_X\circ d + d\circ i_X = (d+i_X)^2 \quad \text{commutes w/ } d$$

degree 0 derivation on $\Omega^\cdot(M)$