Finite Spectral Triple

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References

- Walter van Suijlekom, *Noncommutative Geometry and Particle Physics*, Springer 2014
- T. Krajewski, *Classification of finite spectral triples*, J. Geom. Phys. 28 (1998) 1–30
- Branimir Ćaćić, *Moduli spaces of Dirac operators for finite spectral triples*, arXiv:0902.2068
- L. Dabrowski, G. Dossena, *Product of real spectral triples*, Int. J. Geom. Methods Mod. Phys. 8 (2011) N.8, 1833–1848.
- A.Chamseddine, A.Connes, *Why the Standard Model?* J. Geom. Phys. 58 (2008) 38–47

• Finite space $X = \{p_1, \ldots, p_N\}$ finite set

• Algebra of functions $f : X \to \mathbb{C}$ is $C(X) = \mathbb{C}^N$,
$x = (x_i) = (f(p_i))$

• Noncommutative algebras (space of N pts with inner structure)

$$\mathcal{A} = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$$

• finite dimensional Hilbert space: vector space $\mathcal{H}$ with hermitian inner product $\langle v, w \rangle$

• $\star$-algebra of linear operators $\mathcal{L}(\mathcal{H})$, product = composition, involution = adjoint, norm

$$\|T\| = \sup_{v \,:\, \|v\|=1} \|Tv\|$$

$\sqrt{\lambda}$ largest eigenvalue of $T^* T$

- Representations: $\star$-algebra homomorphism

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$$

irreducible: only $\mathcal{A}$-invariant subspaces are $\{0\}$ and $\mathcal{H}$

- Commutant

$$\pi(\mathcal{A})' = \{ T \in \mathcal{L}(\mathcal{H}) \, : \, \pi(a) T = T \pi(a), \ \forall a \in \mathcal{A}\}$$

$\pi(\mathcal{A})'$ also a $\star$-algebra

- Representation π irreducible iff $\pi(\mathcal{A})'$ scalar multiples of identity
- Unitary equivalence: $(\mathcal{H}_1, \pi_1) \sim (\mathcal{H}_2, \pi_2)$

$$\pi_1(a) = U^* \pi_2(a) U, \ \ \forall a \in \mathcal{A}$$

with unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$

- Structure space: $\hat{\mathcal{A}}$ set of unitary equivalence classes of representations of $\mathcal{A}$

• Modules: algebra $\mathcal{A}$, left $\mathcal{A}$-module: vector space $\mathcal{E}$ with left action $(a_1 a_2)v = a_1(a_2 v)$; right $\mathcal{A}$ module with right action $v(a_1 a_2) = (va_1)a_2$

• (left) $\mathcal{A}$-module homomorphism $\phi : \mathcal{E} \to \mathcal{F}$ with $\phi(av) = a\phi(v)$

• Representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ makes $\mathcal{H}$ into left $\mathcal{A}$-module

• Bimodules: algebras $\mathcal{A}$ and $\mathcal{B}$ is left $\mathcal{A}$-module and right $\mathcal{B}$-module, commuting left $\mathcal{A}$-action and right $\mathcal{B}$-action

• balanced tensor product

$$\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{F} = \mathcal{E} \otimes \mathcal{F} / \{ \sum_i v_i a_i \otimes w_i - v_i \otimes a_i w_i \}$$

$v_i \in \mathcal{E}$, $w_i \in \mathcal{F}$, $a_i \in \mathcal{A}$

• Hilbert bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$ with $\mathcal{B}$-valued inner product

$$\langle \cdot , \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{B}$$

$$\langle v_1, a v_2 \rangle = \langle a^* v_1, v_2 \rangle$$

$$\langle v_1, v_2 b \rangle = \langle v_1, v_2 \rangle b$$

$$\langle v_1, v_2 \rangle^* = \langle v_2, v_1 \rangle$$

$$\langle v, v \rangle \geq 0; \quad \langle v, v \rangle = 0 \text{ iff } v = 0$$

where $b \geq 0$ in $\mathcal{B}$ means $b = h^* h$

• set of Hilbert bimodules $KK_f(\mathcal{A}, \mathcal{B})$

• $\star$-algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ makes $\mathcal{B}$ into a Hilbert bimodule with $\langle b, b' \rangle = b^* b'$ and $\phi(a)$ as $\mathcal{A}$-action

- Kasparov product $KK_f(\mathcal{A},\mathcal{B}) \times KK_f(\mathcal{B},\mathcal{C}) \to KK_f(\mathcal{A},\mathcal{C})$

$$({}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}, {}_{\mathcal{B}}\mathcal{F}_{\mathcal{C}}) \mapsto {}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{F}_{\mathcal{C}}$$

with $\mathcal{C}$-valued inner product

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{\mathcal{C}} = \langle w_1, \langle v_1, v_2 \rangle_{\mathcal{B}} w_2 \rangle_{\mathcal{C}}$$

- Morita equivalence $\mathcal{A} \overset{M}{\simeq} \mathcal{B}$ iff there are $\mathcal{E} \in KK_f(\mathcal{A},\mathcal{B})$ and $\mathcal{F} \in KK_f(\mathcal{B},\mathcal{A})$ such that

$$\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F} = \mathcal{A}, \quad \mathcal{F} \otimes_{\mathcal{A}} \mathcal{E} = \mathcal{B}$$

these give equivalences of the categories of modules by tensoring $- \otimes_{\mathcal{A}} \mathcal{E}$ and $- \otimes_{\mathcal{B}} \mathcal{F}$

• Fact: Algebras $\mathcal{A} = \oplus_{i=1}^N M_{n_i}(\mathbb{C})$ and $\mathcal{B} = \oplus_{j=1}^M M_{m_j}(\mathbb{C})$ Morita equivalent iff $N = M$

• $M_n(\mathbb{C})$ has a unique irreducible representation $\mathbb{C}^n$

• bimodules

$$\mathcal{E} = \oplus_{i=1}^N \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}, \quad \mathcal{F} = \oplus_{i=1}^N \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}$$

with $\mathcal{A}$ acting on the $\mathbb{C}^{n_i}$ factors and $\mathcal{B}$ on the $\mathbb{C}^{m_i}$

$$\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F} = \oplus_i \mathbb{C}^{n_i} \otimes (\mathbb{C}^{m_i} \otimes_{M_{m_i}(\mathbb{C})} \mathbb{C}^{m_i}) \otimes \mathbb{C}^{n_i}$$

$$= \oplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i} = \oplus_i M_{n_i}(\mathbb{C})$$

same form $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{E}$

• $N = \#\hat{\mathcal{A}}$ and $M = \#\hat{\mathcal{B}}$ structure spaces, Morita invariant

Finite Spectral triples as Metric Spaces

- finite space $X = \{p_1, \dots, p_N\}$, metric $d_{ij} = dist(p_i, p_j)$ with $d_{ij} = d_{ji}$, $d_{ij} \geq 0$ and $d_{ij} = 0$ iff $i = j$; $d_{ij} \leq d_{ik} + d_{jk}$
- algebra $\mathcal{A} = C(X) = \mathbb{C}^N$: there is rep $(\mathcal{H}, \pi)$ and $D = D^*$ on $\mathcal{H}$

$$d_{ij} = \sup_{f \in \mathcal{A}} \{ |f(p_i) - f(p_j)| \,:\, \| [D, \pi(f)] \| \leq 1 \}$$

- construct inductively: $N = 2$, $\mathcal{H} = \mathbb{C}^2$

$$\pi(f) = \begin{pmatrix} f(p_1) & 0 \\ 0 & f(p_2) \end{pmatrix}, \quad D = \begin{pmatrix} 0 & d_{12}^{-1} \\ d_{12}^{-1} & 0 \end{pmatrix}$$

$$\| [D, \pi(f)] \| = d_{12}^{-1} \, |f(p_1) - f(p_2)|$$

• given $(\mathcal{H}_N, \pi_N, D_N)$ take

$$\mathcal{H}_{N+1} = \mathcal{H}_N \oplus \bigoplus_{i=1}^{N} \mathbb{C}^2$$

$$\pi_{N+1}(f(p_1), \ldots, f(p_{N+1})) = \pi_N(f(p_1), \ldots, f(p_N)) \oplus$$

$$\oplus \begin{pmatrix} f(p_1) & 0 \\ 0 & f(p_{N+1}) \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} f(p_N) & 0 \\ 0 & f(p_{N+1}) \end{pmatrix}$$

$$D_{N+1} = D_N \oplus \begin{pmatrix} 0 & d_{1,N+1}^{-1} \\ d_{1,N+1}^{-1} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & d_{N,N+1}^{-1} \\ d_{N,N+1}^{-1} & 0 \end{pmatrix}$$

• Finite Spectral Triple: $(\mathcal{A}, \mathcal{H}, D)$ finite dimensional involutive $\mathbb{C}$-algebra, representation $(\mathcal{H}, \pi)$ on a finite dimensional Hilbert space, $D = D^*$ on $\mathcal{H}$
Wedderburn theorem: $\mathcal{A} = \oplus_i M_{n_i}(\mathbb{C})$

- 1-forms: given $(\mathcal{A}, \mathcal{H}, D)$ finite

$$\Omega^1_D(\mathcal{A}) = \{\sum_k a_k[D, b_k] \,:\, a_k, b_k \in \mathcal{A}\}$$

$d : \mathcal{A} \to \Omega^1_D(\mathcal{A})$ with $da = [D, a]$

$$d(ab) = d(a)b + ad(b), \quad d(a^*) = -d(a)^*$$

$\Omega^1_D(\mathcal{A})$ is an $\mathcal{A}$-bimodule

- Example: $(M_n(\mathbb{C}), \mathbb{C}^n, D)$ with $D \neq \lambda I$, then $\Omega^1_D(\mathcal{A}) = M_n(\mathbb{C})$

- Unitary equivalence $(\mathcal{A}, \mathcal{H}_1, D_1) \sim (\mathcal{A}, \mathcal{H}_2, D_2)$: unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$

$$U\pi_1(a)U^* = \pi_2(a), \quad UD_1 U^* = D_2$$

case of inner $u \in U(\mathcal{A})$: gives $uDu^* = D + u[D, u^*]$ shifted by a 1-form

- Morita equivalences of finite spectral triples

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

$\mathcal{E} \in KK_f(\mathcal{B}, \mathcal{A})$ and Dirac operator

$$D'(v \otimes \xi) = v \otimes D\xi + \nabla(v)\xi$$

$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ connection

$$\nabla(va) = \nabla(v)a + v \otimes [D, a]$$

- Role of the connection: if take only $D'(v \otimes \xi) = v \otimes D\xi$ does not preserve ideal defining balanced product $va \otimes \xi - v \otimes a\xi$; this problem corrected precisely by connection

$$D'(va \otimes \xi - v \otimes a\xi) = va \otimes D\xi + \nabla(va)\xi - v \otimes D(a\xi) - \nabla(v)a\xi = 0$$

so D' well defined on quotient by the ideal

- Conclusion: given $(\mathcal{A}, \mathcal{H}, D)$ and $\mathcal{E} \in KK_f(\mathcal{B}, \mathcal{A})$

$$(\mathcal{B}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, D')$$

also a finite spectral triple if the connection ∇ satisfies

$$\langle v_1, \nabla(v_2) \rangle_{\mathcal{A}} - \langle \nabla(v_1), v_2 \rangle_{\mathcal{A}} = d \langle v_1, v_2 \rangle_{\mathcal{A}}$$

compatibility condition: ensures that $D'^{\star} = D'$

Classifying finite spectral triples

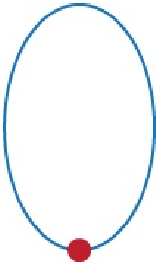
• Algebra $\mathcal{A} = \sum_{i=1}^{N} M_{n_i}(\mathbb{C})$, representation (faithful) $\mathcal{H} = \oplus_{i=1}^{N} \mathbb{C}^{n_1} \otimes V_i$, multiplicity V_i of dim r_i

$$D_{ij} : \mathbb{C}^{n_1} \otimes V_i \to \mathbb{C}^{n_j} \otimes V_j$$

with $D_{ij} = D_{ji}^*$ components of Dirac operator

• Decorated graphs: (V, E) vertices, edges (possible looping edges and multiple edges); $\# V = N$: decorate each vertex v by non-negative integers n_i (rank) and r_i (multiplicity); edge between vertices $v(n_i, r_i)$ and $v(n_j, r_j)$ if $D_{ij} \neq 0$

• Example: $(M_n(\mathbb{C}), \mathbb{C}^n, D = D_e + D_e^*)$, loop with one vertex decorated by n and one edge decorated by D_e



Finite Spectral Triples with Real Structure

- $(\mathcal{A}, \mathcal{H}, D)$ finite spectral triple as before
- even: $\mathbb{Z}/2\mathbb{Z}$-grading γ on $\mathcal{H}$ with $\gamma^* = \gamma$, $\gamma^2 = 1$ and

$$\gamma D + D\gamma = 0, \quad \gamma\pi(a) = \pi(a)\gamma$$

for all $a \in \mathcal{A}$

- anti-unitary: $J : \mathcal{H} \to \mathcal{H}$

$$\langle J\xi_1, J\xi_2\rangle = \langle \xi_2, \xi_1\rangle, \quad \forall \xi_1, \xi_2 \in \mathcal{H}$$

- bimodule: $\pi^0(a) = J\pi(a)^* J^{-1}$ right action of $\mathcal{A}$ on $\mathcal{H}$, with $\pi^0(ab) = \pi^0(b)\pi^0(a)$ and

$$[\pi(a), \pi^0(b)] = 0, \quad \forall a, b \in \mathcal{A}$$

- order one condition: for all $a, b \in \mathcal{A}$

$$[[D, \pi(a)]\pi^0(b)] = 0$$

Notation: usually write $[D, a]$ for $[D, \pi(a)]$ and a^0 for $\pi^0(a)$

- KO-dimension: antilinear isometry $J : \mathcal{H} \to \mathcal{H}$

$$J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon'' \gamma J$$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

- Extended KO-dimension: additional cases that do not correspond to the behavior of classical manifolds but are needed for a good theory of products and locally product-like spectral triples:

n	0_+	0_-	1	2_+	2_-	3	4_+	4_-	5	6_+	6_-	7
ε	1	1	1	-1	1	-1	-1	-1	-1	1	-1	1
ε'	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1
ε''	1	1		-1	-1		1	1		-1	-1	

Products of spectral triples (modelled on manifolds) $J = J_1 \otimes J_2$

1. even/odd $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D = D_1 \otimes 1 + \gamma_1 \otimes D_2)$$

2. odd/even $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2)$

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D = D_1 \otimes \gamma_2 + 1 \otimes D_1)$$

3. even/even $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2)$ use either of the above: the two operators are unitarily equivalent

$$D_1 \otimes 1 + \gamma_1 \otimes D_2 \sim D_1 \otimes \gamma_2 + 1 \otimes D_1$$

4. odd/odd $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}^2, D = D_1 \otimes 1 \otimes \sigma_1 + 1 \otimes D_2 \otimes \sigma_2, \gamma = 1 \otimes 1 \otimes \sigma_3)$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Unitary equivalences: $(\mathcal{A},\mathcal{H}_1,D_1,J_1,\gamma_1)\simeq(\mathcal{A},\mathcal{H}_2,J_2,\gamma_2)$ unitary $U:\mathcal{H}_1\to\mathcal{H}_2$

$$U\pi_1(a)U^*=\pi_2(a),\quad UD_1U^*=D_2,\quad U\gamma_1U^*=\gamma_2,\quad UJ_1U^*=J_2$$

- Morita equivalences: $(\mathcal{A},\mathcal{H},D,J,\gamma)$ and $\mathcal{E}\in KK_f(\mathcal{B},\mathcal{A})$ conjugate module $\mathcal{E}^o=\{\bar{v}\,:\,v\in\mathcal{E}\}$ with $a\bar{v}b=\overline{b^*va^*}$ Hilbert bimodule for $(\mathcal{B}^o,\mathcal{A}^o)$ opposite algebras

$$\mathcal{H}'=\mathcal{E}\otimes_{\mathcal{A}}\mathcal{H}\otimes_{\mathcal{A}^o}\mathcal{E}^o,\quad J'(v\otimes\xi\otimes\bar{w})=v\otimes J\xi\otimes\bar{w}$$

$$D'(v\otimes\xi\otimes\bar{w})=\nabla(v)\xi\otimes\bar{w}+v\otimes D\xi\otimes\bar{w}+v\otimes\xi\bar{\nabla}(\bar{w}),$$

with connection $\nabla:\mathcal{E}\to\mathcal{E}\otimes_{\mathcal{A}}\Omega^1_D(\mathcal{A})$ and $\bar{\nabla}=\tau\circ\nabla$ for $\tau:\mathcal{E}\otimes_{\mathcal{A}}\Omega^1_D(\mathcal{A})\to\Omega^1_D(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E}^o$ by $\tau(v\otimes\omega)=-\omega^*\otimes\bar{v}$

$$\bar{\nabla}(a\bar{v})=[D,a]\otimes\bar{v}+a\bar{\nabla}(\bar{v})$$

right action of 1-forms: $\xi\mapsto\epsilon' J\omega^*J^{-1}\xi$ and $\gamma'=1\otimes\gamma\otimes1$

Classifying Finite Real Spectral Triples: Krajewski diagrams

- again algebra $\mathcal{A} = \oplus_{i=1}^N M_{n_i}(\mathbb{C})$
- the Hilbert space $\mathcal{H}$ is now a bimodule: rep of $\mathcal{A} \otimes \mathcal{A}^o$

$$\mathcal{H} = \oplus_{i,j=1}^N \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j\,o} \otimes V_{ij}$$

multiplicities V_{ij}; irreducible rep $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j\,o}$

- anti-unitary $J : \mathcal{H} \to \mathcal{H}$ with $J^2 = \pm 1$

- $J^2 = 1$: o.n. basis $\{e_k^{ij}\}$ of $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j\,o} \otimes V_{ij}$ with $Je_k^{ij} = e_k^{ij}$
- $J^2 = -1$ o.n. basis $\{e_k^{ij}, f_k^{ji}\}$ with $e_k^{ij} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j\,o} \otimes V_{ij}$ and $f_k^{ji} \in \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i\,o} \otimes V_{ij}$ with $Je_k^{ij} = f_k^{ji}$ and $Jf_k^{ji} = -e_k^{ij}$

- Dirac: $D_{ij,kl}^* = D_{kl,ij}$

$$D_{ij,kl} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j\,o} \otimes V_{ij} \to \mathbb{C}^{n_k} \otimes \mathbb{C}^{n_l\,o} \otimes V_{kl}$$

need compatibility with J and order-one condition

- $JD = \pm DJ$ and $[[D,a]b^0] = 0$: edges in the diagram only horizontal or vertical or looping edges at a vertex: order one condition for a, b diagonal $a = \lambda_1 1_{n_1} \oplus \cdots \oplus \lambda_N 1_{n_N}$ and $b = \mu_1 1_{n_1} \oplus \cdots \oplus \mu_N 1_{n_N}$

$$[[D,a]b^0]_{ij,kl} = D_{ij,kl}(\lambda_i - \lambda_j)(\bar{\mu}_j - \bar{\mu}_i) = 0$$

gives $D_{ij,kl} = 0$ when $i \neq j$ and $k \neq l$ so only vertical and horizontal arrows or looping; compatibility with J relates $D_{ij,kl}$ with $D_{ji,lk}$ (preserves diagonal symmetry of diagram)

Fig. 3.2 The presence of the real structure J implies a symmetry in the diagram along the diagonal

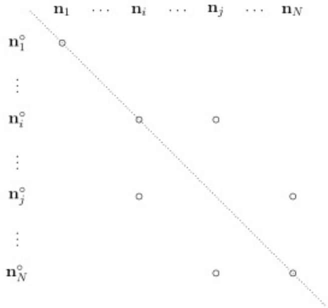
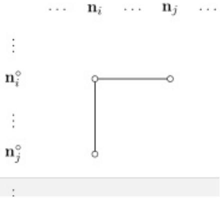


Fig. 3.3 The lines between two nodes represent a non-zero $D_{ii,ji} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i^\circ} \to \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i^\circ}$, as well as its adjoint $D_{ji,ii} : \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i^\circ} \to \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i^\circ}$. The non-zero components $D_{ii,ij}$ and $D_{ij,ii}$ are related to $\pm D_{ii,ji}$ and $\pm D_{ji,ii}$, respectively, according to $JD = \pm DJ$



from Walter van Suijlekom, *Noncommutative Geometry and Particle Physics*, Springer 2014

The case of real algebras

- for $\mathbb{R}$-algebras Wedderburn theorem

$$\mathcal{A} = \oplus_{i=1}^{N} M_{n_i}(\mathbb{F}_i), \quad \mathbb{F}_i = \mathbb{C}, \mathbb{R}, \text{ or } \mathbb{H}$$

quaternions

$$\mathbb{H} = \{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \}$$

$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = M_2(\mathbb{C})$ and $M_k(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} = M_{2k}(\mathbb{C})$

- representation $\mathbb{R}$-linear $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ complex $\mathcal{H}$; one-to-one correspondence with complex representations of $\mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$
- Krajewski diagrams with vertex labels by $(n_i, n_j, v_{ij}, \mathbb{F}_i, \mathbb{F}_j)$ and edges labelled by $D_{ij,kl}$
- irreducible finite real spectral triples classified in

- A. Chamseddine, A. Connes, *Why the Standard Model?* J. Geom. Phys. 58 (2008) 38–47

Moduli spaces of Dirac operators

• in earlier literature on (finite) spectral triples two additional properties are assumed, by analogy with the geometry of manifolds

- Orientability: $\mathcal{A}$-bimodule $(\mathcal{H}, \gamma)$ orientable if $\exists a_1, \dots a_k, b_1, \dots, b_k \in \mathcal{A}$

$$\gamma = \sum_{i=1}^{k} \pi(a_i)\pi^0(b_i)$$

- Poincaré duality: for projections $e, f \in \mathcal{A}$ bilinear form $\mathrm{Tr}(\gamma\pi(e)\pi^0(f))$ is non-degenerate

Note: physically interesting finite spectral triples do not necessarily satisfy these properties (also do not have KO-dim zero)

• structure P on $\mathcal{A}$-bimodule $\mathcal{H}$ (even, real, ...) then $\mathcal{D}(\mathcal{A},\mathcal{H},P)$ is $\mathbb{R}$-vector space of D with $D^* = D$, order one $[[D,\pi(a)],\pi^0(b)] = 0$, and $D\gamma + \gamma D = 0$, $DJ = \epsilon' JD$, ...

• Moduli space of Dirac operators (up to unitary equivalence)

$$\mathcal{M}(\mathcal{A},\mathcal{H},P) = \mathcal{D}(\mathcal{A},\mathcal{H},P)/U_{\mathcal{A}}^{LR}(\mathcal{H},P)$$

where $U_{\mathcal{A}}^{LR}(\mathcal{H},P)$ is $U:\mathcal{H}\to\mathcal{H}$ unitaries compatible with P

• Complete classification of these moduli spaces (depending on KO-dim and other data P) given in
Branimir Ćaćić, *Moduli spaces of Dirac operators for finite spectral triples*, arXiv:0902.2068

• We will discuss in detail moduli space for Standard Model case: physically renormalization group flow is a flow on this moduli space