Noncommutative Geometry, Quantum Statistical Mechanics, and Number Theory

Matilde Marcolli

MAT1314HS Winter 2019, University of Toronto
T 12-2 and W 12 BA6180

References

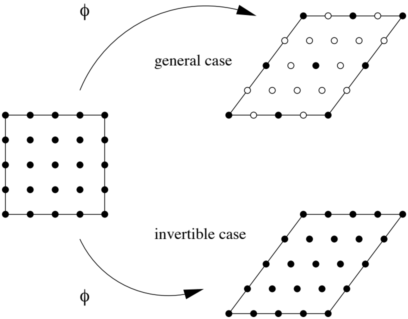
- A. Connes, M. Marcolli, *Noncommutative geometry, quantum fields and motives*, American Mathematical Society, 2008.
- A. Connes, M. Marcolli, *Quantum Statistical Mechanics of* $\mathbb{Q}$*-lattices* in "Frontiers in number theory, physics, and geometry. I", 269–347, Springer, 2006.
- A. Connes, M. Marcolli, N. Ramachandran, *KMS states and complex multiplication*, Selecta Math. (N.S.) 11 (2005), no. 3-4, 325–347.

$\mathbb{Q}$-lattices $\quad (\Lambda, \phi)$ $\mathbb{Q}$-lattice in $\mathbb{R}^n$

lattice $\Lambda \subset \mathbb{R}^n$ + labels of torsion points

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

group homomorphism (invertible $\mathbb{Q}$-lat is isom)



Commensurability

$$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$$

iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and

$$\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$$

$\mathbb{Q}$-lattices / Commensurability bad quotient in a classical sense
$\Rightarrow$ NC space

Quantum Statistical Mechanics (minimalist sketch)

- $\mathcal{A}$ unital C^*-algebra of observables
- σ_t time evolution, $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$
- states $\omega : \mathcal{A} \to \mathbb{C}$ continuous, normalized $\omega(1) = 1$, positive

$$\omega(a^* a) \geq 0$$

- equilibrium states $\omega(\sigma_t(a)) = \omega(a)$ all $t \in \mathbb{R}$
- representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, Hamiltonian H

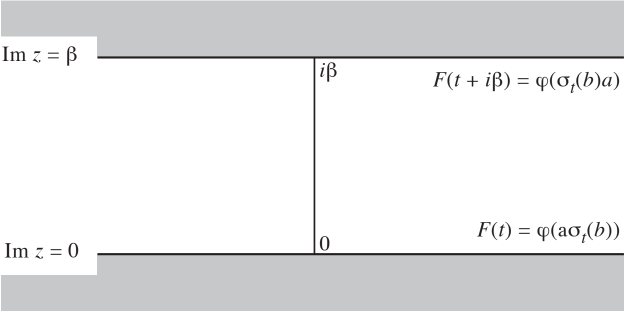
$$\pi(\sigma_t(a)) = e^{itH} \pi(a) e^{-itH}$$

- partition function $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$
- Gibbs states (equilibrium, inverse temperature β):

$$\omega_\beta(a) = \frac{\mathrm{Tr}(\pi(a) e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$$

- Generalization of Gibbs states: KMS states (Kubo–Martin–Schwinger) $\forall a, b \in A$, $\exists$ holomorphic $F_{a,b}$ on strip $I_\beta = \{0 < \Im z < \beta\}$, bounded continuous on ∂I_β,

$$F_{a,b}(t) = \omega(a\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \omega(\sigma_t(b)a)$$



- Fixed $\beta > 0$: KMS$_\beta$ state convex simplex: extremal states (like points in NCG)

Ground states $\beta = \infty,\ T = 0$

At $T > 0$ simplex $\mathrm{KMS}_\beta \rightsquigarrow$ extremal $\mathcal{E}_\beta$
(Points on NC space $\mathcal{A}$)

At $T = 0$: $\mathrm{KMS}_\infty =$ weak limits of KMS_β

$$\varphi_\infty(a) = \lim_{\beta\to\infty} \varphi_\beta(a) \quad \forall a \in \mathcal{A}$$

KMS_β for $0 < \beta < \infty$ simplex (set of extremal points $\mathcal{E}_\beta$) and set KMS_∞ also convex set with extremal set $\mathcal{E}_\infty$

Symmetries $\rightsquigarrow$ action on $\mathcal{E}_\beta$

Automorphisms $G \subset \mathrm{Aut}(\mathcal{A})$, $g\sigma_t = \sigma_t g$

Mod Inner: $u =$ unitary $\sigma_t(u) = u$

$$a \mapsto uau^*$$

Endomorphisms $\rho\sigma_t = \sigma_t\rho$ $e = \rho(1)$
For $\varphi(e) \neq 0$

$$\rho^*(\varphi) = \frac{1}{\varphi(e)}\,\varphi \circ \rho$$

Mod Inner: $u =$ isometry $\sigma_t(u) = \lambda^{it} u$

Action: (on $\mathcal{E}_\infty$: warming up/cooling down)

$$W_\beta(\varphi)(a) = \frac{\mathrm{Tr}(\pi_\varphi(a)\, e^{-\beta H})}{\mathrm{Tr}(\, e^{-\beta H})}$$

Isomorphism of QSM systems: $\varphi : (\mathcal{A}, \sigma) \to (\mathcal{B}, \tau)$

$$\varphi : \mathcal{A} \stackrel{\simeq}{\to} \mathcal{B}, \quad \varphi \circ \sigma = \tau \circ \varphi$$

C^*-algebra isomorphism intertwining time evolution

- Algebraic subalgebras $\mathcal{A}^\dagger \subset \mathcal{A}$ and $\mathcal{B}^\dagger \subset \mathcal{B}$: stronger condition: QSM isomorphism also preserves "algebraic structure"
$$\varphi : \mathcal{A}^\dagger \stackrel{\simeq}{\to} \mathcal{B}^\dagger$$
- Pullback of a state: $\varphi^* \omega(a) = \omega(\varphi(a))$

Bost–Connes QSM system

semigroup crossed product algebra

$$\mathcal{A}_{BC} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$$

- generators $e(r)$, $r \in \mathbb{Q}/\mathbb{Z}$ and μ_n, $n \in \mathbb{N}$ and relations

$$\mu_n \mu_m = \mu_m \mu_n, \quad \mu_m^* \mu_m = 1$$

$$\mu_n \mu_m^* = \mu_m^* \mu_n \quad \text{if} \quad (n, m) = 1$$

$$e(r+s) = e(r)e(s), \quad e(0) = 1$$

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

- time evolution $\sigma_t(f) = f$ and $\sigma_t(\mu_n) = n^{it} \mu_n$

- representations $\pi_\rho : \mathcal{A}_{BC} \to \ell^2(\mathbb{N})$, $\rho \in \hat{\mathbb{Z}}^* = \mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$

$$\pi_\rho(\mu_n)\epsilon_m = \epsilon_{nm}, \quad \pi_\rho(e(r))\epsilon_m = \zeta_r^m \epsilon_m$$

  $\zeta_r = \rho(e(r))$ root of unity
- Hamiltonian $H\epsilon_m = \log(m)\,\epsilon_m$, partition function

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \zeta_{\mathbb{Q}}(\beta)$$

  Riemann zeta function
- Low temperature KMS states: L-series normalized by zeta
- Galois action on zero temperature states (class field theory)

Representations Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$

• $\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ so identify $\hat{\mathbb{Z}}^*$ with all embeddings of abstract roots of unity $\mathbb{Q}/\mathbb{Z}$ as concrete roots of unity in $\mathbb{C}$

• $\alpha \in \hat{\mathbb{Z}}^* = \mathrm{GL}_1(\hat{\mathbb{Z}}) \Leftrightarrow$ embedding $\alpha : \mathbb{Q}^{cycl} \hookrightarrow \mathbb{C}$

$$\pi_\alpha(e(r))\, \epsilon_k = \alpha(\zeta_r^k)\, \epsilon_k$$

$$\pi_\alpha(\mu_n)\, \epsilon_k = \epsilon_{nk}$$

• covariant representation with Hamiltonian H

$$\pi_\alpha(\sigma_t(a)) = e^{itH} \pi_\alpha(a) e^{-itH}$$

for all $a \in \mathcal{A}_{BC}$ and all $t \in \mathbb{R}$

• Hamiltonian $H\, \epsilon_k = \log k \;\; \epsilon_k$

• partition function

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_{k=1}^{\infty} k^{-\beta} = \zeta(\beta)$$

Partition function = Riemann zeta function

Structure of KMS states (Bost–Connes)

- $\beta \leq 1 \Rightarrow$ unique KMS$_\beta$ state
- $\beta > 1 \Rightarrow \mathcal{E}_\beta \cong \hat{\mathbb{Z}}^*$ (free transitive action)

$$\varphi_{\beta,\alpha}(x) = \frac{1}{\zeta(\beta)} \mathrm{Tr}\left(\pi_\alpha(x)\, e^{-\beta H}\right)$$

- $\beta = \infty$ Galois action $\theta : \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \stackrel{\cong}{\to} \hat{\mathbb{Z}}^*$

$$\gamma\, \varphi(x) = \varphi(\theta(\gamma)\, x)$$

$\forall \varphi \in \mathcal{E}_\infty \quad \forall \gamma \in \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$
and $\forall x \in \mathcal{A}_\mathbb{Q}$ (arithmetic subalgebra)
$\varphi(\mathcal{A}_\mathbb{Q}) \subset \mathbb{Q}^{cycl}$ with $\mathcal{A}_\mathbb{Q} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$

Physical interpretation

• $\mathcal{P}$ set of primes, $\ell^2(\mathcal{P})$ and bosonic Fock space $\ell^2(\mathbb{N}) = \mathrm{Sym}(\ell^2(\mathcal{P}))$

• isometries μ_p, μ_p^* creation/annihilation operators on Bosonic Fock space

• $r \in \mathbb{Q}/\mathbb{Z} \rightsquigarrow$ "phase operators" $e(r)$

$$\text{Fock space } e(r)|n\rangle = \alpha(\zeta_r^n)|n\rangle$$

$\zeta_{a/b} = \zeta_b^a$ abstract roots of 1 embedding $\alpha : \mathbb{Q}^{cycl} \hookrightarrow \mathbb{C}$

• Optical phase: $|\theta_{m,N}\rangle = e\left(\frac{m}{N+1}\right) \cdot v_N$ superposition of occupation states $v_N = \frac{1}{(N+1)^{1/2}} \sum_{n=0}^N |n\rangle$

$n \in \mathbb{N}^\times = \mathbb{Z}_{>0} \rightsquigarrow$ changes of scale μ_n

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

1-dimensional $\mathbb{Q}$-lattices

$$(\Lambda, \phi) = (\lambda\,\mathbb{Z}, \lambda\,\rho) \quad \lambda > 0$$

$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$

Up to scaling λ: algebra $C(\hat{\mathbb{Z}}) \simeq C^*(\mathbb{Q}/\mathbb{Z})$

Commensurability Action of $\mathbb{N}$

$$\alpha_n(f)(\rho) = f(n^{-1}\rho)$$

1-dimensional $\mathbb{Q}$-lattices / Commensurability

$$\Rightarrow \quad \text{NC space } C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$$

Hecke algebra $\mathcal{A} = \mathcal{A}_1$

$(\Gamma_0, \Gamma) = (P_{\mathbb{Z}}, P_{\mathbb{Q}})$ $ax + b$ group

- $f : \Gamma_0 \backslash \Gamma / \Gamma_0 \to \mathbb{C}$ (or to $\mathbb{Q}$)

$$(f_1 * f_2)(\gamma) = \sum_{\Gamma_0 \backslash \Gamma} f_1(\gamma\gamma_1^{-1}) f_2(\gamma_1)$$

$f^*(\gamma) := \overline{f(\gamma^{-1})}$

- Regular representation $\ell^2(\Gamma_0 \backslash \Gamma) \Rightarrow$ von Neumann algebra

$\rightsquigarrow$ time evolution

$\gamma \in \Gamma$: $L(\gamma) = \#\Gamma_0\gamma$ and $R(\gamma) = L(\gamma^{-1})$

$$\sigma_t(f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)}\right)^{-it} f(\gamma)$$

$\sigma_t(\mu_n) = n^{it}\mu_n \qquad \sigma_t(e(r)) = e(r)$

NCG and class field theory

• $\mathbb{K}$ = number field $[\mathbb{K}:\mathbb{Q}] = n$, an algebraic closure $\bar{\mathbb{K}} \rightsquigarrow$ group of symmetries $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$

• Max abelian extension $\mathbb{K}^{ab}$:

$$\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}) = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})^{ab}$$

• Kronecker–Weber: $\mathbb{K} = \mathbb{Q}$

$$\mathbb{Q}^{ab} = \mathbb{Q}^{cycl} \quad \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^*$$

Hilbert 12th problem (explicit class field theory)

• Generators of $\mathbb{K}^{ab}$ + action of $\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$

• Solved for: $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-d})$ (imaginary quadratic)

Question: Can NCG say something new?
(at least for real quadratic $\mathbb{Q}(\sqrt{d})$)

Further generalizations: other QSM's with similar properties

- Bost-Connes as GL_1-case of QSM for moduli spaces of $\mathbb{Q}$-lattices up to commensurability (Connes-M.M. 2006) $\Rightarrow \mathrm{GL}_2$-case, modular curves and modular functions
- QSM systems for imaginary quadratic fields (class field theory): Connes-M.M.-Ramachandran
- B.Jacob and Consani-M.M.: QSM systems for function fields (Weil and Goss L-functions as partition functions)
- Ha-Paugam: QSM systems for Shimura varieties $\Rightarrow$ QSM systems for arbitrary number fields (Dedekind zeta function) further studied by Laca-Larsen-Neshveyev

2-dimensional $\mathbb{Q}$-lattices

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}\tau), \lambda\rho)$$

$\lambda \in \mathbb{C}^*$, $\tau \in \mathbb{H}$, $\rho \in M_2(\hat{\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2)$

Up to scale $\lambda \in \mathbb{C}^*$ and isomorphism

$$M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \quad \text{mod } \Gamma = \mathrm{SL}(2, \mathbb{Z})$$

Commensurability action of $\mathrm{GL}_2^+(\mathbb{Q})$ (partially defined)

$\Rightarrow$ NC space

Functions on

$$\mathcal{U} = \{(g, \rho, z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H} | g\rho \in M_2(\hat{\mathbb{Z}})\}$$

invariant under $\Gamma \times \Gamma$ action $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 z)$

Hecke algebra $\mathcal{A} = \mathcal{A}_2$

convolution product

$$(f_1 * f_2)(g,\rho,z) = \sum_{s\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}):s\rho\in M_2(\hat{\mathbb{Z}})} f_1(gs^{-1}, s\rho, s(z)) f_2(s,\rho,z)$$

$$f^*(g,\rho,z) = \overline{f(g^{-1}, g\rho, g(z))}$$

Time evolution

$$\sigma_t(f)(g,\rho,z) = \det(g)^{it}\, f(g,\rho,z)$$

Hilbert spaces for $\rho \in M_2(\hat{\mathbb{Z}})$

$$G_\rho = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) \,:\, g\rho \in M_2(\hat{\mathbb{Z}})\}$$

$$\mathcal{H}_\rho = \ell^2(\Gamma\backslash G_\rho)$$

Representations

$L=(\Lambda,\phi)=(\rho,z)$ gives representation on $\mathcal{H}_\rho$:

$$(\pi_L(f)\xi)(g)=\sum_{s\in\Gamma\backslash G_\rho} f(gs^{-1},s\rho,s(z))\,\xi(s)$$

$L=(\Lambda,\phi)$ invertible $\Rightarrow \mathcal{H}_\rho\cong \ell^2(\Gamma\backslash M_2^+(\mathbb{Z}))$

Hamiltonian

$$H\,\epsilon_m=\log\det(m)\;\epsilon_m$$

$\Rightarrow$ positive energy

Partition function

$\sigma(k)=\sum_{d|k} d$

$$Z(\beta)=\mathrm{Tr}\left(e^{-\beta H}\right)=\sum_{m\in\Gamma\backslash M_2^+(\mathbb{Z})}\det(m)^{-\beta}$$

$$=\sum_{k=1}^{\infty}\sigma(k)\,k^{-\beta}=\zeta(\beta)\zeta(\beta-1)$$

- $\beta < 1$ No KMS states
- $1 < \beta < 2$ Unique
- some special cases at $\beta = 1$, $\beta = 2$ analyzed separately (Laca-Larsen-Neshveyev)
- $\beta > 2 \;\Rightarrow\; \mathcal{E}_\beta \cong \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*$

$$\varphi_{\beta,L}(f) = \frac{1}{Z(\beta)} \sum_{m \in \Gamma\backslash M_2^+(\mathbb{Z})} f(1, m\rho, m(z)) \det(m)^{-\beta}$$

$L = (\rho, z)$
part that's more difficult to show: all $\varphi \in \mathcal{E}_\beta$ of this form (general strategy: show KMS states equivalent to certain measures with a scaling property depending on β on space associated to max abelian subalgebra)

Two phase transitions: poles of $\zeta(\beta)\zeta(\beta - 1)$

Symmetries $2 < \beta \leq \infty$

$\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q})\mathrm{GL}(\hat{\mathbb{Z}})$ acts on $\mathcal{A}_2$

- $\mathrm{GL}(\hat{\mathbb{Z}})$ by automorphisms (deck transformations of coverings of modular curves)

$$\theta_\gamma(f)(g, \rho, z) = f(g, \rho\gamma, z)$$

- $\mathrm{GL}_2^+(\mathbb{Q})$ by endomorphisms

$$\theta_m(f)(g, \rho, z) = \begin{cases} f(g, \rho\tilde{m}^{-1}, z) & \rho \in m\, M_2(\hat{\mathbb{Z}}) \\ 0 & \text{otherwise} \end{cases}$$

$\tilde{m} = \det(m) m^{-1}$

$\mathbb{Q}^* \hookrightarrow \mathrm{GL}_2(\mathbb{A}_f)$ acts by inner

$\Rightarrow \quad S = \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$ symmetries on $\mathcal{E}_\beta$

Action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathcal{E}_\infty$: warming/cooling

Arithmetic subalgebra

Look back at $\mathcal{A}_{\mathbb{Q}} = \mathcal{A}_{1,\mathbb{Q}}$ (Bost–Connes case): homogeneous functions of weight zero on 1-dim $\mathbb{Q}$-lattices

$$\epsilon_{k,a}(\Lambda,\phi) = \sum_{y\in\Lambda+\phi(a)} y^{-k}$$

normalized by covolume $e_{k,a} := c^k\, \epsilon_{k,a}$
($c(\Lambda) \sim |\Lambda|$ covolume with $(2\pi\sqrt{-1})c(\mathbb{Z}) = 1$)
GL_2 case $\mathcal{A}_{\mathbb{Q}} = \mathcal{A}_{2,\mathbb{Q}}$: Eisenstein series

$$E_{2k,a}(\Lambda,\phi) = \sum_{y\in\Lambda+\phi(a)} y^{-2k}$$

and

$$X_a(\Lambda,\phi) = \sum_{y\in\Lambda+\phi(a)} y^{-2} - \sum_{y\in\Lambda}{}' \, y^{-2}$$

normalized to weight zero (by $-2^7 3^5 \frac{g_2(z)g_3(z)}{\Delta(z)}$)

$\Rightarrow$ Modular functions

The modular field

Weierstrass $\wp$-function:
parameterization $w \mapsto (1, \wp(w; \tau, 1), \frac{d}{dw}\wp(w; \tau, 1))$
of elliptic curve $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ by $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$

Fricke functions: (homog weight zero) $v \in \mathbb{Q}^2/\mathbb{Z}^2$

$$f_v(z) = -2^7 3^5 \frac{g_2(z) g_3(z)}{\Delta(z)} \wp(\lambda_z(v); z, 1)$$

$\Delta(z) = g_2^3 - 27 g_3^2$ discriminant, $\lambda_z(v) := v_1 z + v_2$

Generators of the modular field F

(Shimura) $\quad \mathrm{Aut}(F) \cong \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$

$\tau \in \mathbb{H}$ generic $\Rightarrow$ evaluation $f \mapsto f(\tau)$
$\rightsquigarrow$ embedding $F \hookrightarrow \mathbb{C}$ image F_τ

$$\theta_\tau : \mathrm{Gal}(F_\tau/\mathbb{Q}) \stackrel{\simeq}{\to} \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$$

Galois action on $\mathcal{E}_\infty$

State $\varphi = \varphi_{\infty,L} \in \mathcal{E}_\infty$ ($L = (\rho,\tau)$ generic) $\Rightarrow$

$$\theta_\varphi : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$$

$$\theta_\varphi(\gamma) = \rho^{-1}\,\theta_\tau(\gamma)\,\rho$$

$\varphi = \varphi_{\infty,L}$ with $L = (\rho,\tau)$

$\forall f \in \mathcal{A}_{2,\mathbb{Q}}$ and $\forall \gamma \in \mathrm{Gal}(F_\tau/\mathbb{Q})$

$$\gamma\,\varphi(f) = \varphi(\theta_\varphi(\gamma) f)$$

$\varphi(\mathcal{A}_{2,\mathbb{Q}}) \subset F_\tau$

Shimura varieties $Sh(G,X) = G(\mathbb{Q})\backslash G(\mathbb{A}_f) \times X$

$$Sh(\mathrm{GL}_2, \mathbb{H}^{\pm}) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}$$

$$= \mathrm{GL}_2^{+}(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H} = \mathrm{GL}_2^{+}(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*$$

Components: $\pi_0(Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})) = Sh(\mathrm{GL}_1, \{\pm 1\})$

$$Sh(\mathrm{GL}_1, \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A}_f) \times \{\pm 1\} = \mathbb{Q}^*_+ \backslash \mathbb{A}^*_f$$

Commutative points of NC space $\mathrm{GL}_1(\mathbb{Q})\backslash \mathbb{A}_f$ Bost–Connes

$$\mathcal{A} \simeq C_0(\mathbb{A}_f) \rtimes \mathbb{Q}^*_+ \quad \text{Morita equiv}$$

Modular curves $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})$ = adèlic version of modular tower $\varprojlim \Gamma\backslash \mathbb{H}$ congruence subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{Z}) \rightsquigarrow$ congruence subgroup in $\mathrm{SL}(2,\mathbb{Q}) \rightsquigarrow$ components $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})$

NC Shimura varieties

Bost–Connes system:

$$Sh^{(nc)}(\{\pm 1\}, \mathrm{GL}_1) := \mathrm{GL}_1(\mathbb{Q})\backslash(\mathbb{A}_f \times \{\pm 1\})$$

$$= \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}^{\cdot}/\mathbb{R}_+^*$$

$\mathbb{A}^{\cdot} := \mathbb{A}_f \times \mathbb{R}^*$

Compactification:

$$\overline{Sh^{(nc)}}(\{\pm 1\}, \mathrm{GL}_1) = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}_+^*$$

Dual space (crossed product by time evolution σ_t)

$$\mathcal{L} = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A} \to \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}_+^*$$

$\mathbb{R}_+^*$-bundle
Spectral realization of zeros of ζ (Connes)

1-dim $\mathbb{Q}$-lattices (not up to scaling) modulo commensurability

GL_2-system

$$Sh^{(nc)}(\mathbb{H}^\pm, \mathrm{GL}_2) := \mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f)\times\mathbb{H}^\pm)$$

Compactification:

$$\overline{Sh^{(nc)}}(\mathbb{H}^\pm, \mathrm{GL}_2) := \mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f)\times\mathbb{P}^1(\mathbb{C}))$$
$$= \mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A})/\mathbb{C}^*$$

$\mathbb{P}^1(\mathbb{C}) = \mathbb{H}^\pm \cup \mathbb{P}^1(\mathbb{R})$

$\Rightarrow$ Adding NC boundary of modular curves

Dual system: $\mathbb{C}^*$-bundle $\mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A})$
Modular forms (instead of modular functions)
$\Rightarrow$ Modular Hecke algebra (Connes–Moscovici)

Compatibility:

$$\det\times\mathrm{sign} : Sh(\mathbb{H}^\pm, \mathrm{GL}_2) \to Sh(\{\pm 1\}, \mathrm{GL}_1)$$

passing to connected components π_0

Class field theory $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$

$\tau \in \mathbb{H}$ CM point $\mathbb{K} = \mathbb{Q}(\tau)$

Evaluation $F \to F_\tau \subset \mathbb{C}$ not embedding

$$F_\tau \simeq \mathbb{K}^{ab} \quad \{f(\tau),\ f \in F\} \text{ generators}$$

Galois action

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{K}^* & \longrightarrow & \mathrm{GL}_1(\mathbb{A}_{\mathbb{K},f}) & \xrightarrow{\simeq} & \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}) & \longrightarrow & 1 \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & \mathbb{Q}^* & \longrightarrow & \mathrm{GL}_2(\mathbb{A}_f) & \xrightarrow{\simeq} & \mathrm{Aut}(F) & \longrightarrow & 1. \end{array}$$

$\mathbb{A}_{\mathbb{K},f} = \mathbb{A}_f \otimes \mathbb{K}$

$\Rightarrow$ Specialization of GL_2-system at CM points $\rightsquigarrow$ CFT for $\mathbb{K} = \mathbb{Q}(\tau)$

What about $\mathbb{Q}(\sqrt{d})$??
NC boundary of mod curves in $\overline{Sh^{(nc)}}(\mathbb{H}^\pm, \mathrm{GL}_2)$
(Manin's real multiplication program)