The Mathematical Theory of Formal Languages: Part II, Group Theory

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- Group G, with presentation $G = \langle X \mid R \rangle$ (finitely presented)
    - X (finite) set of generators $x_1, \ldots, x_N$
    - R (finite) set of relations: $r \in R$ words in the generators and their inverses

Word problem for G:

- Question: when does a word in the x_j and x_j^{-1} represent the element $1 \in G$?
- When do two words represent the same element?
- Comparing different presentations
- is there an algorithmic solution?

Word problem and formal languages

- for $G = \langle X \,|\, R \rangle$ call $\hat{X} = \{x, x^{-1} \,|\, x \in X\}$ symmetric set of generators
- Language associated to a finitely presented group $G = \langle X \,|\, R \rangle$

$$\mathcal{L}_G = \{ w \in \hat{X}^\star \,|\, w = 1 \in G \}$$

set of words in the generators representing trivial element of G

- What kind of formal language is it?

- Algebraic properties of the group G correspond to properties of the formal language $\mathcal{L}_G$:

1. $\mathcal{L}_G$ is a regular language (Type 3) iff G is finite (Anisimov)
2. $\mathcal{L}_G$ is context-free (Type 2) iff G has a free subgroup of finite index (Muller–Schupp)

- Formal languages and solvability of the word problem:
  - Word problem solvable for G (finitely presented) iff $\mathcal{L}_G$ is a recursive language

Recursive languages (alphabet $\hat{X}$):

• $\mathcal{L}_G$ recursive subset of $\hat{X}^\star$

• equivalently the characteristic function $\chi_{\mathcal{L}_G}$ is a total recursive function

• Total recursive functions are computable by a Turing machine that always halts

• For a recursive language there is a Turing machine that always halts on an input $w \in \hat{X}^\star$: accepts it if $w \in \mathcal{L}_G$, rejects it of $w \notin \mathcal{L}_G$: so word problem is (algorithmically) solvable

Finitely presented groups with unsolvable word problem (Novikov)

• Group G with recursively enumerable presentation: $G = \langle X \mid R \rangle$ with X finite and R recursively enumerable

• Group is recursively presented iff it can be embedded in a finitely presented group (X and R finite)

• Example of recursively presented G with unsolvable word problem

$$G = \langle a, b, c, d \mid a^n b a^n = c^n d c^n, \ \ n \in A \rangle$$

for A recursively enumerable subset $A \subset \mathbb{N}$ that has unsolvable membership problem

• If recursively presented G has unsolvable word problem and embeds into finitely presented H then H also has unsolvable word problem.

Example: finite presentation with unsolvable word problem

- Generators: $X = \{a, b, c, d, e, p, q, r, t, k\}$
- Relations:

$$p^{10}a = ap, \quad p^{10}b = bp, \quad p^{10}c = cp, \quad p^{10}d = dp, \quad p^{10}e = ep$$

$$aq^{10} = qa, \quad bq^{10} = qb, \quad cq^{10} = qc, \quad dq^{10} = qd, \quad eq^{10} = qe$$

$$ra = ar, \quad rb = br, \quad rc = cr, \quad rd = dr, \quad re = er, \quad pt = tp, \quad qt = tq$$

$$pacqr = rpcaq, \quad p^2adq^2r = rp^2daq^2, \quad p^3bcq^3r = rp^3cbq^3$$

$$p^4bdq^4r = rp^4dbq^4, \quad p^5ceq^5r = rp^5ecaq^5, \quad p^6deq^6r = rp^6edbq^6$$

$$p^7cdcq^7r = rp^7cdceq^7, \quad p^8ca^3q^8r = rp^8a^3q^8, \quad p^9da^3q^9r = rp^9a^3q^9$$

$$a^{-3}ta^3k = ka^{-3}ta^3$$

How are such examples constructed?

A technique to construct semigroup presentations with unsolvable word problem:

• G.S. Cijtin, *An associative calculus with an insoluble problem of equivalence*, Trudy Mat. Inst. Steklov, vol. 52 (1957) 172–189

A technique for passing from a semigroup with unsolvable word problem to a group with unsolvable word problem

• V.V. Borisov, *Simple examples of groups with unsolvable word problems*, Mat. Zametki 6 (1969) 521–532

Example above: method applied to simplest known semigroup example

• D.J. Collins, *A simple presentation of a group with unsolvable word problem*, Illinois Journal of Mathematics 30 (1986) N.2, 230–234

Regular language $\Leftrightarrow$ finite group

- If G finite, use standard presentation

$G = \langle x_g,\, g \in G \,|\, x_g x_h = x_{gh} \rangle$

Construct FSA $M = (Q, F, \mathfrak{A}, \tau, q_0)$ with $Q = \{x_g \,|\, g \in G\}$, $\mathfrak{A} = \{x_g^{\pm 1} \,|\, g \in G\}$, $q_0 = x_1$, $F = \{q_0\}$ and transitions τ given by

$$(x_g, x_h, x_{gh}), \quad g, h \in G$$

$$(x_g, x_h^{-1}, x_{gh^{-1}}), \quad g, h \in G$$

The finite state automaton M recognizes $\mathcal{L}_G$

• If G is infinite and X is a finite set of generators for G

For any $n \geq 1$ there is a $g \in G$ such that g not obtained from any word of length $\leq n$ (only finitely many such words and G is infinite)

If M deterministic FSA with alphabet $\hat{X}$ and $n = \#Q$ number of states, take $g \in G$ not represented by any word of length $\leq n$

then there are prefixes w_1 and $w_1 w_2$ of w such that, after reading w_1 and $w_1 w_2$ obtain same state

so M accepts (or rejects) both $w_1 w_1^{-1}$ and $w_1 w_2 w_1^{-1}$ but first is 1 and second is not ($w_2 \neq 1$)

so M cannot recognize $\mathcal{L}_G$

Cayley graph

- Vertices $V(\mathcal{G}_G) = G$ elements of the group

- Edges $E(\mathcal{G}_G) = G \times X$ with edge $e_{g,x}$ oriented with $s(e_{g,x}) = g$ and $t(e_{g,x}) = gx$

- for $x^{-1} \in \hat{X}$ edge with opposite orientation $e_{g,x^{-1}} = \bar{e}_{g,x}$ with $s(e_{g,x^{-1}}) = gx$ and $t(e_{g,x^{-1}}) = gx\,x^{-1} = g$

- word w in the generators $\Rightarrow$ oriented path in $\mathcal{G}_G$ from g to gw

- word $w = 1 \in G$ iff corresponding path in $\mathcal{G}_G$ is closed

- G acts on $\mathcal{G}_G$: acting on $V(\mathcal{G}_G) = G$ and on $E(\mathcal{G}_G) = G \times X$ by left multiplication (translation)

- invariant metric: $d(g,h) =$ minimal length of path from vertex g to vertex h, with $d(ag, ah) = d(g,h)$ for all $a \in G$

Main idea for the context-free case

• X set of generators of G

• if for $y_i \in \hat{X}$, a word $w = y_1 \cdots y_n = 1$ get closed path in the Cayley graph $\mathcal{G}_G$

• consider a polygon $\mathcal{P}$ with boundary this closed path

• obtain a characterization of the context-free property of $\mathcal{L}_G$ in terms of properties of triangulations of this polygon

Plane polygons and triangulations

• a plane polygon $\mathcal{P}$: interior of a simple closed curve given by a finite collections of (smooth) arcs in the plane joined at the endpoints

• triangulation of $\mathcal{P}$: decomposition into triangles (with sides that are arcs): two triangles can meet in a vertex or an edge (or not meet)

• allow 1-gons and 2-gons (as "triangulated")

• triangle in a triangulation is *critical* if has two edges on the boundary of the polygon

• triangulation is *diagonal* if no more vertices than original ones of the polygon

• Combinatorial fact: a diagonal triangulation has at least two critical triangles (for $\mathcal{P}$ with at least two triangles)

K-triangulations

• diagonal triangulation of a polygon $\mathcal{P}$ with boundary a closed path in the Cayley graph $\mathcal{G}_G$

• each edge of the triangulation is labelled by a word in $\hat{X}^\star$

• going around the boundary of each triangle gives a word in $\mathcal{L}_G$ (a word w in $\hat{X}^\star$ with $w = 1 \in G$)

• all words labeling edges of the triangulation have length $\leq K$

Context-free and K-triangulations

Language $\mathcal{L}_G$ is context-free $\Leftrightarrow \exists K$ such that all closed paths in Cayley graph $\mathcal{G}_G$ can be triangulated with a K-triangulation

Idea of argument:

If context-free grammar:
• use production rules for word $w = 1$ (boundary of polygon) to produce a triangulation:

$$S \to AB \overset{\bullet}{\to} w_1 w_2 = w \quad \text{with } A \overset{\bullet}{\to} w_1 \text{ and } B \overset{\bullet}{\to} w_2$$

$\Rightarrow$ a subdivision of polygon in to two arcs: draw an arc in the middle, etc.

If have K-triangulation for all loops in $\mathcal{G}_G$: get a context-free grammar with terminals $\hat{X}$

• for each word $u \in \hat{X}^\star$ of length $\leq K$ variable A_u and for $u = vw$ in G production $A_u \to A_v A_w$ in P

• any word $w = y_1 \cdots y_n$ from boundary of triangles in the triangulation also corresponds to $A_1 \stackrel{\bullet}{\to} A_{y_1} \cdots A_{y_n}$ in the grammar (inductive argument eliminating the critical triangles and reducing size of polygon)

• and productions $A_y \to y$ (terminals); get that the grammar recognizes $\mathcal{L}_G$

accessibility

To link contex-free to the existence of a free subgroup, need a decomposition of the group that preserves both the context-free property and the existence of a free subgroup, so that can do an inductive argument

• HNN-extensions: two subgroups B, C in a group A and an isomorphism $\gamma : B \to C$ (not coming from A)

$$A \star_C B = \langle t, A \,|\, tBt^{-1} = C \rangle$$

means generators as A, additional generator t; relations of A and additional relations $tbt^{-1} = \gamma(b)$ for $b \in B$

• *accessibility series*: (accessibility length n)

$$G = G_0 \supset G_1 \supset \cdots \supset G_n$$

G_i subgroups with $G_i = G_{i+1} \star_K H$ with K finite

• finitely generated G is *accessible* if upper bound on length of any accessibility series (least upper bound = accessibility length)

• assume G context-free and accessible

• inductive argument (induction on accessibility length) on existence of a free finite-index subgroup:
if $n = 0$ have G finite group; if $n > 0$ $G = G_1 \star_K H$, context-free property inherited; inductively: free finite-index subgroup for G_1; show implies free finite-index subgroup for G

• then need to eliminate auxiliary accessibility condition

Context-free $\Leftrightarrow$ free subgroup of finite index

- show that a finitely generated G with $\mathcal{L}_G$ context-free is finitely presented

- then show finitely presented groups are accessible

- Conclusion: equivalent properties for finitely generated G

1. $\mathcal{L}_G$ is a context-free language
2. G has a free subgroup of finite index
3. G has deterministic word problem
(using the fact that free groups do)

Word problem and geometry

• Groups given by explicit presentations arise in geometry/topology as fundamental groups $\pi_1(X)$ of manifolds

Positive results

• Groups with solvable word problem include: negatively curved groups (Gromov hyperbolic), Coxeter groups (reflection groups), braid groups, geometrically finite groups
[all in a larger class of "automatic groups"]

Negative results

• Any finitely presenting group occurs as the fundamental group of a smooth 4-dimensional manifold

• The homeomorphism problem is unsolvable

- A. Markov, *The insolubility of the problem of homeomorphy*, Dokl. Akad. Nauk SSSR 121 (1958) 218–220

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