Tuesday June 1

Unitary equivalences

$$\not{\partial}_{X_\varepsilon} = \frac{\partial}{\partial t}\sigma_0 + e^{t}\frac{\partial}{\partial x}\sigma_1 + e^{-t}\frac{\partial}{\partial y}\sigma_2$$

$$\psi((x,y),t) = \sum_\lambda \psi_\lambda \, e^{2\pi i \langle (a,b),\lambda\rangle}$$

$$= \sum_\lambda \psi_\lambda \, e^{2\pi i \langle \Theta_{-t}(x,y),\lambda\rangle} = \sum_\lambda \psi_\lambda E_\lambda$$

where

$(x,y) = \Theta_t(a,b)$ $\quad t\in[0,\log\varepsilon)$

standard torus

torus $T_t = \mathbb{R}^2/\Lambda_t$

$$\Theta_{-t}(x,y) = (e^{-t}x,\ e^{t}y) \qquad \langle (a,b),\lambda\rangle = a\lambda_1 + b\lambda_2$$

$$\not{\partial}_{X_\varepsilon} E_\lambda = \left(\frac{\partial}{\partial t}\sigma_0 + 2\pi i\,\lambda_1\sigma_1 + 2\pi i\,\lambda_2\sigma_2\right) E_\lambda$$

Thus, passing to Fourier modes in the fiber T^2-direction gives unitary equivalence

$$\hat{\not{\partial}}_{X_\varepsilon}\psi_\lambda = \left(\frac{\partial}{\partial t}\sigma_0 + 2\pi i\,\lambda_1\sigma_1 + 2\pi i\,\lambda_2\sigma_2\right)\psi_\lambda$$

with NC torus action

$$\hat{\pi}(R^\sigma_\eta)\psi_\lambda = \sigma(\eta,\lambda)\,\psi_{\lambda+\eta}$$ in Connes-Landi deformation

Then other unitary equivalence:

$$\mathcal{U}\psi_\lambda = \sigma_\lambda \psi_\lambda \quad \text{where}$$

σ_λ = prod of Pauli matrices σ_i $i=1,2$
where σ_i is in σ_λ iff corresp. $\lambda_i < 0$

$$\Rightarrow \quad \mathcal{U}\hat{\partial}_{\chi_\varepsilon}\mathcal{U}^* = \mathrm{sign}(N(\lambda))\left(\frac{\partial}{\partial t}\sigma_0 + 2\pi i|\lambda_1|\sigma_1 + 2\pi i|\lambda_2|\sigma_2\right)$$

and with

$$\mathcal{U}\hat{\pi}(R_\eta^\sigma)\mathcal{U}^* : \sigma_\lambda\psi_\lambda \longmapsto \sigma_{\lambda+\eta}\psi_{\lambda+\eta}$$

Then one more unitary transformation:

$\mathcal{F}_V$ = fundam domain of action of V on lattice Λ

Write then $\lambda \in \Lambda$ as $\lambda = A_\varepsilon^k(\mu)$ some $\mu \in \mathcal{F}_V$
some $k \in \mathbb{Z}$

then for $\lambda \neq 0$ time shift

$$\tilde{\mathcal{U}}(\sigma_\lambda\psi_\lambda)(t) = \sigma_\lambda\psi_\lambda\left(t - \log\frac{|\mu_1|}{|N(\mu)|^{1/2}}\right)$$

$$\Rightarrow \quad \tilde{\psi}_\lambda := \tilde{\mathcal{U}}(\sigma_\lambda\psi_\lambda) = \sigma_\lambda \psi_{|N(\lambda)|^{1/2}(\mathrm{sign}(\lambda_1)\varepsilon^k,\ \mathrm{sign}(\lambda_2)\varepsilon^{-k})}$$

This gives a unitarily equivalent spectral triple (3)

$$\tilde{\not{D}} = \tilde{\not{D}}^{(0)} + \sum_{\mu \in \frac{\Lambda \setminus \{0\}}{V}} \tilde{\not{D}}^{\mu} \quad \text{where}$$

$$\tilde{\not{D}}^{\mu} \tilde{\psi}_{A_\varepsilon^k(\mu)} = \operatorname{sign}(N(\mu)) \, |N(\mu)|^{1/2} \cdot \left(|N(\mu)|^{-1/2} \frac{\partial}{\partial t} \sigma_0 + 2\pi i \varepsilon^k \sigma_1 + 2\pi i \varepsilon^{-k} \sigma_2 \right) \tilde{\psi}_{A_\varepsilon^k(\mu)}$$

$$\tilde{\pi}(R^\sigma_\eta) \tilde{\psi}_\lambda = \tilde{\psi}_{\lambda+\eta}$$

$$\tilde{\not{D}}^{\mu} = D_\mu B_\mu \quad \text{where} \quad D_\mu \tilde{\psi}_{A_\varepsilon^k(\mu)} = \operatorname{sign}(N(\mu)) |N(\mu)|^{1/2} \tilde{\psi}_{A_\varepsilon^k(\mu)}$$

$$B_\mu \tilde{\psi}_{A_\varepsilon^k(\mu)} = \left(|N(\mu)|^{-1/2} \frac{\partial}{\partial t} \sigma_0 + 2\pi i \varepsilon^k \sigma_1 + 2\pi i \varepsilon^{-k} \sigma_2 \right) \tilde{\psi}_{A_\varepsilon^k(\mu)}$$

Relation to NC torus

$$\psi_\lambda \longmapsto 2\pi i \, (\lambda_1 \sigma_1 + \lambda_2 \sigma_2) \, \psi_\lambda \quad \text{is}$$

$$D_{\theta,\theta'} = \begin{pmatrix} 0 & \delta_{\theta'} - i\delta_\theta \\ \delta_{\theta'} + i\delta_\theta & 0 \end{pmatrix}$$

$$\delta_\theta \psi_{n,m} \longmapsto (n + m\theta) \psi_{n,m}$$
$$\delta_{\theta'} \psi_{n,m} \longmapsto (n + m\theta') \psi_{n,m}$$

Under same unitary equivalences

$$\cancel{D}_{\theta,\theta',0} = \sum_{\mu \in \frac{\Lambda \setminus \{0\}}{V}} D^{\mu}_{\theta,\theta'}$$

$$\cancel{D}^{\mu}_{\theta,\theta'} \psi_{A^k_\varepsilon(\mu)} = (\lambda_1 \sigma_1 + \lambda_2 \sigma_2) \psi_{A^k_\varepsilon(\mu)}$$

becomes after unitary equivalences

$$\tilde{\cancel{D}}^{\mu}_{\theta,\theta'} \tilde{\psi}_{A^k_\varepsilon(\mu)} = \operatorname{sign}(N(\mu)) \, |N(\mu)|^{1/2} \, (\varepsilon^k \sigma_1 + \varepsilon^{-k} \sigma_2) \tilde{\psi}_{A^k_\varepsilon(\mu)}$$

$$\tilde{\cancel{D}}^{\mu}_{\theta,\theta'} = D^{\mu}_{\theta} B_{\theta}$$

$$D^{\mu}_{\theta} \tilde{\psi}_{A^k_\varepsilon(\mu)} = \operatorname{sign}(N(\mu)) \, |N(\mu)|^{1/2} \, \tilde{\psi}_{A^k_\varepsilon(\mu)}$$

$$B_{\theta} \tilde{\psi}_{A^k_\varepsilon(\mu)} = (\varepsilon^k \sigma_1 + \varepsilon^{-k} \sigma_2) \tilde{\psi}_{A^k_\varepsilon(\mu)}$$

$$\zeta_{\cancel{D}_{\theta,\theta'}}(s) = 2 Z_\varepsilon\left(\frac{s}{2}\right) \sum_{\mu} |N(\mu)|^{-s/2}$$

$$\eta_{\cancel{D}_{\theta,\theta'}}(s) = L\left(\Lambda, V, \frac{s}{2}\right) Z_\varepsilon\left(\frac{s}{2}\right)$$

with $Z_\varepsilon(s) = \sum_k (\varepsilon^{2k} + \varepsilon^{-2k})^{-s/2}$

Lorentzian spectral triples

(another look at NC tori w/ real multipl. and the Shimizu L-function)

$\Lambda \subset \mathbb{R}^2$ as before $\quad \mathbb{K} = \mathbb{Q}(\sqrt{d}) \quad \alpha_i : \mathbb{K} \hookrightarrow \mathbb{R}$

$V = \varepsilon^{\mathbb{Z}} \quad \lambda \mapsto A_\varepsilon^k(\lambda) = (\varepsilon^k \lambda_1, \bar{\varepsilon}^{k} \lambda_2)$

$x \in \mathbb{K} \Rightarrow c'(x) = x'$ Galois conjugate

$\lambda_2 = c(\lambda_1)$

$N(\lambda) = \lambda_1 \lambda_2 = (n + m\theta)(n + m\theta')$

quadratic form <u>but</u> not pos. def.

<u>Note</u>: think wave operator $\Box = p_1^2 - p_2^2$ instead of Laplace operator $p_1^2 + p_2^2 = \Delta$ (in momentum variables)

$$\mathcal{D}_\lambda = \begin{pmatrix} 0 & \mathcal{D}_\lambda^+ \\ \mathcal{D}_\lambda^- & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix}$$

first order (Dirac) factorization

$$\mathcal{D}_\lambda^2 = \Box_\lambda \qquad \Box_\lambda = \begin{pmatrix} N(\lambda) & 0 \\ 0 & N(\lambda) \end{pmatrix}$$

Assemble together these Fourier modes

$$\mathcal{H} = \ell^2(\Lambda) \oplus \ell^2(\Lambda) \qquad \mathcal{D}\, e_{\pm,\lambda} = \mathcal{D}_\lambda\, e_{\pm,\lambda}$$

$\mathcal{D}\gamma = -\gamma \mathcal{D}$ w/ grading $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathcal{H}$

and $\quad [\mathcal{D}, R^{\sigma}_{\eta}]\, e_{\pm\lambda} = \sigma(\lambda,\eta)\, \eta_{\mp}\, e_{\mp\eta+\lambda}$

where $\eta_1 = \eta_+$ & $\eta_2 = \eta_-$

So far properties similar to usual Dirac of spectral triples on NC tori but other properties different

1) $\mathcal{D}$ not self-adjoint!

but invariant under a different involution

$\mathcal{D} = c(\mathcal{D}^t)$ $\qquad c$ = Galois involution on $\mathbb{K}$

[So see this only if working on $\mathbb{K}$ not on $\mathbb{R}$ or $\mathbb{C}$]

2) More serious problem: $\mathcal{D}$ has infinite multiplicities in the spectrum

(hence cannot have compact resolvent as for spectral triple) : typical problem in Lorentzian geom. infinite non-compact group of isometries (here V)

Idea: use Krein spaces instead of Hilbert spaces & Wick rotation to Hilbert spaces

here: variation on this idea: want to keep arithm. structure so not Hilb/Krein spaces but analogous vect. spaces over $\mathbb{K}$ w/ bilin forms

Arithmetic Krein spaces:

$c : \mathbb{K} \to \mathbb{K}$ Galois invol. $c : x \mapsto x'$

$\mathcal{V} = \mathbb{K}$-vector space

$T : \mathcal{V} \to \mathcal{V}$ is c-linear if

$$T(av + bw) = c(a)\,T(v) + c(b)\,T(w)$$

Lorentzian pairing on a $\mathbb{K}$-vector space $\mathcal{V}$

$$(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$$

c-lin. in first, linear in second var.

Krein $\mathbb{K}$-space: $(\mathcal{V}, (\cdot,\cdot))$ s.t.

$\exists$ $k : \mathcal{V} \to \mathcal{V}$ c-lin. involution s.t.

$(k\cdot,\cdot)$ satisfies

(1) $(k\cdot,\cdot) = c(\cdot, k\cdot)$

(2) $\forall$ $v \neq 0$ in $\mathcal{V}$ $(kv, v) \in \mathbb{K}$ is totally positive

soy then $(k\cdot,\cdot)$ "pos-def." inn prod (in $\mathbb{K}$-sense)

Krein $\mathbb{K}$-adjoint: $T^{\dagger}$

$$(v, Tw) = (T^{\dagger}v, w)$$

The c-lin. invol. is the Wick rotation κ

Given $\mathcal{V}$, $\langle \cdot,\cdot \rangle = (\kappa \cdot, \cdot)$

$\Rightarrow$ real Hilbert spaces

$$\mathcal{V}_{\alpha_i} = \mathcal{V} \otimes_{\alpha_i(\mathbb{K})} \mathbb{R}$$

$$\langle v, w \rangle = \tfrac{1}{2} \alpha_1 \big((\kappa v, w) + (v, \kappa w) \big)$$

$$= \tfrac{1}{2} \alpha_2 \big((\kappa v, w) + (v, \kappa w) \big)$$

gives inner prod

$$\langle v, w \rangle = \tfrac{1}{2} \alpha_1 (\kappa v, w) + \tfrac{1}{2} \alpha_2 (\kappa v, w)$$

positive definite in usual sense since $(\kappa v, v)$ totally pos.

T $\mathbb{K}$-lin op. on $\mathcal{V}$

$$M_{\alpha_i}(T) \geq -\infty$$

$$M_{\alpha_i}(T) = \inf_{(v,v)=1} \alpha_i (Tv, Tv)$$

Krein $\mathbb{K}$-triple $(\mathcal{A}, \mathcal{V}, \mathcal{D})$

$\mathcal{A}$ invol. $\mathbb{K}$-algebra

$\mathcal{V}$ Krein $\mathbb{K}$-space $(\cdot,\cdot)$

$\mathcal{D}$ densely defined $\mathbb{K}$-linear

$\mathcal{D}^{\dagger} = \mathcal{D}$ Krein self-adj.

the "bounded commutator" condition becomes

$C_a = [\mathcal{D}, \pi(a)]$ has

$$M_{\alpha_i}(C_a) > -\infty$$

w/ repres. of $\mathcal{A}$

$\pi(a^*) = \pi(a)^{\dagger}$ Krein adjoint

$$M_{\alpha_i}(\pi(a)) > -\infty$$

Then a Lorentzian $\mathbb{K}$-spectral triple $(\mathcal{A}, \mathcal{V}, \mathcal{D})$ is as above with

(1) $\exists$ densely def. $\mathbb{K}$-linear op.

$$U : \mathcal{V} \to \mathcal{V} \qquad (Uv, Uv) = (v, v)$$

$$U^{+} = U^{-1} \quad \text{s.t.}$$

$$U^{+} \mathcal{D} U = \mathcal{D}$$

(2) $C_{a,U} = [\mathcal{D}, \pi_U(a)]$ has $M_{\alpha_i}(C_{a,U}) > -\infty$

$$\pi_U(a) = U^{+} \pi(a) U$$

(3) U unbounded self-adjoint $U = U^{K}$ on associated real Hilbert space $\langle \cdot, \cdot \rangle$

(4) $(\mathcal{A}, \mathcal{V}, \mathcal{D})$ satisfies "p-summability"

$$\sum_n |\langle U e_n, |\mathcal{D}^2| U e_n \rangle|^{-s/2} < \infty \qquad \forall s \geq p$$

on real Hilbert space o.n. basis & $\langle , \rangle$

(Krein isometries need not be bounded operators!)

Now back to our example e_λ $\lambda \in \Lambda$ basis of $\mathcal{V}_\Lambda$ over $\mathbb{K}$

$$(v, w) := \sum_\lambda c(a_\lambda) b_\lambda \qquad \text{for } v = \sum a_\lambda e_\lambda \;;\; w = \sum b_\lambda e_\lambda$$

$$K(v) = \sum_\lambda c(a_\lambda) e_\lambda \qquad \text{Krein involution}$$

$\alpha_i : \langle v, v\rangle = \sum_\lambda \alpha_i(a_\lambda^2)$ where

$\langle v, w\rangle = (\kappa v, w) = c(v, \kappa w)$

$\mathbb{k}[\Lambda]$ acting on V_Λ : $R_\lambda e_\eta = e_{\lambda+\eta}$

$M_{\alpha_i}(R_\lambda) > -\infty$ & induce bounded op's on associated real Hilbert spaces

$(R_\lambda v, R_\lambda w) = (v, w)$ Krein isometries

analog of twisted group rings:

$\omega \in \mathbb{K}^*$ with $N(\omega) = \omega\omega' = 1$

$\tilde{w}(\lambda, \eta) = \omega^{(n,m)\wedge(r,k)}$ cocycle

$R_\lambda^{\tilde{\omega}} e_\eta = \tilde{\omega}(\lambda, \eta)\, e_{\lambda+\eta}$

but unbounded op's on associated Hilbert space

$R_\lambda^{\tilde{\omega}} e_{\eta,\pm} = A_\omega^{(r,k)\wedge(n,m)} e_{\lambda+\eta,\pm}$

Lorentzian Dirac operator

$$D_{\mathbb{K}} e_{\lambda,\pm} = D_{\mathbb{K},\lambda} e_{\lambda,\pm} = \begin{pmatrix} 0 & \Delta_\lambda^+ \\ \Delta_\lambda^- & 0 \end{pmatrix} e_{\lambda,\pm} = \begin{pmatrix} 0 & \ell \\ c(\ell) & 0 \end{pmatrix} e_{\lambda,\pm}$$

$\lambda = (\alpha_1(\ell), \alpha_2(\ell))$

$$D_\lambda = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} \qquad c(D_\lambda) = \begin{pmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{pmatrix}$$

Then Krein triple with involution:

$$T_\varepsilon e_{\lambda,\pm} = \begin{pmatrix} \varepsilon^{\rho(\lambda)} & 0 \\ 0 & \varepsilon^{-\rho(\lambda)} \end{pmatrix} e_{\lambda,\pm} \text{ for } A_\varepsilon^{\rho(\lambda)}(\mu) = \lambda \qquad U = T_\varepsilon J$$

$J e_{\lambda,\pm} = e_{J(\lambda),\pm}$ $\quad J(\lambda) = A_\varepsilon^{-k}(\mu)$ for $\lambda = A_\varepsilon^k(\mu)$

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Then

$$\sum_{\lambda \neq 0} \langle U_\varepsilon e_{\lambda,\pm}, |D_K^2| U_\varepsilon e_{\lambda,\pm} \rangle^{-s/2}$$

$$= \sum_{\lambda \neq 0} \left(\varepsilon^{2\rho(\lambda)} + \varepsilon^{-2\rho(\lambda)} \right)^{-s/2} |N(\lambda)|^{-s/2}$$

$$= \sum_{k \in \mathbb{Z}} \left(\varepsilon^{2k} + \varepsilon^{-2k} \right)^{-s/2} \sum_{\mu \in \frac{\Lambda \setminus \{0\}}{V}} |N(\mu)|^{-s/2} \quad \text{zeta function}$$

$$Z_\varepsilon\left(\tfrac{s}{2}\right) L\left(\Lambda, V, \tfrac{s}{2}\right) = \sum_n \operatorname{sgn}\left(\langle U e_n, D^2 U e_n \rangle\right) \cdot |\langle U e_n, |D^2| U e_n \rangle|^{-s/2}$$

Similarly for η function $\quad =: \eta_D(s)$ Lorentzian
