Hilbert modules (generalize Hilbert spaces)

right (pre) Hilbert module over C^*-alg. $\mathcal{A}$

right $\mathcal{A}$-module $\mathcal{E}$

$\xi \mapsto \xi \cdot a$ right action of $\mathcal{A}$

$\mathcal{A} \to \mathrm{End}(\mathcal{E})$

with $\mathcal{A}$-valued inner prod

$\langle , \rangle_{\mathcal{A}} : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ conjugate lin. in first var, lin in second

$$\langle \xi_1, \xi_2 a \rangle_{\mathcal{A}} = \langle \xi_1, \xi_2 \rangle_{\mathcal{A}} \cdot a \qquad (*)$$

$\xi(ab) = (\xi a)b$

$\xi(a+b) = \xi a + \xi b$

$(\xi + \eta)a = \xi a + \eta a$

morphisms

$\rho : \mathcal{E} \to \mathcal{E}'$

$\rho(\xi a) = \rho(\xi) a$

$$\langle \xi_1, \xi_2 \rangle_{\mathcal{A}}^* = \langle \xi_2, \xi_1 \rangle_{\mathcal{A}}$$

$$\langle \xi, \xi \rangle_{\mathcal{A}} \geq 0 \quad \text{and} \quad \langle \xi, \xi \rangle_{\mathcal{A}} = 0 \text{ iff } \xi = 0$$

in sense of span of a^*a pos. coeff.

in partic. $\langle \xi, \xi \rangle$ self adjoint

Cauchy-Schwartz inequality:

$$\langle \eta, \xi \rangle_{\mathcal{A}}^* \langle \eta, \xi \rangle_{\mathcal{A}} \leq \| \langle \eta, \eta \rangle_{\mathcal{A}} \| \cdot \langle \xi, \xi \rangle_{\mathcal{A}}$$

hence

$$\| \langle \eta, \xi \rangle_{\mathcal{A}} \|^2 \leq \| \langle \eta, \eta \rangle_{\mathcal{A}} \| \cdot \| \langle \xi, \xi \rangle \|_{\mathcal{A}}$$

but notice that (unlike Hilbert spaces) if $\langle \xi_1, \xi_2 \rangle_{\mathcal{A}} = 0$ in general $\|\xi_1 + \xi_2\|_{\mathcal{A}}^2 \neq \|\xi_1\|_{\mathcal{A}}^2 + \|\xi_2\|_{\mathcal{A}}^2$

$\leq$ holds not reverse

$\Rightarrow$ using norm on $\mathcal{A}$ obtain norm on $\mathcal{E}$

$$\| \xi \|_{\mathcal{A}} := \sqrt{\| \langle \xi, \xi \rangle_{\mathcal{A}} \|}$$

right-Hilbert module: completion of $\mathcal{E}$ in this norm

right-Hilbert modules for $\mathcal{A} = \mathbb{C}$: Hilbert spaces

left - Hilbert module : similar $\mathcal{E}$ left $\mathcal{A}$-module
same properties for $\mathcal{A}$-valued ${}_{\mathcal{A}}\langle , \rangle$ except

$${}_{\mathcal{A}}\langle a\xi_1, \xi_2 \rangle = a \, {}_{\mathcal{A}}\langle \xi_1, \xi_2 \rangle \quad \text{replacing right-version}$$

(take right for simplicity)

$\mathcal{E}$ right $\mathcal{A}$-Hilbert module

$$\{ \langle \xi_1, \xi_2 \rangle_{\mathcal{A}} \ , \forall \xi_1, \xi_2 \in \mathcal{E} \} \subset \mathcal{A}$$

$\mathbb{I}_{\mathcal{E}} \subset \mathcal{A}$ $\quad$ $\mathbb{I}_{\mathcal{E}} =$ closure of linear span of these

is an ideal in $\mathcal{A}$ because of ⊛

if $\mathbb{I}_{\mathcal{E}} = \mathcal{A}$ then say $\mathcal{E}$ is a <u>full</u> Hilbert module

Operators on Hilbert modules

$T : \mathcal{E} \to \mathcal{E}$ an $\mathcal{A}$-linear map

does <u>not</u> (unlike Hilbert spaces) always admit adjoint

T is <u>adjointable</u> if $\exists$ $T^* : \mathcal{E} \to \mathcal{E}$ such that

$$\langle T^*\xi_1, \xi_2 \rangle_{\mathcal{A}} = \langle \xi_1, T\xi_2 \rangle_{\mathcal{A}} \quad \forall \xi_1, \xi_2 \in \mathcal{E}$$

$\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ = all continuous adjointable $\mathcal{A}$-linear operators on $\mathcal{E}$ (endomorphisms)

operator norm $\|T\| = \sup_{\|\xi\|_{\mathcal{A}} \le 1} \|T\xi\|_{\mathcal{A}}$ $\quad$ (C^*-algebra since

$\langle T\xi, T\xi \rangle_{\mathcal{A}} \le \|T\|^2 \langle \xi, \xi \rangle_{\mathcal{A}}$

(complete in norm if $\mathcal{E}$ is)

$(TS)^* = S^* T^*$

$(T^*)^* = T$

On a Hilbert space $\mathcal{H}$ have $\mathcal{B}(\mathcal{H})$ bounded operators and important subalgebra $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ of <u>compact</u> operators: norm closure of finite rank operators

in Hilbert modules

$$P_{\xi_1,\xi_2}(\xi) = \xi_1 \cdot \langle \xi_2, \xi \rangle_A$$

or $|\xi_1\rangle\langle\xi_2|$ in Dirac's bra-ket notation

adjointable

$$P^*_{\xi_1,\xi_2} = P_{\xi_2,\xi_1}$$

bounded $\quad \| P_{\xi_1,\xi_2} \|_A \leq \|\xi_1\|_A \, \|\xi_2\|_A$

$T \in \mathrm{End}_A(\mathcal{E})$

$$T \cdot P_{\xi_1,\xi_2} = P_{T\xi_1,\xi_2}$$

$$P_{\xi_1,\xi_2} \cdot T = P_{\xi_1, T^*\xi_2}$$

$$P_{\eta_1,\eta_2} P_{\xi_1,\xi_2} = P_{\eta_1 \cdot \langle \eta_2, \xi_1 \rangle_A, \xi_2} = P_{\eta_1, \xi_2 \langle \xi_1, \eta_2 \rangle_A}$$

$$P_{\eta_1\eta_2} P_{\xi_1\xi_2}(\xi) = P_{\eta_1\eta_2}\, \xi_1 \langle \xi_2, \xi \rangle_A = \eta_1 \langle \eta_2, \xi_1 \langle \xi_2, \xi \rangle_A \rangle_A$$

$$= \eta_1 \langle \eta_2, \xi_1 \rangle_A \langle \xi_2, \xi \rangle_A$$

&

$$((\langle \eta_2, \xi \rangle_A \langle \xi_2, \xi \rangle_A)^*)^* = (\langle \xi, \xi_2 \rangle_A \langle \xi_1, \eta_2 \rangle_A)^* = \langle \xi, \xi_2 \langle \xi_1, \eta_2 \rangle_A \rangle_A^*$$

$$= \langle \xi_2 \langle \xi_1, \eta_2 \rangle_A, \xi \rangle_A \qquad \doteq P_{\eta_1, \xi_2 \langle \xi_1, \eta_2 \rangle_A}$$

Two-sided ideal span of P_{ξ_1,ξ_2} $\quad \mathrm{End}^0_A(\mathcal{E})$ compact endomorphisms

Example 1: $\mathcal{A}$ is a Hilbert module on itself

$$\langle\,,\rangle_{\mathcal{A}} : \mathcal{A}\times\mathcal{A} \to \mathcal{A} \qquad \langle a,b\rangle_{\mathcal{A}} := a^* b$$

norm $\|a\|_{\mathcal{A}} = \sqrt{\|\langle a,a\rangle_{\mathcal{A}}\|} = \sqrt{\|a^*a\|} = \sqrt{\|a\|^2} = \|a\|$

agrees with original C^*-norm

$\mathcal{A}$ unital : $\mathrm{End}^0_{\mathcal{A}}(\mathcal{A}) \cong \mathrm{End}_{\mathcal{A}}(\mathcal{A}) \cong \mathcal{A}$

$\sum \lambda_k P_{a_k, b_k} \longmapsto \sum \lambda_k a_k b_k^*$ gives isom.

Example 2: $\mathcal{E} = \mathcal{A}^N = \underbrace{\mathcal{A}\oplus\cdots\oplus\mathcal{A}}_{N \text{ times}}$

right module $(a_1,\dots,a_n)a = (a_1 a,\dots,a_n a)$

$$\langle (a_1,\dots,a_n), (b_1,\dots,b_n)\rangle_{\mathcal{A}} = \sum_{i=1}^n a_i^* b_i$$

$$\|(a_1,\dots,a_n)\|_{\mathcal{A}} = \|\sum_{k=1}^n a_k^* a_k\|$$

e_k unit vectors in $\mathbb{C}^N$ form basis of $\mathcal{A}^N$

$(a_1,\dots,a_n) = \sum_{k=1}^{N} e_k \cdot a_k$ $\qquad$ $\mathrm{End}_{\mathcal{A}}(\mathcal{A}^N) = M_n(\mathcal{A}) \cong \mathrm{End}^0_{\mathcal{A}}(\mathcal{A}^N)$

Commutative case $\mathcal{A} = C(X)$ $\qquad$ $E \to X$ complex vector bundle

$\langle\,,\rangle_{E_p} : E_p\times E_p \to \mathbb{C}$
hermitian scalar product

$\mathbb{C}^n \simeq E_p$ fiber, $E_p \to p\in X$

$U_\alpha \subset X$ open covering

$E|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^n$

$\mathcal{E} = \Gamma(X,E)$ sections

$s_\alpha : U_\alpha \to E|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^n$

$(x, s_\alpha(x))$

on $U_\alpha \cap U_\beta$

$g_{\alpha\beta} : U_\alpha\cap U_\beta \to Gl_n(\mathbb{C})$ gluing local patches

$C(X) \ni \langle s_1, s_2\rangle(x) = \langle s_1(x), s_2(x)\rangle$

$s_\beta(x) = g_{\beta\alpha}(x)\cdot s_\alpha(x)$

$\mathrm{End}_{C(M)}(\mathcal{E}) \cong \mathrm{End}^0_{C(M)}(\mathcal{E}) \cong \Gamma(\mathrm{End}(E))$

$\gamma : E \to E$, $E \to X \leftarrow E$

C^*-alg. of continuous sections of endomorphisms bundle

If X non compact $C_0(X) = \mathcal{A}$

$\Gamma_0(X, E)$ sections vanishing at ∞

$\mathrm{End}_{C_0(M)}(\Gamma_0(X,E)) = \Gamma_b(X, \mathrm{End}\,E)$ bounded sections of $\mathrm{End}\,E$

$\mathrm{End}^0_{C_0(M)}(\Gamma_0(X,E)) = \Gamma_0(X, \mathrm{End}\,E)$ sections vanishing at ∞

<u>Finite projective modules:</u>

free module $\mathcal{E} = \mathcal{A}^I = \bigoplus_{I} \mathcal{A}$ $\Leftrightarrow$ $\exists\ \xi_i \in \mathcal{E}$ $i \in I$

("trivial vector bundles") $\xi = \sum_i \xi_i a_i$ uniquely

<u>finite (finitely generated)</u>:

$\exists$ surjective $\mathcal{A}$-module map $\mathcal{A}^k \to \mathcal{E}$

i.e. all $\xi \in \mathcal{E}$ can be written (non uniq.) in form

$\xi = \sum_{i=1}^{k} \xi_i a_i$ some $a_i \in \mathcal{A}$

<u>Projective modules:</u>

$\mathcal{E}$ s.t. $\exists\ \mathcal{E}'$ module s.t. $\mathcal{E} \oplus \mathcal{E}' \cong \mathcal{A}^I$ free

(direct summand of free)

$\rightsquigarrow$ finite projective

Equivalent to projective $\mathcal{P}$ proj. module

any surjection $\mathcal{E} \to \mathcal{P} \to 0$ of modules

splits $\begin{tikzcd} \mathcal{E} \arrow[r, shift right] & \mathcal{P} \arrow[l, hook', shift right] \arrow[r] & 0 \end{tikzcd}$

composition $=$ id

if

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, dashed, "\exists"'] \arrow[d, "\text{morphism of } \mathcal{A}\text{-mod}"] & \\ \mathcal{E}' \arrow[r, "\text{surj.}"', "\text{morphism of } \mathcal{A}\text{-mod}"' below] & \mathcal{E} \arrow[r] & 0 \end{tikzcd}$$

For Hilbert modules

(1) $\mathcal{E}$ right Hilbert mod over $\mathcal{A}$ unital C^*-alg.

if $1_{\mathcal{E}}$ identity endom. $1_{\mathcal{E}} \in \mathrm{End}^0(\mathcal{E})$

(hence $\mathrm{End}^0(\mathcal{E}) = \mathrm{End}(\mathcal{E})$)

then $\mathcal{E}$ is finite projective

(2) if $\mathcal{E}$ fin proj. module on $\mathcal{A}$ $\exists$ $\langle , \rangle_{\mathcal{A}}$ that makes $\mathcal{E}$ into right (pre) Hilbert module for which $1_{\mathcal{E}} \in \mathrm{End}^0(\mathcal{E})$

(any two choices $\langle \xi, \eta \rangle_{A,1}$ $\langle \xi, \eta \rangle_{A,2}$ are related by

$$\langle \xi, \eta \rangle_{\mathcal{A},1} = \langle T\xi, T\eta \rangle_{\mathcal{A},2}$$ for some invertible endomorphism T)

Pf: have finite sets

1) $\{\xi_k\}_{k=1}^{N}$ $\{\eta_k\}_{k=1}^{N}$ with $1_{\mathcal{E}} = \sum_k P_{\xi_k, \eta_k}$

$(1_{\mathcal{E}} \in \mathrm{End}^0(\mathcal{E}))$

because these orthog. (rank one) projections so convergence in norm
$\Rightarrow$ equal ~~[illegible]~~ $= \sum_k P_{\xi_k, \eta_k}$ finitely many

$\forall \xi \in \mathcal{E}$

$\xi = 1_{\mathcal{E}} \xi = \sum_k P_{\xi_k \eta_k} \xi$

$= \sum \xi_k \langle \eta_k, \xi \rangle_{\mathcal{A}}$

finitely gen. by ξ_k

can embed $\mathcal{E} \xrightarrow{\lambda} \mathcal{A}^N$ using $\lambda(\xi) = (\langle \eta_1, \xi \rangle_{\mathcal{A}}, \dots, \langle \eta_N, \xi \rangle_{\mathcal{A}})$

and surjection $\rho : \mathcal{A}^N \to \mathcal{E}$ $\rho((a_1, \dots, a_N)) = \sum_k \xi_k a_k$

$$\rho \circ \lambda(\xi) = \sum_k \xi_k \langle \xi_k, \xi \rangle_{\mathcal{A}} = \sum_k P_{\xi_k, \xi_k}(\xi) = \xi$$

$p = \lambda \circ \rho$ identifies $\underline{\underline{\mathcal{E} \cong p\mathcal{A}^N}}$
projection

2) $\mathcal{E} \oplus \mathcal{E}' \cong \mathcal{A}^N$
restrict $\langle , \rangle_{\mathcal{A}}$ from $\mathcal{A}^N$ to $\mathcal{E}$ $\Rightarrow$ Hilbert mod.

have $p : \mathcal{A}^N \to \mathcal{E}$ surjection
$\{e_k\}$ basis $\Rightarrow$ $p(e_k) = \xi_k$ $\Rightarrow$ $\mathbb{1}_{\mathcal{E}} = \sum_k P_{\xi_k, \xi_k}$
$\cap$
$\mathrm{End}^\circ_{\mathcal{A}}(\mathcal{E})$

<u>Serre-Swan theorem:</u>

$\mathcal{A} = C^\infty(X)$ X smooth mfld

$\mathcal{E}$ module on $\mathcal{A}$ finite projective $\iff$ $\mathcal{E} \cong \Gamma(X, E)$ smooth sections of complex vector bundle

<u>Pf</u>:

$\Rightarrow$ $\mathcal{E} = p\mathcal{A}^N$ $p \in M_n(\mathcal{A})$ $p^2 = p^* = p$

$\mathcal{A}^N = \Gamma(X, \mathcal{E}_0)$ $\mathcal{E}_0 = X \times \mathbb{C}^N$ trivial vector bdle

$p(sf) = p(s)f$ module map $p : \mathcal{A}^N \to \mathcal{E}$ surj

$I_x = \{f \in \mathcal{A} : f(x) = 0\}$ ideal, $x \in M$ submodule $\mathcal{A}^N I_x$ preserved by p

$s \mapsto s(x)$ linear isom

$$\mathcal{A}^N / \mathcal{A}^N I_x \xrightarrow{\cong} (X \times \mathbb{C}^N)_x = (\mathcal{E}_0)_x$$ fiber at x

$$\pi : \underset{\mathcal{E}_0}{\underset{\parallel}{X \times \mathbb{C}^N}} \longrightarrow \underset{\mathcal{E}_0}{\underset{\parallel}{X \times \mathbb{C}^N}}$$

$\pi(S(x)) = p(S)(x)$ $\qquad$ $p = \pi \circ S$ bundle homom.

$p^2 = p \implies \pi^2 = \pi$

if $\dim \pi((X \times \mathbb{C}^N)_x) = K$ (locally constant) $\quad \exists\ K$ lin. indep smooth sections $S_1 \ldots S_K \in \mathcal{A}^N$ near $x \in X$

spanning range of π

then π acts identically $\quad \pi S_j(x) = S_j(x)$

$\mathcal{E}_0 = X \times \mathbb{C}^N = \mathcal{E} \oplus \ker(\pi)$

with $\Gamma(X, E) = \{ \pi \circ s \mid s \in \Gamma(X, \mathbb{C}^N) \} = \operatorname{Im}\{ p : \mathcal{A}^N \to \mathcal{A}^N \} = \mathcal{E}$

$\operatorname{End}_{\mathcal{A}}(\mathcal{E}) \simeq p M_n(\mathcal{A}) p$ $\quad$ compressed with the projection

for $\mathcal{E} = p\mathcal{A}^N$
