1.2 From Lagrangian to effective action

In the case of a scalar field theory, one replaces the expression $\frac{1}{2}x^2 + P(x)$ of the one-dimensional toy model we saw in the previous section with a non-linear functional, the Lagrangian density, defined on a configuration space of classical fields. Here we give only a very brief account of the basics of perturbative quantum field theory. A more detailed presentation, aimed at giving a self contained introduction to mathematicians, can be found in the book [Connes and Marcolli (2008)].

In the scalar case the classical fields are (smooth) functions on a spacetime manifolds, say $\phi \in \mathcal{C}^\infty(\mathbb{R}^D, \mathbb{R})$, and the Lagrangian density is given by an expression of the form

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{P}(\phi), \tag{1.15}$$

where $(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ for $g^{\mu\nu}$ the Lorentzian metric of signature $(1, -1, -1, \ldots, -1)$ on $\mathbb{R}^D$ and a summation over repeated indices understood. The *interaction term* $\mathcal{P}(\phi)$ in the Lagrangian is a polynomial in the field ϕ of degree $\deg \mathcal{P} \geq 3$. Thus, when one talks about a scalar field theory one means the choice of the data of the Lagrangian density and the spacetime dimension D. We can assume for simplicity that $\mathcal{P}(\phi) = \frac{\lambda}{k!}\phi^k$. We will give explicit examples using the special case of the ϕ^3 theory in dimension $D = 6$: while this is not a physically significant example because of the unstable equilibrium point of the potential at $\phi = 0$, it is both sufficiently simple and sufficiently generic with respect to the renormalization properties (*i.e.* non superrenormalizable, unlike the more physical ϕ^4 in dimension $D = 4$) .

To the Lagrangian density one associates a classical action functional

$$S_L(\phi) = \int_{\mathbb{R}^D} \mathcal{L}(\phi) d^D x. \tag{1.16}$$

The subscript L here stays for the Lorentzian signature of the metric and we'll drop it when we pass to the Euclidean version. This classical action is written as the sum of two terms $S_L(\phi) = S_{free,L}(\phi) + S_{int,L}(\phi)$, where the free field part is

$$S_{free,L}(\phi) = \int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 \right) d^D x$$

and the interaction part is given by

$$S_{int,L}(\phi) = -\int_{\mathbb{R}^D} P(\phi) d^D x.$$

The *probability amplitude* associated to the classical action is the expression

$$e^{i\frac{S_L(\phi)}{\hbar}}, \tag{1.17}$$

where $\hbar = h/2\pi$ is Planck's constant. In the following we follow the convention of taking units where $\hbar = 1$ so that we do not have to write explicitly the powers of $\hbar$ in the terms of the expansions. An observable of a scalar field theory is a functional on the configuration space of the classical fields, which we write as $\mathcal{O}(\phi)$. The *expectation value* of an observable is defined to be the functional integral

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int \mathcal{O}(\phi) e^{iS_L(\phi)} \mathcal{D}[\phi]}{\int e^{iS_L(\phi)} \mathcal{D}[\phi]}, \tag{1.18}$$

where the integration is supposed to take place on the configuration space of all classical fields. In particular, one has the N-points Green functions, defined here as

$$G_{N,L}(x_1, \ldots, x_N) = \frac{\int \phi(x_1) \cdots \phi(x_N)\, e^{iS_L(\phi)} \mathcal{D}[\phi]}{\int e^{iS_L(\phi)} \mathcal{D}[\phi]}, \tag{1.19}$$

for which the generating function is given again by a functional integral with source term

$$\int e^{iS_L(\phi) + \langle J, \phi \rangle} \mathcal{D}[\phi], \tag{1.20}$$

where J is a linear functional (a distribution) on the space of classical fields and $\langle J, \phi \rangle = J(\phi)$ is the pairing of the space of fields and its dual. If $J = J(x)$ is itself a smooth function then $\langle J, \phi \rangle = \int_{\mathbb{R}^D} J(x)\phi(x) d^D x$.

Although the notation of (1.18) and (1.19) is suggestive of what the computation of expectation values should be, there are in fact formidable obstacles in trying to make sense rigorously of the functional integral involved. Despite the successes of constructive quantum field theory in several important cases, in general the integral is ill defined mathematically. This is, in itself, not an obstacle to doing quantum field theory, as long as one regards the expression (1.18) as a shorthand for a corresponding asymptotic expansion, obtained by analogy to the finite dimensional case we have seen previously.

A closer similarity between (1.20) and (1.4) appears when one passes to Euclidean signature by a Wick rotation to imaginary time $t \mapsto it$. This has the effect of switching the signature of the metric to $(1, 1, \ldots, 1)$, after collecting a minus sign, which turns the probability amplitude into the Euclidean version

$$e^{iS_L(\phi)} \mapsto e^{-S(\phi)}, \tag{1.21}$$

with the Euclidean action

$$S(\phi) = \int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{P}(\phi) \right) d^D x. \tag{1.22}$$

Thus, in the Euclidean version we are computing functional integrals of the form

$$G_N(x_1, \ldots, x_N) = \frac{\int \phi(x_1) \cdots \phi(x_N)\, e^{-S(\phi)}\, \mathcal{D}[\phi]}{\int e^{-S(\phi)}\, \mathcal{D}[\phi]}, \tag{1.23}$$

for which the generating function resembles (1.4) in the form

$$Z[J] = \int e^{-\int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{P}(\phi) + J(x)\phi(x) \right) d^D x}\, \mathcal{D}[\phi], \tag{1.24}$$

satisfying

$$\frac{Z[J]}{Z[0]} = \sum_{N=0}^{\infty} \frac{1}{N!} \int J(x_1) \cdots J(x_N)\, G_N(x_1, \ldots, x_N)\, d^D x_1 \cdots d^D x_N, \tag{1.25}$$

for

$$Z[0] = \int e^{-\int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{P}(\phi) \right) d^D x}\, \mathcal{D}[\phi]. \tag{1.26}$$

In order to make sense of this functional integral, one uses an analog of the asymptotic expansion (1.6), where one expands out the exponential of the interaction term $S_{int}(\phi) = \int_{\mathbb{R}^D} \mathcal{P}(x)\, d^D x$ of the Euclidean action and one follows the same formal rules about integration by parts of the final dimensional case to write the label the terms of the expansion by graphs. What is needed in order to write the contribution of a given graph to the asymptotic series is to specify the rules that associate the analogs of the powers of λ, J and a^{-1} to the vertices, external and internal edges of the graph. These are provided by the *Feynman rules* of the theory.

1.3 Feynman rules

By analogy to what we saw in the 1-dimensional model, where one writes the Green functions (1.11) in terms of integrals of the form (1.12), and the latter in terms of sums over graphs as in (1.13), one writes the Green functions (1.23) also in terms of an asymptotic series whose terms are parameterized by graphs,

$$\mathcal{G}_N(p_1, \ldots, p_N) = \sum_{\Gamma} \frac{V(\Gamma, p_1, \ldots, p_N)}{\#\mathrm{Aut}(\Gamma)}, \tag{1.27}$$

where $\mathcal{G}(p_1, \dots, p_N)$ is the Green function in momentum space, *i.e.* the Fourier transform

$$\mathcal{G}_N(p_1, \dots, p_N) = \int G_N(x_1, \dots, x_N) e^{i(p_1 x_1 + \cdots + p_N x_N)} \frac{d^D p_1}{(2\pi)^D} \cdots \frac{d^D p_N}{(2\pi)^D}. \tag{1.28}$$

The reason for writing the contributions of Feynman integrals in momentum space is that in physics one does not only think of the Feynman graphs as computational devices that do the bookkeeping of terms in integration by parts of polynomials under a Gaussian measure, but one can think of a diagram as representing a (part of) a physical process, where certain particles with assigned momenta (external edges) interact (vertices) by creation and annihilation of virtual particles (internal edges). The momenta flowing through the graph then represent the physical process. In fact, it is clear from this point of view that what has physical meaning is not so much an individual graph but the collection of all graphs with given external edges and assigned external momenta, and among them the subset of all those with a fixed number of loops. The latter specifies the order in the perturbative expansion one is looking at. The terms $V(\Gamma, p_1, \dots, p_N)$ are constructed according to the Feynman rules as follows.

- Each internal edge $e \in E_{int}(\Gamma)$ contributes a momentum variable $k_e \in \mathbb{R}^D$ so that

$$V(\Gamma, p_1, \dots, p_N) = \int \mathcal{I}_\Gamma(p_1, \dots, p_N, k_1, \dots, k_n) \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_n}{(2\pi)^D}, \tag{1.29}$$

  for $n = \# E_{int}(\Gamma)$.
- Each vertex $v \in V(\Gamma)$ contributes a factor of $\lambda_v (2\pi)^D$, where λ_v is the coupling constant of the monomial in the Lagrangian of order equal to the valence of v and a conservation law for all the momenta that flow through that vertex,

$$\delta_v := \delta\Big(\sum_{s(e)=v} k_e - \sum_{t(e)=v} k_e \Big), \tag{1.30}$$

  written after chosing an orientation of the edges of the graph.
- Each internal edge contributes an inverse propagator, that is, a term of the form q_e^{-1}, where q_e is a quadratic form, which in the case of a scalar field in the Euclidean signature is given by

$$q_e(k_e) = k_e^2 + m^2. \tag{1.31}$$

- Each external edge $e \in E_{ext}(\Gamma)$ contributes a propagator $q_e(p_e)^{-1}$, with $q_e(p_e) = p_e^2 + m^2$. The external momenta are assigned so that they satisfy the conservation law $\sum_e p_e = 0$, when summed over the oriented external edges.
- The integrand $\mathcal{I}_\Gamma(p_1, \dots, p_N, k_1, \dots, k_n)$ is then a product

$$\prod_{v \in V(\Gamma)} \lambda_v (2\pi)^D \delta_v \prod_{e \in E_{int}(\Gamma)} q_e(k_e)^{-1} \prod_{e \in E_{ext}(\Gamma)} q_e(p_e)^{-1}. \tag{1.32}$$

We can then write the Feynman integral associated to a Feynman graph Γ of the given theory in the form

$$V(\Gamma, p_1, \dots, p_N) = \varepsilon(p_1, \dots, p_N)\, U(\Gamma, p_1, \dots, p_N), \tag{1.33}$$

where the factor $\varepsilon(p)$ is the product of the inverse propagators of the external edges

$$\varepsilon(p_1, \dots, p_N) = \prod_{e \in E_{ext}(\Gamma)} q_e(p_e)^{-1}, \tag{1.34}$$

while the factor $U(\Gamma, p)$ is given by

$$U(\Gamma, p_1, \dots, p_N) = C \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1(k_1) \cdots q_n(k_n)} \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_n}{(2\pi)^D}, \tag{1.35}$$

with $C = \prod_{v \in V(\Gamma)} \lambda_v (2\pi)^D$.

1.4 Simplifying graphs: vacuum bubbles, connected graphs

There are some useful simplifications that can be done in the combinatorics of graphs that appear in the formal series (1.27).

The basic property that makes these simplifications possible is the multiplicative form (1.32) of the Feynman integrand $\mathcal{I}_\Gamma(p_1, \dots, p_N, k_1, \dots, k_n)$. This implies the following property.

Lemma 1.4.1. *The Feynman integral $V(\Gamma, p_1, \dots, p_N)$ is multiplicative on connected components of the graph Γ.*

Proof. This follows immediately from the form (1.32) and (1.33) with (1.34), (1.35) of the Feynman integral. In fact, if the graph Γ has different connected components, no linear relations arise between momentum variables of the edges of different components (as these have no common vertices) and the corresponding integrals split as a product. □

Moreover, one also has the following multiplicative form of the symmetry factors of graphs.

Lemma 1.4.2. *For a graph Γ that is a union of connected components Γ_j with multiplicities n_j (i.e. there are n_j connected components of Γ all isomorphic to the same graph Γ_j), the symmetry factor splits multiplicatively on components according to the formula*

$$\#\mathrm{Aut}(\Gamma) = \prod_j (n_j)! \prod_j \#\mathrm{Aut}(\Gamma_j)^{n_j}. \tag{1.36}$$

Proof. The factorials come from the symmetries of the graph Γ that permute topologically equivalent components. All symmetries of Γ are obtained by composing this type of symmetries with symmetries of each component. □

One then has a first useful observation on the combinatorics of the graphs that appear in the asymptotic expansion of the Green functions.

Lemma 1.4.3. *The graphs of* (1.27) *do not contain* vacuum bubbles.

Proof. Recall that a vacuum bubble is a graph with no external edges. As we have seen in the finite dimensional toy model, these correspond to the terms with $J = 0$ in the asymptotic series. Thus, when one writes the expansion (1.25) into Green functions, and then the expansion (1.27) of the latter into Feynman integrals of graphs, the expansion of the functional integral $Z[J]$ would count the contribution of all graphs including components that are vacuum bubbles as in the case of (1.8) in the finite dimensional case. The expansion of $Z[0]$ on the other hand only has contributions from the vacuum bubble graphs, and the multiplicative properties of Lemma 1.4.1 and Lemma 1.4.2 then imply that the expansion for $Z[J]/Z[0]$ only has contributions from graphs with no vacuum bubbles. □

One can then pass from multi-connected to connected graphs by rewriting the functional integral $Z[J]$ in an equivalent form in terms of

$$W[J] = \log\left(\frac{Z[J]}{Z[0]}\right). \tag{1.37}$$

One can again write a formal asymptotic series for $W[J]$ as

$$W[J] = \sum_{N=0}^{\infty} \frac{1}{N!} \int J(x_1)\cdots J(x_N)\, G_{N,c}(x_1,\ldots,x_N)\, d^D x_1 \cdots d^D x_N, \tag{1.38}$$

where now the Green functions $G_{N,c}(x_1,\dots,x_N)$ will also have an expansion on graphs of the form (1.27), where, however, only a smaller class of graphs will be involved.

Lemma 1.4.4. *The* connected *Green functions* $G_{N,c}(x_1,\dots,x_N)$ *of* (1.38) *have an expansion*

$$\mathcal{G}_{N,c}(p_1,\dots,p_N) = \sum_{\Gamma \text{ connected}} \frac{V(\Gamma,p_1,\dots,p_N)}{\#\mathrm{Aut}(\Gamma)}, \tag{1.39}$$

where $\mathcal{G}_{N,c}(p_1,\dots,p_N)$ *is the Fourier transform*

$$\mathcal{G}_{N,c}(p_1,\dots,p_N) = \int G_{N,c}(x_1,\dots,x_N) e^{i(p_1x_1+\cdots+p_Nx_N)} \frac{d^Dp_1}{(2\pi)^D}\cdots\frac{d^Dp_N}{(2\pi)^D} \tag{1.40}$$

and the $V(\Gamma,p_1,\dots,p_N)$ *in* (1.39) *are computed as in* (1.33).

Proof. We only sketch briefly why the result holds. More detailed expositions can be found in standard Quantum Field Theory textbooks (for example in §of [LeBellac]). Suppose that Γ is a disjoint union of connected components $\Gamma = \cup_j \Gamma_j$, with multiplicities n_j and with N_j external edges, so that $\sum_j n_j N_j = N$. Then by Lemma 1.4.1 and 1.4.2 we get

$$\frac{V(\Gamma,p_1,\dots,p_N)}{\#\mathrm{Aut}(\Gamma)} = \prod_j \frac{V(\Gamma_j,p_1,\dots,p_{N_j})^{n_j}}{(n_j)!\#\mathrm{Aut}(\Gamma_j)^{n_j}}.$$

Thus, we can write

$$\frac{Z[J]}{Z[0]} = \sum_N \frac{1}{N!}\int J(x_1)\cdots J(x_N)\, G_N(x_1,\dots,x_N)\prod_i d^Dx_i$$

$$= \sum_N \sum_{\sum_j n_jN_j=N} \prod_j \frac{1}{n_j}\left(\int J(x_1)\cdots J(x_{N_j}) G_{N_j,c}(x_1,\dots,x_{N_j})\right)^{n_j}$$

$$= \exp\left(\sum_N \frac{1}{N!}\int J(x_1)\cdots J(x_N)\, G_{N,c}(x_1,\dots,x_N)\right). \qquad \square$$

1.5 One-particle-irreducible graphs

Further simplifications of the combinatorics of graphs can be obtained by passing to the *1PI effective action*, or higher loop versions like *2PI effective action*, etc. We discuss here briefly only the 1PI effective action, though

we will later need to return to discussing higher connectivity conditions on Feynman graphs. We first recall the following notions of connectivity of graphs.

Definition 1.5.1. The notion of k-connectivity of graphs is given as follows:

- A graph is k-edge-connected if it cannot be disconnected by removal of any set of k edges.
- A graph is k-vertex-connected if it cannot be disconnected by removal of any set of $k-1$ vertices.

Here what one means by removing a vertex from a graph is the following. Given a graph Γ and a vertex $v \in V(\Gamma)$, the graph $\Gamma \smallsetminus v$ is the graph with vertex set $V(\Gamma) \smallsetminus \{v\}$ and edge set $E(\Gamma) \smallsetminus \{e : v \in \partial(e)\}$, *i.e.* the graph obtained by removing from Γ the star of the vertex v. Thus, 1-vertex-connected and 1-edge-connected simply mean connected, while for $k \geq 2$ the condition of being k-vertex-connected is stronger than that of being k-edge-connected. The terminology more commonly in use in the physics literature is the following.

Definition 1.5.2. For $k \geq 2$ a $(k+1)$-edge-connected graph is also called k-particle-irreducible (kPI). For $k = 1$, a 2-edge-connected graph that is not a tree is called one-particle-irreducible (1PI) graph. These cannot be disconnected by removal of a single (internal) edge.

Notice that trees are all considered *not* to be 1PI, even though a tree consisting of just n edges attached to a single valence n vertex cannot be disconnected by removal of a single edge (such edges are not internal though in this case).

Lemma 1.5.3. *Any connected graph can be obtained from a tree, after replacing the vertices by 1PI graphs with a number of external edges equal to the valence of the vertex.*

Proof. If the connected graph Γ is 1PI the tree consists of a single vertex with a number of edges attached equal to the number of external edges of the graph. Suppose the graph is not 1PI. Find an edge that disconnects the graph. Look at each component and again repeat the operations finding edges that disconnect them further until one is left with a collection of 1PI graphs, $\Gamma_1, \ldots, \Gamma_n$. It suffices then to show that the graph obtained from Γ by shrinking each Γ_i to a vertex is a tree. Since each of the internal edges

that remain in this graph was an edge whose removal increased the number of connected components, this must still be true in the graph obtained after collapsing all the Γ_i. A graph that is disconnected by the removal of any one internal edge is a tree with at least one internal edge. □

This suggests that there should be a way to further simplify the combinatorics of graphs in the asymptotic expansion of the functional integrals, by counting separately the contributions of trees and that of 1PI graphs, and getting back from these the contributions of all connected graphs.

This is done by passing to the *1PI effective action*, which is defined as the Legendre transform of the functional $W[J]$, namely

$$S_{eff}(\phi) = (\langle \phi, J \rangle - W[J])\,|_{J=J(\phi)}, \tag{1.41}$$

evaluated at a stationary J, that is, a solution of the variational equation

$$\frac{\delta}{\delta J}\,(\langle \phi, J \rangle - W[J]) = 0.$$

The asymptotic expansion of the effective action collects the contributions of all the 1PI graphs, so that the semiclassical calculations (*i.e.* involving only graphs that are trees) done with the effective action recover the full quantum corrections to the classical action given by all the connected graphs that appear in the expansion of $W[J]$. We do not prove here how one gets the asymptotic expansion for the effective action and we refer the interested reader to [Connes and Marcolli (2008)] and [LeBellac].

In the following we restrict our attention to Feynman integrals of graphs that are at least 1PI.

1.6 The problem of renormalization

So far we have treated the integrals $U(\Gamma, p_1, \dots, p_N)$ of (1.35) as purely formal expressions. However, if one tries to assign to such integrals a numerical value, one soon realizes that most of them are in fact divergent. This was historically one of the main problems in the development of perturbative quantum field theory, namely the *renormalization problem*: how to extract in a consistent and physically significant manner finite values from the divergent integrals (1.35) that appear in the asymptotic expansion of the functional integrals of quantum field theory.

The problem of renormalization consists of three main aspects:

- Regularization

- Subtraction
- Renormalization

Regularization consists of a procedure that replaces the divergent integrals (1.35) by functions of some regularization parameters, in such a way that the resulting function has a pole or a divergence for particular values or limits of the additional parameters that correspond to the original divergent integral, but has finite values for other values of the regularization parameters. Subtraction then consists of removing the divergent part of the regularized integrals by a uniform procedure (such as removing the polar part of a Laurent series in the main example we use below). This is not all there is yet. Renormalization means being able to perform the subtraction procedure by modifying the parameters in the Lagrangian (which become themselves functions of the regularization parameter). A physical theory is called *renormalizable* if this last step is possible by adding at most finitely many new terms to the Lagrangian. A theory is non-renormalizable if, in order to correct by regularization and subtraction all the divergences arising from all the divergent graph integrals in the asymptotic series, one needs to add infinitely many new parameters to the theory.

To understand more clearly the last point, it is important to stress the fact that the parameters that appear in the Lagrangian, such as masses and couling constants of the interaction terms, are not physical observable, nor are the actual masses and physical quantities that one can measure in experiments. In fact, the parameters in the Lagrangian can be modified without affecting the physics one observes. This is what makes it possible, in a renormalizable theory, to correct for divergent graphs by readjusting the parameters in the Lagrangian. Thus, if one introduces a regularization of the divergent Feynman integrals in terms of a complex parameter z (as we discuss below) or in terms of a cutoff Λ, then the Lagrangian can be modified by changing the coefficients to

$$\mathcal{L}(\phi) = \left(\frac{1+\delta Z}{2}(\partial\phi)^2 + \frac{m^2+\delta m^2}{2}\phi^2 + \frac{\lambda+\delta\lambda}{k!}\phi^k \right), \tag{1.42}$$

where the functions δZ, δm^2 and $\delta\lambda$ depend on the regularization parameter. They consist, in fact, of a formal series of contributions coming from all the divergent graphs of the theory. Notice that a theory is still renormalizable if, in addition to the modifications of the coefficients of the terms initially present in the Lagrangian, to compensate for divergent graphs one needs to also add a *finite* number of other terms, *i.e.* other monomials $\frac{\delta\lambda_i}{(k_i)!}\phi^{k_i}$, that were not initially present. In fact, one can just think of this

as being the effect of having chosen an arbitrary value = 0 for the coefficients of these terms in the initial Lagrangian. However, a theory is no longer renormalizable if an infinite number of additional terms is needed to compensate for the divergences.

1.7 Gamma functions, Schwinger and Feynman parameters

We digress momentarily to recall some useful formulae involving Gamma functions, which are extensively used in Feynman integral coputations and are the basis of both the dimensional regularization procedure we describe below and the parametric representation of Feynman integrals that we discuss later and which is the basis for the relation between Feynman integrals and periods of algebraic varieties.

First recall that the Gamma function is defined by the integral

$$\Gamma(s+1) = \int_0^\infty t^s e^{-s} dt. \tag{1.43}$$

It satisfies $\Gamma(s+1) = s\,\Gamma(s)$, hence it extends the factorial, namely $\Gamma(n+1) = n!$ for nonnegative integers. The function $\Gamma(s)$ defined in this way extends to a meromorphic function with poles at all the non-positive integers.

As we are going to see in more detail below, a typical way of dealing with divergences in Feynman integrals is to first identify them with poles of some meromorphic function and typically a product of Gamma functions.

The first useful operation on Feynman integrals is the introduction of *Schwinger parameters*. These are based on the very simple identity

$$\frac{1}{q} = \int_0^\infty e^{-sq} ds. \tag{1.44}$$

This allows the reformulation of integrals where quadratic forms q_i in the momentum variables appear in the denominator in terms of Gaussian integrals where quadratic forms appear in the exponent. In terms of Schwinger parameters, one writes the denominator of a Feynman integral of the form (1.35) as

$$\frac{1}{q_1 \cdots q_n} = \int_{\mathbb{R}^n_+} e^{-(s_1 q_1 + \cdots + s_n q_n)} ds_1 \cdots ds_n. \tag{1.45}$$

This is a special case of the more general useful identity

$$\frac{1}{q_1^{k_1} \cdots q_n^{k_n}} = \frac{1}{\Gamma(k_1)\cdots\Gamma(k_n)} \int_{\mathbb{R}^n_+} e^{-(s_1 q_1 + \cdots + s_n q_n)}\, s_1^{k_1-1} \cdots s_n^{k_n-1}\, ds_1 \cdots ds_n. \tag{1.46}$$

Another related way to reformulate the expression (1.35) of Feynman integrals is based on the *Feynman parameters*. Here the basic example, analogous to (1.44) is the *Feynman trick*

$$\frac{1}{ab} = \int_0^1 \frac{1}{(ta + (1-t)b)^2} dt. \tag{1.47}$$

The more general expression analogous to (1.45) is obtained by considering the general formula (1.46) and performing a change of variables $s_i = St_i$ with $S = s_1 + \cdots + s_n$ so as to obtain

$$\frac{1}{q_1^{k_1} \cdots q_n^{k_n}} = \frac{\Gamma(k_1 + \cdots + k_n)}{\Gamma(k_1) \cdots \Gamma(k_n)} \int_{[0,1]^n} \frac{t_1^{k_1 - 1} \cdots t_n^{k_n - 1} \, \delta(1 - \sum_i t_i)}{(t_1 q_1 + \cdots + t_n q_n)^n} \, dt_1 \cdots dt_n. \tag{1.48}$$

In the particular case of the denominators of (1.35) this gives

$$\frac{1}{q_1 \cdots q_n} = (n-1)! \int_{[0,1]^n} \frac{\delta(1 - \sum_i t_i)}{(t_1 q_1 + \cdots + t_n q_n)^n} \, dt_1 \cdots dt_n. \tag{1.49}$$

Thus, the integration is performed on the n-dimensional *topological simplex*

$$\sigma_n = \{t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+ \mid \sum_i t_i = 1\}. \tag{1.50}$$

1.8 Dimensional Regularization and Minimal Subtraction