Lecture 12: Semantics and Compositionality
Ma 191c: Mathematical Models of Generative Linguistics

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this part based on

- Matilde Marcolli, Robert C. Berwick, Noam Chomsky, *Syntax-semantics interface: an algebraic model*, arXiv:2311.06189

also included in the book:
Matilde Marcolli, Noam Chomsky, Robert C. Berwick, "Mathematical structure of syntactic Merge", MIT Press.

Pietroski's semantics and Merge

- independent existence *within* semantics of a *Combine* binary operation that parallels the functioning of Merge in syntax
- describing a "Merge for i-concepts" (compositional structure in semantics)
- in Pietroski's formulation $Combine = Label \circ Conc$ two operations
- in original formulation compatible with Old Minimalism, so Merge is not symmetric (planar trees)
- *Conc* is *concatenation of strings*
- but this is not compatible with a map from syntax if *free symmetric Merge*

Conc operation on strings

- $\Sigma^*(\mathcal{SO}_0)$ set of ordered sequences of elements in $\mathcal{SO}_0$ of arbitrary (finite) length
- associative non-commutative binary operation

$$\mathrm{Conc} : \Sigma^*(\mathcal{SO}_0) \times \Sigma^*(\mathcal{SO}_0) \to \Sigma^*(\mathcal{SO}_0)$$

$$(\alpha, \beta) \mapsto \mathrm{Conc}(\alpha, \beta) = \alpha^\wedge \beta = \alpha\beta$$

  combines ordered sets α and β so string β follows string α.
- planar binary rooted tree $\tilde{T} \in \mathfrak{T}^{pl}_{\mathcal{SO}_0} \Rightarrow$ ordered set of leaves $L(\tilde{T}) \in \Sigma^*(\mathcal{SO}_0)$

strings and noncommutative Merge

- free non-commutative non-associative magma

$$\mathfrak{T}^{pl}_{\mathcal{SO}_0} = \mathcal{SO}^{nc} = \mathrm{Magma}_{na,nc}(\mathcal{SO}_0, \mathfrak{M}^{nc})$$

- Π the canonical projection (morphism of magmas)

$$\Pi : \mathfrak{T}^{pl}_{\mathcal{SO}_0} \to \mathfrak{T}_{\mathcal{SO}_0}$$

- abstract tree $T \in \mathfrak{T}_{\mathcal{SO}_0}$ and $\tilde{T}$ any choice of planar tree $\tilde{T} \in \Pi^{-1}(T)$, in fiber $\Pi^{-1}(T)$ of projection
- map $L : \mathfrak{T}^{pl}_{\mathcal{SO}_0} \to \Sigma^*(\mathcal{SO}_0)$ with $L : \tilde{T} \mapsto L(\tilde{T})$ satisfies

$$L(\mathfrak{M}^{nc}(\tilde{T}_1, \tilde{T}_2)) = \mathrm{Conc}(L(\tilde{T}_1), L(\tilde{T}_2))$$

- Note: $\mathfrak{M}^{nc}$ non-associative while Conc associative, so map L kills the associators of $\mathfrak{M}^{nc}$.

Conc and linearization algorithms

- abstract trees that are produced by free symmetric Merge

$$\mathcal{SO} = \mathrm{Magma}_{na,c}(\mathcal{SO}_0, \mathfrak{M}) = \mathfrak{T}_{\mathcal{SO}_0},$$

- there is no possible *morphism of magmas* $\mathcal{SO} \to \mathcal{SO}^{nc}$
- in general we have for any section $\sigma : \mathcal{SO} \to \mathcal{SO}^{nc}$ with $\Pi \circ \sigma = \mathrm{id}$

$$\mathfrak{M}^{nc}(\sigma(T_1), \sigma(T_2)) \neq \sigma(\mathfrak{M}(T_1, T_2))$$

- a choice of a section $\sigma : \mathcal{SO} \to \mathcal{SO}^{nc}$ is a linearization algorithm
- obstruction to a consistent definition of Conc on the image of a "linearization algorithm"

$$L(\sigma(\mathfrak{M}(T_1, T_2))) \neq \mathrm{Conc}(L(\sigma(T_1)), L(\sigma(T_2))) = L(\mathfrak{M}^{nc}(\sigma(T_1), \sigma(T_2)))$$

no linearization with well defined Conc

- show that, given any section $\sigma : \mathcal{SO} \to \mathcal{SO}^{nc}$, $\exists\ T_1, T_2$

$$L(\sigma(\mathfrak{M}(T_1, T_2))) \neq \mathrm{Conc}(L(\sigma(T_1)), L(\sigma(T_2)))$$

- σ cannot be a morphism of magmas so $\exists\ T_1, T_2$

$$\sigma(\mathfrak{M}(T_1, T_2)) \neq \mathfrak{M}^{nc}(\sigma(T_1), \sigma(T_2)).$$

- planar trees $\sigma(\mathfrak{M}(T_1, T_2))$ and $\mathfrak{M}^{nc}(\sigma(T_1), \sigma(T_2))$ in same fiber of Π over $T = \mathfrak{M}(T_1, T_2)$
- how to *characterize* the sources of non-well-behaved concatenations?
- in terms of geometry of *associahedra*

- projection Π collapses the 2^{n-1} different planar structures, with $n = \#L(T)$
- the planar trees $\sigma(\mathfrak{M}(T_1, T_2))$ and $\mathfrak{M}^{nc}(\sigma(T_1), \sigma(T_2))$ have same ordered set of leaves (concatenation *well behaved*) iff they differ by an associator (a change of bracketing on the *same* ordered set)
- a given ordered set of leaves realized by C_{n-1} possible planar structures, vertices of the associahedron K_n
- Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

combinatorial counting

- combinatorial identity

$$n!\, C_{n-1} = 2^{n-1}\, (2n-3)!!$$

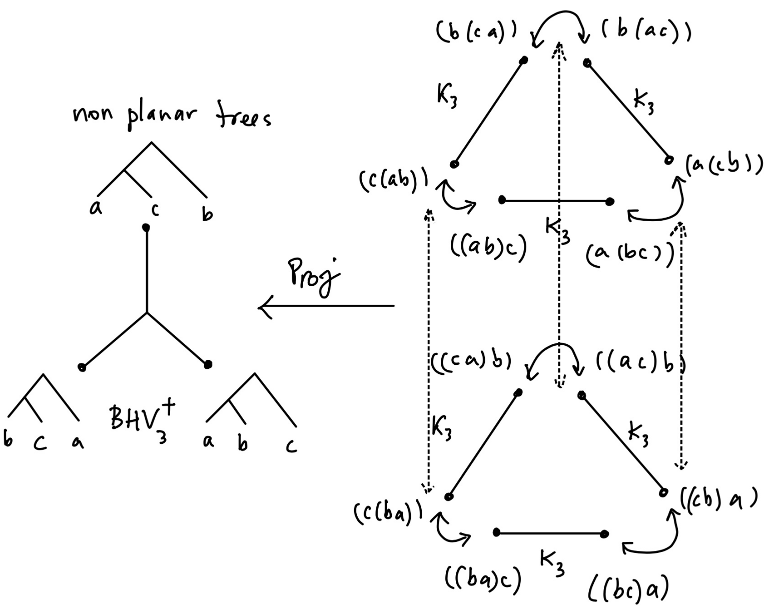
- C_{n-1} possible bracketing on a fixed ordered set
- number of planar trees $\tilde{T}$ with the same ordered set $L(\tilde{T})$
- $(2n-3)!!$ counts number of different abstract (non-planar) binary rooted trees on n labelled leaves
- number of abstract trees T with same non-ordered set $L(T)$
- 2^{n-1} counts number of possible planar structures on each T
- $n!$ total number possible orderings of non-ordered set $L(T)$ of n-leaves

- planar binary rooted tree $\tilde{T}$: symmetric group S_n acts on the leaves producing other planar trees $\tau : \tilde{T} \mapsto \tau(\tilde{T})$
- same tree structure but the labels of leaves permuted
- $\tilde{T}$ at a vertex of one of the assiciahedra, orbit under S_n determines a vertex in each other associahedra ($n!$ of them)
- consider group $G_{\tilde{T}}$ of transformations generated by the involutions γ_v that flip subtrees $\tilde{T}_{v,L}$ and $\tilde{T}_{v,R}$ below v
- $\# G_{\tilde{T}} = 2^{n-1}$ the number of possible planar structures on T
- normal subgroup $\mathrm{Aut}(\tilde{T})$ transformations in $G_{\tilde{T}}$ preserving $\tilde{T}$
- orbit-stabilizer theorem

$$\#\mathrm{Orbit}_{G_{\tilde{T}}}(\tilde{T}) = \frac{\# G_{\tilde{T}}}{\#\mathrm{Aut}(\tilde{T})}$$

- this gives

$$\sum_T \#(G_{\tilde{T}}/\mathrm{Aut}(\tilde{T})) = \sum_T \frac{2^{n-1}}{\#\mathrm{Aut}(\tilde{T})} = C_{n-1} = \frac{(2n-3)!!}{n!}$$



non planar trees
a
c
b
b
c
a
BHV_3^+
a
b
c
Proj
(b(ca))
(b(ac))
K_3
K_3
(c(ab))
(a(cb))
((ab)c)
K_3
(a(bc))
((ca)b)
((ac)b)
K_3
K_3
(c(ba))
((cb)a)
((ba)c)
K_3
((bc)a)

- given a linearization algorithm $\sigma : \mathcal{SO} \to \mathcal{SO}^{nc}$ and

$$\tilde{T}^\sigma := \sigma(T) = \sigma(\mathfrak{M}(T_1, T_2)) \quad \text{and} \quad \tilde{T} := \mathfrak{M}^{nc}(\sigma(T_1), \sigma(T_2))$$

- in same fiber over T so $\exists\, \gamma \in G_{\tilde{T}}$ such that $\gamma(\tilde{T}) = \tilde{T}^\sigma$
- incompatibility between the linearization algorithm and the asymmetric Merge are pairs $(\tilde{T}^\sigma, \tilde{T})$ for which $\gamma \in G_{\tilde{T}}$ with $\gamma(\tilde{T}) = \tilde{T}^\sigma$ is not in the subgroup $\mathrm{Aut}(\tilde{T})$
- these are the sources of the non-well-behaved concatenations
- better: *reformulate* concatenation for free symmetric Merge

Combine operation and free symmetric Merge

- has to receive map from syntax, compatible with Merge (magma homomorphism) so have to define *Concatenate* in symmetric form
- again *Combine* = *Label* ∘ *Concatenate* two operations

$$\mathit{Concatenate}(\alpha,\beta) = \{\alpha,\beta\} = \overset{\frown}{\alpha \quad \beta}$$

$$\mathit{Combine}(\alpha,\beta) = \mathit{Label} \circ \mathit{Concatenate}(\alpha,\beta)$$

$$= \mathit{Label}(\overset{\frown}{\alpha \quad \beta}) = \underset{\overset{\frown}{\alpha \quad \beta}}{h(\alpha,\beta)}$$

- binary operation *Combine* is not symmetric because of the *head* label
- if compositional operation takes place in semantic space $\mathcal{S}$, then $\mathcal{S}$ needs to have own computational system (at least partially defined): two systems each with "Merge" type operation, one for syntax one for semantics
- different from other conceptual spaces (perceptual manifolds for vision)

Main claim: Merge suffices

all computational structure is on the side of syntax

- start with map $s : \mathcal{SO}_0 \to \mathcal{S}$ extended to $s : \mathrm{Dom}(h) \to \mathcal{S}$ (using some property like geodesic convexity on $\mathcal{S}$)
- the i-concept *Combine*(α, β) where $\alpha = s(T_1)$ and $\beta = s(T_2)$ is well defined if $T = \mathfrak{M}(T_1, T_2) \in \mathrm{Dom}(h)$ and given by

$$Combine(\alpha, \beta) := s(\mathfrak{M}(T_1, T_2)) \in \mathcal{S}$$

  with $s(T)$ constructed using geodesic arcs and a semantic proximity $\mathcal{P}$ as discussed before
- Note: no need to separate *Combine* into *Label* and *Concatenate*
- check that this is OK with some potential issues (idempotents, rule out improper inferences)

idempotents

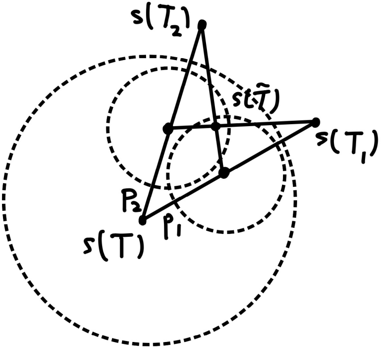
- on the semantics side expect possible idempotent structures $Combine(\alpha, \alpha) = \alpha$
- never have this on the syntax side: $\mathfrak{M}(T, T) = \widehat{T \quad T} \neq T$
- is this a problem with $Combine(\alpha, \beta) := s(\mathfrak{M}(T_1, T_2))$?
- no: it just means $s : \mathrm{Dom}(h) \to \mathcal{S}$ not always an embedding
- location of the point $s(\mathfrak{M}(T, T'))$ on geodesic arc between $s(T)$ and $s(T')$ depends on $\mathbb{P}(s(T), s(T'))$
- if $\mathbb{P}(s, s') = 0$ or $\mathbb{P}(s, s') = 1$ obtain cases where

$$Combine(\alpha, \beta) = \alpha \quad \text{or} \quad Combine(\alpha, \beta) = \beta$$

even if $\mathfrak{M}(T, T') \neq T$ and $\mathfrak{M}(T, T') \neq T'$

inference: Example (adjuncts to verb)

- consider sentences: "John ate a sandwich in the basement" and "John ate a sandwich at noon",
- these two sentences clearly do *not* imply that "John ate a sandwich in the basement at noon"
- one can see this in terms of the construction of $s(\mathfrak{M}(T_1, T_2))$



Heim-Kratzer semantics (often considered in generative linguistics)

Semantic *types* inductive construction

- type τ and set D_τ of possible "denotations"
- e type of *individual* D_e set of individuals
- t is type of *truth values* $D_t = \{0, 1\}$ set of truth values
- $\langle e, t\rangle$ type of *functions* $D_{\langle e,t\rangle} = \{f : D_e \to D_t\}$
- inductively σ and τ types, then $\langle \sigma, \tau\rangle$ type of functions

$$D_{\langle \sigma,\tau\rangle} = \{f : D_\sigma \to D_\tau\}$$

semantic space in this setting is

$$\mathcal{S} = \bigcup_\tau D_\tau$$

interpretability

- assume $s : \mathcal{SO}_0 \to \mathcal{S}$, want to extend to $s : \mathrm{Dom}(h) \subset \mathcal{SO} \to \mathcal{S}$
- HK prescription:
  1. for $T = \mathfrak{M}(T_1, T_2)$ if $s(T_1) = [[T_1]]$ in $\mathcal{S}$ (or $s(T_2)$) is a function $s(T_1) \in D_{\langle \sigma, \tau \rangle}$ and the other $s(T_2) \in D_\sigma$
  $$s(T) = [[T]] := s(T_1)(s(T_2)) = [[T_1]]([[T_2]]) \in D_\tau$$
  2. if neither $s(T_1)$ nor $s(T_2)$ is a function that takes the type of the other as input, then $T \notin \mathrm{Dom}(s)$ and T is non-interpretable
- non-interpretability can be due to a mismatch of function/input at some internal vertex while other substructures interpretable: again use a Birkhoff factorization to extract where problems occur

- Boolean semiring $\mathfrak{B} = (\{0,1\}, \max, \cdot)$, identity Rota-Baxter $R = id$
- character from Hopf algebra of workspaces

$$\phi(T) = \begin{cases} 1 & T \text{ is interpretable in HK semantics} \\ 0 & \text{otherwise.} \end{cases}$$

- Bogolyubov preparation

$$\tilde{\phi}(T) = \max\{\phi(T), \phi(F_{\underline{v}_1})\phi(T/^d F_{\underline{v}_1}), \ldots, \phi(F_{\underline{v}_N})\phi(F_{\underline{v}_{N-1}}/^d F_{\underline{v}_N})\cdots\phi(T/^d F_{\underline{v}_1})\}$$

- maximizers of $\tilde{\phi}(T)$ are chains of accessible terms where all the substructures and the respective quotients are themselves interpretable (even when $\phi(T) = 0$ and full T non-interpretable)

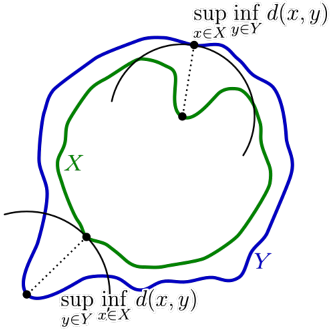
topological HK semantic types

- assume D_e is geodesically convex compact Riemannian manifold as our previous semantic spaces (proximity.interpolation)
- $D_t = \{0,1\}$ is a discrete set
- continuous functions $f : D_\tau \to D_t = \{0,1\}$ too small: only detects connected components of D_τ
- Note: all functions $f : D_e \to D_t = \{0,1\}$ is 2^{D_e} power set (characteristic functions of subsets)
- subset of 2^{D_e} that topologized nicely: characteristic functions of *compact* subsets
- instead of all functions $f : D_e \to D_t = \{0,1\}$ take set of compact subsets $\mathcal{K}(D_e)$ (identified with their characteristic functions)

- set $\mathcal{K}(D_e)$ of compact subsets with Hausdorff metric

$$d_H(A,B) = \max\{\sup_a d(a,B), \sup_b d(A,b)\}$$

(or Vietoris topology if D_e just topological space)



Hausdorff metric and Vietoris topology

- more general case where X just topological space
- the set $\mathcal{K}(X)$ of compact subsets topologized by Vietoris topology
- generated by "hit-and-miss" sets (sets that meet a given open set and sets that miss its complement):
- for U varying over the open sets of X

$$\begin{aligned} \mathcal{V}_{+,U} = & \ \{K \in \mathcal{K}(X) \,|\, K \cap U \neq \emptyset\} \\ \mathcal{V}_{-,U} = & \ \{K \in \mathcal{K}(X) \,|\, K \subset U\}\,. \end{aligned}$$

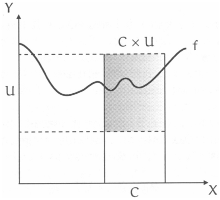
- when the Hausdorff metric exists, it induces the Vietoris topology

topologizing spaces of functions

compact open topology

- generated by sets $\mathcal{U}_{K,U}$, for $K \subset X$ compact and $U \subset Y$ open,

$$\mathcal{U}_{K,U} = \{ f \in \mathcal{C}(X,Y) \,|\, f(K) \subset U \}$$



- if X is compact and Y is metric induced by

$$d_{\mathcal{C}(X,Y)}(f,g) = \sup_{x \in X} \{ d_Y(f(x), g(x)) \}$$

- Note: even if start with best possible properties for D_e (metric, compact, connected, complete) these do not extend inductively to function spaces (can be non-compact, non-metrizable) to $\mathcal{S}$ has weaker topological properties

- topological space
$$\mathcal{S}^c = \bigcup_\tau D^c_\tau$$
- assuming that D_e is already a topological space, which is metric, complete, and compact $D^c_e = D_e$
- $D^c_{\langle e,t\rangle} = \mathcal{K}(D_e) \subset D_{\langle e,t\rangle}$ characteristic functions of closed (hence compact since D_e is compact) subsets of D_e
- with Hausdorff metric $(\mathcal{K}(D_e), d_H)$ is both complete and compact since D_e is
- D_e compact, $D^c_{\langle e,t\rangle}$ metric $\Rightarrow D^c_{\langle e,t\rangle}$ metrizable
- next step space of continuous functions
$$D^c_{\langle e,\langle e,t\rangle\rangle} = \mathcal{C}(D_e, D^c_{\langle e,t\rangle}) = \{f : D_e \to D^c_{\langle e,t\rangle} \mid f \text{ continuous}\}$$
with compact-open topology
- D_e is compact and $D^c_{\langle e,t\rangle}$ is metric, $D^c_{\langle e,t\rangle}$ is metrizable

- similarly also metrizable

$$D^c_{\langle e^n,t\rangle} := D^c_{\underbrace{\langle e,\langle e,\cdots,\langle e,t\rangle\cdots\rangle}_{n-\text{times}}} = \mathcal{C}(D_e, D^c_{\underbrace{\langle e,\langle e,\cdots,\langle e,t\rangle\cdots\rangle}_{(n-1)-\text{times}}})$$

- several inductive types maintain metrizability (not compactness)
- but other inductive types can also lose metrizability
- two arbitrary choices τ_1, τ_2 of types, $\tau_2 \neq t$
- form the type $\tau = \langle \tau_1, \tau_2 \rangle$, with

$$D^c_{\langle \tau_1,\tau_2\rangle} = \mathcal{C}(D^c_{\tau_1}, D^c_{\tau_2}) = \{f : D^c_{\tau_1} \to D^c_{\tau_2} \,|\, f \text{ continuous}\} \subset D_{\langle \tau_1,\tau_2\rangle},$$

  with the compact-open topology
- now $D^c_{\tau_1}$ in general non-compact, so even though $D^c_{\tau_2}$ metrizable, the metrizability property need not extend to $D^c_{\langle \tau_1,\tau_2\rangle}$

probes and characters

- probe Υ in topological Heim–Kratzer semantics
- collection $\Upsilon = \{\Upsilon_\tau\}_\tau$ of compact subsets $\Upsilon_\tau \subset D^c_\tau$
- Boolean character from probe

$$\phi_{\Upsilon,s}(T) = \begin{cases} \chi_{\Upsilon_\tau}(s(T)) & \text{if } s(T) \in D^c_\tau \\ 0 & \text{otherwise,} \end{cases}$$

- $\chi_{\Upsilon_\tau} \in D^c_{\langle \tau, t\rangle}$ characteristic function of compact set Υ_τ
- $s(T) \in \mathcal{S}$ is the HK interpretation of T when T is interpretable
- $\phi_{\Upsilon,s}(F) = \prod_a \phi_{\Upsilon,s}(T_a)$ for $F = \sqcup_a T_a$
- Bogolyubov preparation $\tilde{\phi}_{\Upsilon,s}$ identifies nested chains of substructures where the probe evaluates *True* on all the terms

fuzzy topological HK types

- fuzzy set (X, f) with $f : X \to [0,1]$
- fuzzy truth values $D_f = [0,1]$
- $\langle e, f\rangle$ type with $D^f_{\langle e,f\rangle} = \mathcal{C}(D_e, [0,1])$ continuous functions with compact-open topology
- semantic probes $\{\upsilon_\tau : D^f_\tau \to [0,1]\}$ fuzzy set structures
- $\upsilon_\tau(s(T)) \in [0,1]$ (Viterbi parsing) for $s(T) \in D^f_\tau \subset \mathcal{S}$ zero otherwise

fuzzy interpretability

nterpretability is assignment $s : \mathcal{SO}_0 \to \mathcal{S}^{c,\mathfrak{f}}$ that extends to partially defined $s : \mathrm{Dom}(s) \subset \mathcal{SO} \to \mathcal{S}^{c,\mathfrak{f}}$ following same rules as original Heim–Kratzer interpretability, but with the sets D_τ replaced by topological spaces $D_\tau^{c,\mathfrak{f}}$.

probes and Viterbi parsing

- fuzzy semantic probes: collections $\upsilon = \{\upsilon_T\}_T$ of continuous fuzzy set structures $\upsilon_T : D_T^{c,\mathfrak{f}} \to [0,1]$ on $D_T^{c,\mathfrak{f}}$,

$$\upsilon_T \in \mathcal{C}(D_T^{c,\mathfrak{f}}, D_{\mathfrak{f}}^{c,\mathfrak{f}}).$$

- Viterbi semiring

$$\mathcal{P} = ([0,1], \max, \cdot, 0, 1)$$

- associated character $\phi_{\upsilon,s,\mathfrak{f}}$ with values in $\mathcal{P}$

$$\phi_{\upsilon,s,\mathfrak{f}}(T) = \begin{cases} \upsilon_T(s(T)) & \text{if } s(T) \in D_T^{c,\mathfrak{f}} \\ 0 & \text{otherwise.} \end{cases}$$

- Rota-Baxter threshold operators c_λ on $\mathcal{P}$
- Birkhoff factorization identifies as maximizers those accessible terms $T_v \subset T$ with values $\phi_{\upsilon,s,\mathfrak{f}}(T_v)$ above a threshold λ, meaning with a sufficently large degree of confidence as their assigned fuzzy truth values

Some references

- P.M. Pietroski, *Conjoining Meanings. Semantics Without Truth Values*, Oxford University Press, 2018.
- I. Heim, A. Kratzer, *Semantics in Generative Grammar*, Blackwell Publishing, 1998.
- Noam Chomsky, *Studies on Semantics in Generative Grammar*, Mouton, 1972.

Semantics: from Topology to Logic

- proposed basic structure of semantics to be topological: proximity, relatedness, distance, interpolation
- in general argued that semantics should also account for logical operations
- assuming only the basic model: where does the *logic* part comes from?
- idea: open sets in a topological space define a (Brower) logic
- Boolean and Heyting algebras

concrete Boolean algebra

- non-empty set of subsets of a given set X that is closed under the operations of union, intersection, and complement
- e.g. power set $\mathcal{P}(X)$: set of all subsets of X
- intersection = logical AND, notations:

$$A \wedge B = A \cdot B = A \text{ AND } B := A \cap B$$

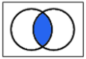
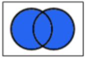
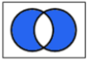
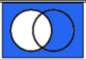
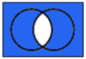
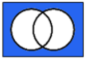
- union = logical OR, notations:

$$A \vee B = A + B = A \text{ OR } B := A \cup B$$

- complement = negation:

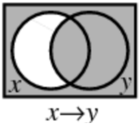
$$\bar{A} = \neg A = \text{NOT } A := A^c = X \smallsetminus A$$

- other logical gates, such as XOR, NAND, NOR, XNOR are compositions of these

Expression	Symbol	Venn diagram	Boolean algebra	Values: A	B	Output
AND			$A \cdot B$	0 0 1 1	0 1 0 1	0 0 0 1
OR			$A + B$	0 0 1 1	0 1 0 1	0 1 1 1
XOR			$A \oplus B$	0 0 1 1	0 1 0 1	0 1 1 0
NOT			$\overline{A}$	0 1		1 0
NAND			$\overline{A \cdot B}$	0 0 1 1	0 1 0 1	1 1 1 0
NOR			$\overline{A + B}$	0 0 1 1	0 1 0 1	1 0 0 0
XNOR			$\overline{A \oplus B}$	0 0 1 1	0 1 0 1	1 0 0 1

other logical operations: implication

$$A \to B = \neg A \vee B := A^c \cup B$$



$\to$ (x \ y)	0	1
0	1	1
1	0	1

abstract Boolean algebra

- partially ordered set $(\mathcal{B}, \leq)$
- two associative and commutative operations: *join* $\vee$ and *meet* $\wedge$
  - if $A_1 \leq A_2$ and $B_1 \leq B_2$ then $A_1 \vee B_1 \leq A_2 \vee B_2$ and $A_1 \wedge B_1 \leq A_2 \wedge B_2$
  - idempotent, $A \wedge A = 1$ and $A \vee A = A$ for all $A \in \mathcal{B}$
  - absorption law, for all $A, B \in \mathcal{B}$

$$A \vee (A \wedge B) = A \quad \text{and} \quad A \wedge (A \vee B) = A$$

  - distributive law, for all $A, B, C \in \mathcal{B}$

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

- also $\mathcal{B}$ has *complemented property*: $\exists$ least element 0, greatest element 1

$$A \vee 0 = A \quad \text{and} \quad A \vee 1 = 1$$

$$A \wedge 0 = 0 \quad \text{and} \quad A \wedge 1 = A$$

  and every A in $\mathcal{B}$ has complement $c(A) \in \mathcal{B}$ with $A \vee c(A) = 1$ and $A \wedge c(A) = 0$

Boolean algebras satisfy De Morgan law

$$\neg A \wedge \neg B = \neg(A \vee B) \quad \text{and} \quad \neg A \vee \neg B = \neg(A \wedge B)$$

other properties of classical Boolean logic

- *cancellation of double negations*

$$\neg\neg A = A$$

- *law of excluded middle*

$$\neg A \vee A = 1$$

- *law of non-contradiction*

$$A \wedge \neg A = 0$$

law of excluded middle is the complemented property:
$(X \smallsetminus A) \cup A = X$

Constructive logic (Intuitionistic/Brouwer logic)

- in classical logic propositions are assigned a truth value (law of excluded middle: this is always possible)
- in constructive logic propositions are only assigned a true value if a constructive proof (an algorithm) is available: proposition is "inhabited" by a proof
- law of excluded middle and the cancellation of double negation do not hold universally in constructive logic (but can hold in specific cases)
- Boolean algebras are replaced by *Heyting algebras*

abstract Heyting algebra

- partially ordered set $(\mathcal{B}, \leq)$
- operations of join $\vee$ and meet $\wedge$ as before, with minimal and maximal elements 0 and 1
- instead of the complemented property one only assumes the existence of a "relative pseudo-complement": for any $A, B \in \mathcal{B}$ $\exists$ greatest element $C = \psi(A, B) \in \mathcal{B}$ such that

$$A \wedge C \leq B$$

- this replaces the Boolean definition of logical implication with

$$A \to B := \psi(A, B)$$

- pseudo-complement

$$\psi c(A) := (A \to 0)$$

so negation defined as

$$\neg A := \psi c(A) = (A \to 0)$$

other properties of constructive logic

- *law of non-contradiction* $A \wedge \neg A = 0$ holds as in Boolean case
- *law of the excluded middle* $A \vee \neg A = 1$ no longer holds with $\neg A$ the pseudo-complement
- pseudo-complement and implication (relative pseudo-complement) satisfy

$$(\neg A \vee B) \leq (A \to B)$$

  (it is Boolean iff this $\leq$ is everywhere $=$)
- only one De Morgan law holds:

$$\neg(A \vee B) = \neg A \wedge \neg B$$

- the other is replaced by weaker form

$$\neg(A \wedge B) = \neg\neg(\neg A \vee \neg B)$$

  where double negation in general does not cancel concrete Heyting algebras:

- open sets in a topological space $(X, \mathcal{T}_X)$
- implication (relative pseudo-complement) given by

$$A \to B := \mathrm{Int}(A^c \cup B)$$

Int = interior, A^c = set-theoretic complement
- pseudo-complement is $\psi c(A) = \mathrm{Int}(A^c)$

Note: other important parts of mathematical logic depend on constructive logic:

- Martin-Löf type theory
- homotopy type theory

type theory closely related to inductive construction of types used in Heim Kratzer semantics

punchline: having a topological space as the basis of semantics guarantees to also have a logic (constructive) through the Heyting algebra of its open sets