Direct sum; tensor products; direct limits; multiplier algebras

Hilbert C^*-modules; strong Morita equivalence
finite projective modules

GNS representation; states, weights

Direct sum: A_i C^*-alg's $A = A_1 \oplus \cdots \oplus A_n$ as algebras w/ norm
$\|(a_1, \ldots, a_n)\| = \max\{\|a_i\|\}$

infinite families

direct sum: (c_0-directed sum)

sequences $\{a_i\}_{i\in\mathbb{N}}$ $a_i \in A_i$ s.t. $\|a_i\| \xrightarrow[i\to\infty]{} 0$

direct product: (ℓ^∞-direct sum)

sequences $\{a_i\}_{i\in\mathbb{N}}$ $a_i \in A_i$ s.t. $\|a_i\|$ bounded

$\|a\| = \sup_i \|a_i\|$

Continuous field of C^*-algebras:

X top. space $x \mapsto A_x$ A_x C^*-algebra

$\exists\, \Gamma \subset \prod_x A_x$ s.t. Γ ~~[illegible]~~ $*$-algebra

$\Gamma \ni a$: $a(x)$ dense in A_x for each x

$x \mapsto \|a(x)\|$ continuous for all $a \in \Gamma$

Γ loc. unif. closed

$\Rightarrow$ C^*-alg. by $\{a \in \Gamma$ s.t. $x \mapsto \|a(x)\|$ vanishes at $\infty\}$
with $\sup_{x\in X} \|a(x)\|$ norm

Tensor products :

Univ. property V, W vector spaces

$$\begin{tikzcd} V\times W \arrow[r, "\exists\,\varphi \text{ bilinear}"] \arrow[dr, "\psi"'] & V\otimes W \arrow[d] \\ & Z \end{tikzcd}$$

bilinear map

$\varphi(v,w) = v\otimes w$ elementary tensors

spanning $V\otimes W$

(non-unique decomp into elementary tensors

$u = \sum v_i \otimes w_i$

can do so that v_i lin. indep then w_i uniquely determined

V^* = dual vector space $= \mathrm{Hom}(V, \mathbb{C})$

$(f,g)\in V^* \times W^* \rightsquigarrow \exists!\, h \in (V\otimes W)^*$

$f(v)\,g(w) = h(v\otimes w)$

1) V, W with $\langle,\rangle_V$ & $\langle,\rangle_W$ inner products

$\Rightarrow \langle v\otimes w, v'\otimes w'\rangle = \langle v, v'\rangle_V \langle w, w'\rangle_W$ defines $\langle,\rangle$ on $V\otimes W$

2) if A, B algebras then product △

$(a\otimes b)(a'\otimes b') = aa'\otimes bb'$

involution $(a\otimes b)^* = a^*\otimes b^*$

3) Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2 \Rightarrow \mathcal{H}_1\otimes\mathcal{H}_2$ = completion of vector space tensor product in $\langle,\rangle$ inner prod.

4) A, B C^*-algebras need $\otimes$ so that C^*-identity of norms still holds (not unique way)

via representations: $\pi: A \to B(\mathcal{H})$

$\pi': B \to B(\mathcal{H}')$

$\alpha(a\otimes b) = \pi(a)\otimes\pi'(b)$

$$\begin{tikzcd} A\otimes B \arrow[r, "\alpha"] & B(\mathcal{H})\otimes B(\mathcal{H}') \arrow[d, "\Phi"] \\ & B(\mathcal{H}\otimes\mathcal{H}') \end{tikzcd}$$

$\Rightarrow$ prod as algebras

$\Phi(S\otimes T)(\xi_1\otimes\xi_2) = S\xi_1\otimes T\xi_2$

is C^*alg. homom.

Examples:

$$C_0(X) \otimes C_0(Y) \simeq C_0(X \times Y)$$

$$C_0(X) \otimes A \simeq C_0(X, A)$$

$$M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \simeq M_{nk}(\mathbb{C})$$

$$M_n(\mathbb{C}) \otimes A \simeq M_n(A)$$

Inductive system of C^*algebras

directed set I $\{A_i\}_{i \in I}$ A_i C^*-algebras

$*$-homomorphisms $\varphi_{ij} : A_i \longrightarrow A_j$ whenever $i < j$ in I

with composition $(\varphi_{ij} \cdot \varphi_{jk}) = \varphi_{ik}$

$$A = \varinjlim_{i \in I} A_i$$ direct limit C^*-alg

take $A_0 \subset \prod_i A_i$

$\ni$ $a = \{a_i\}_{i \in I}$ s.t. $\exists\, i_0 :$ $\varphi_{ij}(a_i) = a_j$ $\forall\ i_0 < i < j$

$\Rightarrow \|a\| = \lim_{i \in I} \|a_i\|$ exists (the φ_{ij} have to be norm decreasing $\|\varphi_{ij}(a_i)\| \leq \|a_i\|$ same as for multiplicative functionals)

and $\|\cdot\|$ satisfies C^*-id.

$\Rightarrow A$ = completion in this norm of A_0

Example AF-algebras: (approximately finite dimensional) direct limits of sequences of fin dim C^*-algebras

Bratteli diagrams (and moves relating them)

Multiplier algebras (More general compactifications than one-point)

double centralizer :

$$(L, R) \qquad L, R : A \longrightarrow A$$

with $\quad R(a)b = aL(b) \qquad \forall a, b \in A$

e.g. if A, B with $A \hookrightarrow B$ $\quad A$ (two sided) ideal in B

L_b, R_b left & right multipl on A by an element of B

→ becomes associativity of prod in B

$$R_b(a)c = (ab)c = a(bc) = aL_b(c)$$

L, R are necessarily bounded, linear and with

$$L(ab) = L(a)b \quad \text{and} \quad R(ab) = aR(b)$$

$$xL(a+b) = R(x)(a+b) = R(x)a + R(x)b$$
$$= xL(a) + xL(b)$$

(note $xc = 0 \quad \forall\, x \in A \Rightarrow c^*c = 0 \Rightarrow c = 0$)

$M(A)$ = multiplier algebra

$$(L,R)(L',R') = (LL', R'R); \quad \text{invol } (L,R)^* = (R^*, L^*)$$

So that $T : A \to A \qquad T^*(a) = (T(a^*))^*$;

$$\|(L,R)\| = \|L\| = \|R\|$$

Since $\|L(b)\| = \sup\{\|aL(b)\| : \|a\| \le 1\} = \sup\{\|R(a)b\| : \|a\| \le 1\}$
$\le \|R\| \cdot \|b\|$

and conversely

– In what sense is $M(A)$ a compactification?

$a \mapsto (L_a, R_a)$ isometric $*$-homom

$A \hookrightarrow M(A)$

image is a two-sided "essential ideal"

$\left(Am = 0 \text{ or } mA = 0 \text{ iff } m = 0 \right)$

$M(C_0(X)) = C_b(X)$
continuous bounded functions on X

– $M(A)$ has unit (id, id)

– if A unital then $A = M(A)$

$\left(L(x) = L(1 \cdot x) = L(1) \cdot x \right)$

$\Rightarrow (L, R) = (L_a, R_a) \quad a = L(1) = R(1)$

– if A commutative $M(A)$ also

– $M(A)$ is maximal among C^*-alg's containing A as ~~essential~~ closed ideal: if $A \hookrightarrow B$ then B in $M(A)$

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & B \arrow[d, "\exists"] \\ & M(A) \end{tikzcd}$$

$M(c_0) = \ell^\infty \not\hookrightarrow c = C(\mathbb{N} \cup \{\infty\})$
$c_0 \hookrightarrow c$

(Stone-Čech compactification) → without the covariant functoriality !!

State space of A C^*alg.

$$\Phi(A) := \{ \varphi : A \to \mathbb{C} \text{ state} \}$$

$\Phi(A) \subset A^*$ subset of unit ball in op. $\|\cdot\|$
weak*-closed
$\Rightarrow$ (Banach-Alaoglu) compact

$\cup$
$M(A)$
contains comp. space of characters
(which may be trivial = pt)

$\Phi(A)$ convex:

$\varphi_1 , \varphi_2 : A \to \mathbb{C}$ states

$\lambda_1, \lambda_2 \in \mathbb{R}_+$ $\quad \lambda_1 + \lambda_2 = 1$

$\lambda_1 \varphi_1 + \lambda_2 \varphi_2 = \varphi : A \to \mathbb{C}$ also a state

$\varphi(1) = 1$ since $\lambda_1 + \lambda_2 = 1$

$$\varphi(a^* a) = \lambda_1 \varphi_1(a^* a) + \lambda_2 \varphi_2(a^* a) \geq 0$$

so positivity also still holds

So it is a convex compact topol. space

more general convex combinations:

$\int_\Sigma \varphi_\alpha \, d\mu(\alpha)$

$\Sigma \subset \Phi(A)$

with $\int_\Sigma d\mu(\alpha) = 1$

$\rightsquigarrow$ Extremal points of convex
(those that cannot be further decomposed
$\varphi = \lambda_1 \varphi_1 + \lambda_2 \varphi_2$ iff $\lambda_1 = 1$ or $\lambda_2 = 1$)

Note: if think of $\Phi(A)$ as set of probability measures on NC space A, then extremal ones are "like" measures supported on points
Another way to describe points in NCG

<u>States:</u> A C^*-algebra

<u>Positive cone</u> $A_+ = \{a \in A : \sigma_A(a) \geq 0\}$

$\lambda a \in A_+$ for $a \in A_+$ $\lambda \in \mathbb{R}_+^*$
$a, b \in A_+ \Rightarrow \frac{1}{2}(a+b) \in A_+$ $(\Rightarrow a+b \in A_+)$ } cone

for $A = C(X)$ $f \geq 0$ usual sense

in particular $a^*a \in A_+$ for all $a \in A$

(or can take A_+ to be cone gen. by a^*a elements & their combinations with $\mathbb{R}_+$ coefficients)

<u>State:</u> $\varphi : A \to \mathbb{C}$ (continuous) linear functional
which is positive on A_+ $\varphi : A_+ \to \mathbb{R}_+$
and of norm $= 1$ $\|\varphi\| = 1$
(for unital A require $\varphi(1) = 1$)

<u>Note</u>: if φ multiplicative functional (as in GN)
then automatically positive $\varphi(a^*a) = \varphi(a^*)\varphi(a) = \overline{\varphi(a)}\varphi(a) \geq 0$

So all multiplicative functionals define states
but in general states are <u>NOT</u> multiplicative

Multipl. functionals	States
$\Updownarrow$	$\Updownarrow$
"Points"	"Measures" on an NC space

Example: $A = C_0(X)$ $\quad d\mu$ = Borel measure

$$\varphi(f) = \int_X f\, d\mu \quad \text{is a state on } A$$ Normalized: {probability measure}

$A = M_n(\mathbb{C})$ $\quad \rho$ = positive (spectrum positive) $\rho = t^*t$

$$\varphi(a) = \frac{Tr(\rho a)}{Tr(\rho)} \quad \text{is a state}$$ (normalization to have $\|\varphi\|$, $\varphi(1) = 1$)

Note: Cauchy-Schwarz inequality

$$|\varphi(a^*b)|^2 \le \varphi(a^*a)\varphi(b^*b)$$

$\varphi(a^*b)$ behaves "like" an inner product $\langle a, b\rangle$

(same proof as classical

$$\varphi((a+\lambda b)^*(a+\lambda b)) \ge 0, \quad \lambda = \frac{t\,\varphi(b^*a)}{|\varphi(b^*a)|}, \quad t \in \mathbb{R}$$

$$\Rightarrow \varphi(a^*a) = 0 \text{ iff } \varphi(ba) = 0 \quad \forall b \in A$$

(means Ker φ (on a^*a type elts) will define an ideal even though φ not multipl.)

GNS representation: $\varphi: A \to \mathbb{C}$ state

$\exists\ \mathcal{H}_\varphi$ Hilbert space $\quad \pi_\varphi: A \to B(\mathcal{H}_\varphi)$ representation

$\xi_\varphi \in \mathcal{H}_\varphi$ vector s.t. $\pi_\varphi(A)\cdot\xi_\varphi$ dense in $\mathcal{H}_\varphi$ ← cyclic vector

$(\langle \xi_\varphi, \pi(a)\xi_\varphi\rangle = \varphi(a))$

~~Define V_φ = vector space~~ $\quad I_\varphi = \{a \in A : \varphi(a^*a) = 0\}$ $\quad$ left ideal in A

$\xi_a = a + I_\varphi$ equivalence class

$$\langle \xi_a, \xi_b\rangle := \varphi(a^*b) \quad \text{inner product on } A/I_\varphi$$ $= \mathcal{H}_\varphi$ ← completion in this $\langle,\rangle$

representation

$$\pi_\varphi(a)\xi_b = \xi_{ab} = ab + I_\varphi$$

$$\|\pi_\varphi(a)\| \le \|a\|$$

$\xi_\varphi = 1 + \mathcal{H}_\varphi$ for unital A (else approx un. $\xi_\varphi = \lim u_\lambda + I_\varphi$)

<u>Thm</u>: $\varphi : A \to \mathbb{C}$ state

then GNS representation $\pi_\varphi : A \to B(\mathcal{H}_\varphi)$
is irreducible <u>iff</u> φ is an extremal point
of $\Phi(A)$

<u>Preliminary observation</u>:

φ_1, φ_2 $\varphi_1 \leq \varphi_2$ (i.e. $\varphi_2 - \varphi_1$ positivity condition of states)
linear functional

$\Updownarrow$

$\exists\ H \in B(\mathcal{H}_{\varphi_2})$ s.t. $\varphi_1(a) = \langle H \pi_{\varphi_2}(a) \xi_{\varphi_2}, \xi_{\varphi_2} \rangle_{\mathcal{H}_{\varphi_2}}$

bilinear form $\varphi_1(a^* b)$ bounded positive defined on a dense subset of $\mathcal{H}_{\varphi_2}$ $\{\varphi_2 a\}$

$$\ell(\pi_{\varphi_2}(a)\xi_{\varphi_2}, \pi_{\varphi_2}(b)\xi_{\varphi_2}) := \varphi_1(a^* b)$$

$\Rightarrow$ $\exists$ bounded op H s.t.

$$\ell(\ ,\ ) = \langle\ , H\ \rangle_{\mathcal{H}_{\varphi_2}}$$

Also H commutes with $\pi_{\varphi_2}(a)$:

$$\langle \pi_{\varphi_2}(b)\xi_2,\ H \pi_{\varphi_2}(c)\pi_{\varphi_2}(a)\xi_2 \rangle$$

$$= \varphi_1(b^*(ca)) = \varphi_1((c^* b^*)^* a) = \langle \pi_{\varphi_2}(b)\xi_2, \pi_{\varphi_2}(c) H \pi_{\varphi_2}(a)\xi_2 \rangle$$

Irreducible representations and pure states

First observation on GNS repres:

Given $\pi : A \to B(\mathcal{H})$
and a unit cyclic vectors
$\xi \in \mathcal{H}$ $\|\xi\| = 1$ $\pi(A)\xi$ dense in $\mathcal{H}$

$\Rightarrow$ state $\varphi_\xi : A \to \mathbb{C}$

by $\varphi_\xi(a) = \langle \xi, \pi(a)\xi \rangle$

$\Rightarrow$ GNS rep π_φ of this state

$\pi(a)\xi \mapsto \pi_\varphi(a)\xi_\varphi$ gives a unitary
equivalence $U : \mathcal{H} \to \mathcal{H}_\varphi$

s.t. $\pi(a) = U^* \pi_\varphi(a) U$

$\pi : A \to B(\mathcal{H})$ <u>irreducible</u>
if there is <u>no</u> closed subspace $\mathcal{H}' \subset \mathcal{H}$
other than $\{0\}$ or $\mathcal{H}$
which is invariant under all the $\pi(a)$, $a \in A$

<u>i.e.</u> any $T \in B(\mathcal{H})$ that commutes with all $\pi(a)$, $a \in A$
is a scalar $\lambda \in \mathbb{C} \cdot 1$.

<u>Note</u>: this implies in commutative case $C_0(X)$
the only <u>irreducible</u> rep's can be 1-dimensional
<u>i.e.</u> <u>characters</u> $\varphi \in M(C_0(X))$ multiplicative lin. functionals

use different notation
M used for mult. lin functionals
but also for multipliers
Use $\chi(A)$ for characters $M(A)$ for multipliers

If $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2 \quad \varphi_i \in \Phi(A) \quad 0 \le \lambda \le 1$

$\Rightarrow \lambda\varphi_1 = \langle \xi_\varphi , H\pi_\varphi(a)\xi_\varphi \rangle$ for some H

$[H, \pi_\varphi(a)] = 0$

if π_φ irreducible $H = x \cdot 1 \quad x \in \mathbb{C}$

$\Rightarrow \lambda\varphi_1 = x\varphi$ normalization $\varphi(1) = 1$, $\varphi_1(1) = 1$ gives same

Conversely. if φ extremal state

suppose $\exists\, H \in B(H_\varphi) \quad [H, \pi_\varphi(a)] = 0$

then $\langle \xi_\varphi , H\pi_\varphi(a)\xi_\varphi \rangle$ defines a state $\varphi_H < \varphi$

and also $\langle \xi_\varphi , (1-H)\pi_\varphi(a)\xi_\varphi \rangle$ does φ_{1-H}

$\varphi = \varphi_H + \varphi_{1-H} \qquad \dfrac{\varphi_H}{\|\varphi_H\|} , \dfrac{\varphi_{1-H}}{\|\varphi_{1-H}\|}$ are states

$\|\varphi_H\| = \varphi_H(1)$ so $\varphi_H(1) + \varphi_{1-H}(1) = \varphi(1) = 1$

$\varphi = \|\varphi_H\| \left(\dfrac{\varphi_H}{\|\varphi_H\|}\right) + \|\varphi_{1-H}\| \left(\dfrac{\varphi_{1-H}}{\|\varphi_{1-H}\|}\right)$ would be a decomposition

since φ extremal must be

$\varphi = \dfrac{\varphi_H}{\|\varphi_H\|}$ and $\|\varphi_H\| = 1$ or other

$\Rightarrow H = \|\varphi_H\| \cdot 1 \Rightarrow \pi_\varphi$ irreducible

Extreme pts of $\Phi(A)$ called pure states

Universal representation of a C^* algebra

$$\pi : A \to B(\mathcal{H})$$

$$\mathcal{H} = \oplus \mathcal{H}_\varphi \qquad \varphi \in \Phi(A)$$

$$\pi = \oplus \pi_\varphi \qquad \text{faithful repres.}$$

Weak closure $\pi(A)''$ in $B(\mathcal{H})$
is enveloping von Neumann algebra (measurable functions on the NC space A)

Problem with using $\Phi(A)$ $P(A)$ as notion of points ~~& submanifolds~~ of A
"too big" tends to be non-locally-compact

while $\chi(A)$ tends to be too small
(a single pt)

Need some good intermediate notion of the "classical points" of a noncommutative space

↳ Use dynamical information

NC spaces are always dynamical
(time evolution)

↑
from a result of Connes on von Neumann algebras in the '70s

Will only look for those "pure states" that are equilibrium states of dynamics.