Strong Morita equivalence of C^*-algebras

A, B C^*-alg's

$\exists$ $\mathcal{E}$ right Hilbert module over A
w/ $\langle , \rangle_A$ inner prod (A-valued)
left Hilbert module over B
w/ ${}_B\langle , \rangle$ inner prod (B-valued)

and compatibility conditions:

(1) $\mathcal{E}$ is <u>full</u> both as right and as left Hilb. mod.

(2) $$ {}_B\langle \eta, \xi \rangle \cdot \zeta = \eta \cdot \langle \xi, \zeta \rangle_A \qquad \forall \eta, \xi, \zeta \in \mathcal{E} $$

compatibility of the l-r structures

(3) left action of B on $\mathcal{E}$ by operators bounded for $\langle , \rangle_A$:

$$ \langle b\eta, b\eta \rangle_A \leq \|b\|^2 \langle \eta, \eta \rangle_A $$

as pos. elts in A

right action of A on $\mathcal{E}$ bounded in ${}_B\langle , \rangle$:

$$ {}_B\langle \eta a, \eta a \rangle \leq \|a\|^2 \, {}_B\langle \eta, \eta \rangle $$

Example: $\mathcal{E}$ right full Hilb. mod over $\mathcal{A}$

$$\mathcal{A} \underset{M}{\simeq} \mathrm{End}^0_{\mathcal{A}}(\mathcal{E}) \quad \text{compact endom.}$$

(if $\mathcal{E}$ fin. proj. $\mathcal{A} \underset{M}{\simeq} \mathrm{End}_{\mathcal{A}}(\mathcal{E})$)

in fact a right $\mathcal{A}$-Hilb. mod $\mathcal{E}$ is also a left Hilb-mod for $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$ with

$${}_{\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})}\langle \xi_1, \xi_2 \rangle := P_{\xi_1 \xi_2}$$

(check it has properties of a left Hilb mod str.)

because $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$ compact endom are closure of $P_{\xi,\eta}$'s

$\Rightarrow$ full as left $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$-mod

- $${}_{\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})}\langle \xi_1, \xi_2 \rangle \cdot \xi = P_{\xi_1 \xi_2}(\xi) = \xi_1 \langle \xi_2, \xi \rangle_{\mathcal{A}}$$

compatibility

- $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$ acts by bounded op's in $\langle , \rangle_{\mathcal{A}}$ by def.

- $$\langle {}_{\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})}\langle \eta a, \eta a \rangle \xi, \xi \rangle_{\mathcal{A}} = \langle \eta a \langle \eta a, \xi \rangle_{\mathcal{A}}, \xi \rangle_{\mathcal{A}}$$

$$= \langle \eta a a^* \langle \eta, \xi \rangle_{\mathcal{A}} \cdot \xi \rangle_{\mathcal{A}}$$

$$= \langle \eta, \xi \rangle^*_{\mathcal{A}} a a^* \langle \eta, \xi \rangle_{\mathcal{A}}$$

$$\leq \|a\|^2 \langle \eta, \xi \rangle^*_{\mathcal{A}} \langle \eta, \xi \rangle_{\mathcal{A}}$$

$$\|a\|^2 \cdot \langle \eta \langle \eta, \xi \rangle_{\mathcal{A}}, \xi \rangle_{\mathcal{A}}$$

$$\|a\|^2 \; \langle {}_{\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})}\langle \eta, \eta \rangle \xi, \xi \rangle_{\mathcal{A}}$$

Back to noncommutative tori!

A. Connes "C*-algèbres et géométrie différentielle" C R A Sci. 290 (1980) N.13 599-604.
(M. Rieffel "C*-algebras associated with irrational rotations" Pacific J. Math Vol 93 (1981) N.2 415 - 429)

Noncommutative torus $\mathcal{A}_\theta$ generators U, V

$$UV = e^{2\pi i\theta} VU$$

Subalgebra of smooth functions $\mathcal{A}_\theta^\infty$

$$a = \sum_{n,m} a_{n,m} U^n V^m \qquad a_{n,m} \in \mathcal{S}(\mathbb{Z}^2)$$

C*-completion of $\mathcal{A}_\theta^\infty$ is C*-alg. $\mathcal{A}_\theta$

von Neumann trace $\tau(\sum_{n,m} a_{n,m} U^n V^m) = a_{0,0}$

torus action $T^2 \ni (\lambda, \mu)$

$$\rho_{(\lambda,\mu)}(U) = \lambda U \qquad \rho_{(\lambda,\mu)}(V) = \mu V$$

$|\lambda| = |\mu| = 1$

Expectation:

$\Phi : \mathcal{A} \to \mathcal{B} \subset \mathcal{A}$ positive unital mapping $\mathcal{A}$ onto a subalgebra

idempotent $\Phi^2 = \Phi$

$$\Phi(a) = \int_{T^2} \rho_{(\lambda,\mu)}(a)\, dt_1 dt_2 \qquad \lambda = e^{2\pi i t_1} \quad \mu = e^{2\pi i t_2}$$

$(t_1, t_2) \in [0,1] \times [0,1]$

Conjugate $\mathcal{E}$:

if $\mathcal{E}$ B-$\mathcal{A}$ bimodule giving strong Morita equivalence

$\tilde{\mathcal{E}}$ ($= \mathcal{E}$ as add. grp. (vector space))

with $\quad a \cdot \tilde{\xi} = \widetilde{\xi a^*} \qquad {}_{\mathcal{A}}\langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle = \langle \xi_1, \xi_2 \rangle_{\mathcal{A}}$

$\quad \tilde{\xi} \cdot b = \widetilde{b^* \xi} \qquad \langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle_B = {}_B\langle \xi_1, \xi_2 \rangle$

Use $\mathcal{E}, \tilde{\mathcal{E}}$ to transfer back and forth representations

$$\pi : \mathcal{A} \to B(\mathcal{H})$$

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \quad : \quad \eta a \otimes_{\mathcal{A}} \xi = \eta \otimes_{\mathcal{A}} \pi(a)\xi$$

$$\langle \eta_1 \otimes_{\mathcal{A}} \xi_1 , \eta_2 \otimes_{\mathcal{A}} \xi_2 \rangle_{\mathcal{H}'} = \langle \xi_1, \langle \eta_1, \eta_2 \rangle_{\mathcal{A}} \cdot \xi_2 \rangle_{\mathcal{H}}$$

$\pi_B : B \to B(\mathcal{H}')$ representation

$$\pi_B(b)(\eta \otimes_{\mathcal{A}} \xi) = (b\eta) \otimes_{\mathcal{A}} \xi$$

going back with $\tilde{\mathcal{E}}$ gives a rep of $\mathcal{A}$ unitarily equiv. to $\pi_{\mathcal{A}} : \mathcal{A} \to B(\mathcal{H})$

$\Rightarrow$ $(\mathcal{E}, \tilde{\mathcal{E}})$ identify (unitary equiv. classes of) rep's of $\mathcal{A}$ and rep's of $\mathcal{B}$

τ is unique trace on A_θ

if $\tilde{\tau}$ another trace

$$\tilde{\tau}(a) = \tilde{\tau}(U^j a U^{-j})$$

so $\tilde{\tau}(a) = \tilde{\tau}\left(\lim_n \frac{1}{2n+1} \sum U^j a U^{-j}\right)$

$$= \tilde{\tau}(\tau_1(a))$$

Same way

$$\tilde{\tau}(a) = \tilde{\tau}(\tau_2(a))$$
$$= \tilde{\tau}(\tau_2 \tau_1(a))$$
$$= \tilde{\tau}(\tau(a))$$
$$= \tau(a)\,\tilde{\tau}(1) = \tau(a)$$

Note: $\tau_1(a) = \int_0^1 \rho_{e^{2\pi i t}, 1}(a)\, dt$

$$\Rightarrow \boxed{\tau_1(a) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} V^j a V^{-j}}$$

$\left(a = \sum_{n,m} a_{n,m} U^n V^m \right.$

$\left. \Rightarrow \tau_1(a) = \sum_n a_{n0} U^n \right)$

because

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} V^j U^k V^l V^{-j}$$
$$= \lim \frac{1}{2n+1} \sum e^{2\pi i j l \theta} U^k V^l$$
$$= \lim \frac{1}{2n+1} \left(\frac{\sin(2n+1)\pi l\theta}{\sin \pi l \theta} \right) U^k V^l$$
$$= \delta_{l0}\, U^k = \tau_1(U^k V^l)$$

$(\tau_1 : A_\theta \to C(S^1))$

Note: $\tau : A_\theta \to \mathbb{C}$

values on projections should be in $[0,1]$ (type II_1 case)

Explicit construction of projections in A_θ (Rieffel, Boca)

$$\boxed{\text{Range of } \tau \text{ on projections is } (\mathbb{Z} + \mathbb{Z}\theta) \cap [0,1]}$$

"rank" of a finite projective module $\mathcal{E}$ over A_θ $\quad \mathcal{E} = p A_\theta^n \quad p^2 = p^* = p \in M_n(A_\theta)$

$$rk(\mathcal{E}) := Tr_\tau(p) = (\tau \otimes Tr)(p)$$

for $M_n(A_\theta) = A_\theta \otimes_{\mathbb{C}} M_n(\mathbb{C})$

$\mathcal{E}^\infty$ module over A_θ^∞ fin. proj.

$\Rightarrow \quad \mathcal{E} = \mathcal{E}^\infty \otimes_{A_\theta^\infty} A_\theta$

as in commut. case cat. of smooth & contin. vector bundles same (Serre-Swan) in <u>NC tori</u> case also (Connes)

$c, d \in \mathbb{Z} \quad c > 0 \quad \Rightarrow \quad \mathcal{E}_{d,c}(\theta)$ right A_θ-module w/ action

$$\mathcal{E}_{d,c}(\theta) = S(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z}) = S(\mathbb{R})^c$$

$$\begin{cases} (fU)(x,\alpha) = f\left(x - \frac{c\theta + d}{c}, \alpha - 1\right) \\ (fV)(x,\alpha) = \exp\left(2\pi i\left(x - \frac{\alpha d}{c}\right)\right) f(x,\alpha) \end{cases}$$

(Connes '80): $\operatorname{rk} \mathcal{E}_{d,c}(\theta) = |c\theta + d|$

any right mod of this rank isom to $\mathcal{E}_{d,c}(\theta)$

[illegible]

$g \in SL_2(\mathbb{Z}) \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

write $\mathcal{E}_g := \mathcal{E}_{d,c}(\theta)$ "basic modules"

also <u>left</u> action by $A_{g\theta}$

$$(\tilde{U} f)(x,\alpha) = f\left(x - \frac{1}{c}, \alpha - a\right)$$

$$(\tilde{V} f)(x,\alpha) = \exp\left(2\pi i\left(\frac{x}{c\theta + d} - \frac{\alpha}{c}\right)\right) f(x,\alpha)$$

The bimodule $\mathcal{E}_g(\theta)$ gives strong Morita equivalence

(two actions commute)

$$\mathcal{E}_g(\theta) \otimes_{A_\theta} \cdots \otimes_{A_\theta} \mathcal{E}_g(\theta) = \mathcal{E}_{g^n}(\theta)$$

$$\langle f, g \rangle_{A_\theta} = \sum_{(m,n)} \langle f, U^m V^n g \rangle U^m V^n$$