Noncommutative spaces: geometry and dynamics

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Beyond the case of almost-commutative geometries

measure theory	von Neumann algebras
topology	C^*–algebras
smooth structures	smooth subalgebras
Riemannian geometry	spectral triples

Methods of constructing noncommutative spaces:

1. quotients
2. deformations
3. spaces defined by global properties

Von Neumann Algebras

• Hilbert space $\mathcal{H}$ (infinite dimensional, separable, over $\mathbb{C}$) algebra of bounded operators $\mathcal{B}(\mathcal{H})$ with operator norm

• Commutant of $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$:

$$\mathcal{M}' := \{ T \in \mathcal{B}(\mathcal{H}) : \; TS = ST, \forall S \in \mathcal{M} \}.$$

• von Neumann algebra: $\mathcal{M} = \mathcal{M}''$ (double commutant)
$\Leftrightarrow$ weakly closed

• Center: $Z(\mathcal{M}) = L^\infty(X, \mu)$.

• If $Z(\mathcal{M}) = \mathbb{C}$: factor

C^*-algebras

- involutive ($*$ anti-isomorphism) Banach algebra (complete in norm, $\|ab\| \leq \|a\| \cdot \|b\|$, $\|a^* a\| = \|a\|^2$)

- Gel'fand–Naimark correspondence: locally compact Hausdorff topological space $\Leftrightarrow$ commutative C^*-algebra:

$$X \Leftrightarrow C_0(X)$$

- representation of a C^*-algebra $\mathcal{A}$

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

C^*-algebra homomorphism

- state: continuous linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with positivity $\varphi(a^* a) \geq 0$ for all $a \in \mathcal{A}$ and $\varphi(1) = 1$

GNS representation

• cyclic vector ξ for a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a C^*-algebra if set $\{\pi(a)\xi \,:\, a \in \mathcal{A}\}$ dense in $\mathcal{H}$

• state from unit norm cyclic vector $\varphi(a) = \langle \pi(a)\xi, \xi \rangle$

• given a state $\varphi : \mathcal{A} \to \mathbb{C}$ construct a representation (GNS) where state is as above

• define $\langle a, b \rangle = \varphi(a^* b)$ for $a, b \in \mathcal{A}$

• $\mathcal{N} = \{a \in \mathcal{A} : \varphi(a^* a) = 0\}$ linear subspace but for C^*-algebras also a *left* ideal in $\mathcal{A}$

• $\mathcal{H} = \mathcal{A}/\mathcal{N}$ with $\langle a, b \rangle = \varphi(a^* b)$ Hilbert space

• the representation $\pi(a) b + \mathcal{N} = ab + \mathcal{N}$

• cyclic vector $\xi = 1 + \mathcal{N}$ unit of $\mathcal{A}$

Noncommutativity from quotients

Algebra of functions for a quotient space $X = Y/\sim$:

- Functions on Y with $f(a) = f(b)$ for $a \sim b$. **Poor!**
- Functions f_{ab} on the graph of the equivalence relation. **Good!**

• For sufficiently *nice* quotients: Morita equivalent

• simplest example: $Y = [0,1] \times \{0,1\}$; equivalence $(x,0) \sim (x,1)$ for $x \in (0,1)$. First method: constant functions $\mathbb{C}$; second method:

$$\{f \in C([0,1]) \otimes M_2(\mathbb{C}) : \ f(0) \text{ and } f(1) \text{ diagonal } \}$$

Morita equivalence

C^*-algebras $\mathcal{A}_1 \sim \mathcal{A}_2$: $\exists$ bimodule $\mathcal{M}$, right Hilbert $\mathcal{A}_1$ module with $\langle\cdot,\cdot\rangle_{\mathcal{A}_1}$, left Hilbert $\mathcal{A}_2$-module with $\langle\cdot,\cdot\rangle_{\mathcal{A}_2}$:

- All $\mathcal{A}_i$ as closure of span

$$\{\langle\xi_1,\xi_2\rangle_{\mathcal{A}_i} : \xi_1,\xi_2 \in \mathcal{M}\}$$

- $\forall \xi_1,\xi_2,\xi_3 \in \mathcal{M}$.

$$\langle\xi_1,\xi_2\rangle_{\mathcal{A}_1}\xi_3 = \xi_1\langle\xi_2,\xi_3\rangle_{\mathcal{A}_2}$$

- $\mathcal{A}_1$ and $\mathcal{A}_2$ act on $\mathcal{M}$ by bounded operators

Isomorphism class of noncommutative spaces $\Leftrightarrow$ Morita equivalence class of C^*-algebras

Variables and infinitesimals

real variable	self–adjoint oper.
complex variable	operator
infinitesimal	compact oper.
infinitesimal of order α	comp. op. with eigenvalues $\sim O(n^{-\alpha})$

$$f(T) = \frac{1}{2\pi i} \int_{\partial D} f(z)(zI - T)^{-1} \, dz.$$

- T self adjoint $\Rightarrow$ f measurable function;
- $T \in \mathcal{B}(\mathcal{H}) \Rightarrow f$ holomorphic on neighborhood of $\mathrm{Spec}(T)$.
- T compact: for any $\epsilon > 0$, there exists a finite dimensional subspace $E \subset H$ such that $\| T|_{E^\perp} \| < \epsilon$.

Spectral triple $(\mathcal{A}, \mathcal{H}, D)$

- $\mathcal{A} = \mathrm{C}^*$-algebra
- $\mathcal{H}$ Hilbert space: $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
- D unbounded self–adjoint operator on $\mathcal{H}$
- $(D - \lambda)^{-1}$ compact operator, $\forall \lambda \notin \mathbb{R}$
- $[D, a]$ bounded operator, $\forall a \in \mathcal{A}_0 \subset \mathcal{A}$, dense involutive subalgebra of $\mathcal{A}$.

Riemannian spin manifold X: $\mathcal{A} = C(X)$, $\mathcal{H} = L^2$-spinors, $D =$ Dirac operator, $\mathcal{A}_0 = C^\infty(X)$

Zeta functions

• spectral triple $(\mathcal{A}, \mathcal{H}, D) \Rightarrow$ family of zeta functions: for $a \in \mathcal{A}_0 \cup [D, \mathcal{A}_0]$

$$\zeta_{a,D}(z) := \mathrm{Tr}(a|D|^{-z}) = \sum_\lambda \mathrm{Tr}(a\,\Pi(\lambda, |D|))\lambda^{-z}$$

Dimension of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$

- Simpler definition: dimension n (*n–summable*) if $|D|^{-n}$ infinitesimal of order one: $\lambda_k(|D|^{-n}) = O(k^{-1})$
- Refined definition: **dimension spectrum** $\Sigma \subset \mathbb{C}$: set of poles of the zeta functions $\zeta_{a,D}(z)$. (all zetas extend holomorphically to $\mathbb{C} \setminus \Sigma$)
- in sufficiently nice cases (almost commutative geometries) poles of $\zeta_D(s) = \zeta_{1,D}(s)$ suffice

Spectral triples and Morita equivalence

$(\mathcal{A}_1, \mathcal{H}, D)$ = spectral triple

$\mathcal{A}_1 \sim \mathcal{A}_2$ Morita equivalent: bimodule $\mathcal{M}$ (fin. proj. right Hilbert module over $\mathcal{A}_1$)

$\mathcal{A}_1$-bimodule:

$$\Omega^1_D = \text{gen. by } \{a_1[D, b_1] : a_1, b_1 \in \mathcal{A}_1\}$$

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}_1} \Omega^1_D$$

$\nabla(\xi a_1) = (\nabla\xi)a_1 + \xi \otimes [D, a_1]$, $\forall \xi \in \mathcal{M}$, $a_1 \in \mathcal{A}_1$ and $\forall \xi_1, \xi_2 \in \mathcal{M}$

$$\langle \xi_1, \nabla\xi_2 \rangle_{\mathcal{A}_1} - \langle \nabla\xi_1, \xi_2 \rangle_{\mathcal{A}_1} = [D, \langle \xi_1, \xi_2 \rangle_{\mathcal{A}_1}].$$

$\Rightarrow$ spectral triple $(\mathcal{A}_2, \tilde{\mathcal{H}}, \tilde{D})$

Spectral triple $(\mathcal{A}_2, \tilde{\mathcal{H}}, \tilde{D})$

Hilbert space $\tilde{\mathcal{H}} = \mathcal{M} \otimes_{\mathcal{A}_1} \mathcal{H}$

action

$$a_2 \, (\xi \otimes_{\mathcal{A}_1} x) := (a_2 \xi) \otimes_{\mathcal{A}_1} x$$

Dirac operator

$$\tilde{D}(\xi \otimes x) = \xi \otimes D(x) + (\nabla \xi) x.$$

Remark

need Hermitian connection ∇, because commutators $[D, \rho(a)]$ for $a \in \mathcal{A}_1$ are non-trivial $\Rightarrow 1 \otimes D$ would not be well defined on $\tilde{\mathcal{H}}$

Example: Fractal string (already seen in Apollonian packings)

Ω bounded open in $\mathbb{R}$ (e.g. complement of Cantor set Λ in $[0,1]$)
$\mathcal{L} = \{\ell_k\}_{k\geq 1}$ lengths of connected components of Ω with

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k \cdots > 0.$$

Geometric zeta function (Lapidus and van Frankenhuysen)

$$\zeta_{\mathcal{L}}(s) := \sum_k \ell_k^s$$

Cantor set: spectral triple

$\Lambda =$ middle-third Cantor set: $\zeta_L(s) = \frac{3^{-s}}{1-2\cdot 3^{-s}}$

algebra commutative C^*-algebra $C(\Lambda)$.

Hilbert space: $E = \{x_{k,\pm}\}$ endpoints of intervals $J_k \subset \Omega = [0,1] \smallsetminus \Lambda$, with $x_{k,+} > x_{k,-}$

$$\mathcal{H} := \ell^2(E)$$

action $\mathrm{C}(\Lambda)$ acts on $\mathcal{H}$

$$f \cdot \xi(x) = f(x)\xi(x), \quad \forall f \in \mathrm{C}(\Lambda),\ \forall \xi \in \mathcal{H},\ \forall x \in E.$$

sign operator subspace $\mathcal{H}_k$ of coordinates $\xi(x_{k,+})$ and $\xi(x_{k,-})$,

$$F|_{\mathcal{H}_k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Dirac operator

$$D|_{\mathcal{H}_k} \begin{pmatrix} \xi(x_{k,+}) \\ \xi(x_{k,-}) \end{pmatrix} = \ell_k^{-1} \cdot \begin{pmatrix} \xi(x_{k,-}) \\ \xi(x_{k,+}) \end{pmatrix}$$

- verify $[D,a]$ bounded for $a \in \mathcal{A}_0$:

$$[D,f]|_{\mathcal{H}_k} = \frac{(f(x_{k,+}) - f(x_{k,-}))}{\ell_k} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

for f Lipschitz: $\|[D,f]\| \leq C(f)$
take dense $\mathcal{A}_0 \subset C(\Lambda)$ to be locally constant or more generally Lipschitz functions

- same for any self-similar set in $\mathbb{R}$ (Cantor-like)

Zeta function

$$\mathrm{Tr}(|D|^{-s}) = 2\zeta_L(s) = \sum_{k\geq 1} 2^k 3^{-sk} = \frac{2\cdot 3^{-s}}{1-2\cdot 3^{-s}}$$

Dimension spectrum

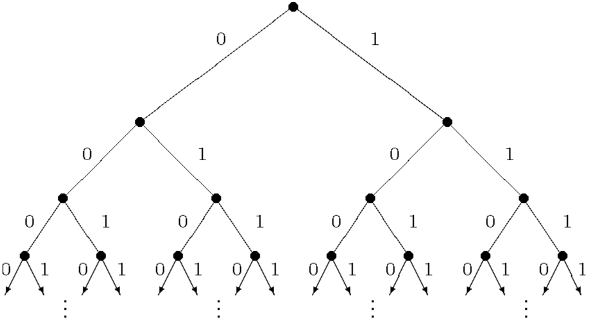
$$\Sigma = \left\{ \frac{\log 2}{\log 3} + \frac{2\pi i n}{\log 3} \right\}_{n\in\mathbb{Z}}$$

Example: AF algebras (noncommutative Cantor sets)

C^*-algebras approximated by finite dimensional algebras (direct limits of a direct system of finite dimensional algebras)

determined by Bratteli diagram: $\mathcal{F}_k$ = fin dim algebras $\phi_{k,k+1}$ embeddings with specified *multiplicities*

$C(\Lambda)$ = commutative AF algebra corresponding to the diagram

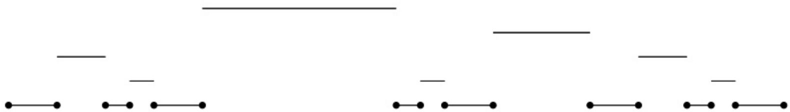


Example: Fibonacci spectral triple

Fibonacci AF algebra $\mathcal{F}_n = \mathcal{M}_{F_{n+1}} \oplus \mathcal{M}_{F_n}$, embeddings from partial embedding

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Fibonacci Cantor set from the interval $I = [0,4]$ remove F_{n+1} open intervals $J_{n,j}$ of lengths $\ell_n = 1/2^n$, according to the rule:



Hilbert space E of endpoints $x_{n,j,\pm}$ of the intervals $J_{n,j}$:
$\mathcal{H} = \ell^2(E)$, completion of

$$\begin{array}{ccccccccccc} \mathbb{C} & \oplus & \mathbb{C} & \oplus & \mathbb{C}^2 & \oplus & \mathbb{C}^3 & \oplus & \mathbb{C}^5 & \cdots \\ & \oplus & \mathbb{C} & \oplus & \mathbb{C} & \oplus & \mathbb{C}^2 & \oplus & \mathbb{C}^3 & \cdots \end{array}$$

Action of $\mathcal{M}_{F_{n+1}} \oplus \mathcal{M}_{F_n} \Rightarrow$ of AF algebra

Sign on subspace $\mathcal{H}_{n,j}$ spanned by $\xi(x_{n,j,\pm})$

$$F|_{\mathcal{H}_{n,j}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Dirac operator

$$D|_{\mathcal{H}_{n,j}} \begin{pmatrix} \xi(x_{n,j,+}) \\ (x_{n,j,-}) \end{pmatrix} = \ell_n^{-1} \begin{pmatrix} \xi(x_{n,j,-}) \\ \xi(x_{n,j,+}) \end{pmatrix}$$

$\Rightarrow$ spectral triple with zeta function

$$Tr(|D|^{-s}) = 2\zeta_F(s) = \frac{2}{1 - 2^{-s} - 4^{-s}}$$

geometric zeta function $\zeta_F(s) = \sum_n F_{n+1} 2^{-ns}$

- bounded commutators condition: $[D, a]$ bounded for $a \in \mathcal{A}_0$:

$$[D, U]|_{\mathcal{H}_{n,j}} \begin{pmatrix} \xi(x_{n,j,+}) \\ \xi(x_{n,j,-}) \end{pmatrix} = \ell_n^{-1} \begin{pmatrix} (A_{n,+} - A_{n,-})\xi(x_{n,j,-}) \\ (A_{n,-} - A_{n,+})\xi(x_{n,j,+}) \end{pmatrix}.$$

$\Rightarrow$ for $U \in \cup_k \mathcal{F}_k$ (dense subalgebra)

Dimension spectrum with $\phi = \frac{1+\sqrt{5}}{2}$

$$\left\{ \frac{\log \phi}{\log 2} + \frac{2\pi i n}{\log 2} \right\}_{n \in \mathbb{Z}} \cup \left\{ -\frac{\log \phi}{\log 2} + \frac{2\pi i (n + 1/2)}{\log 2} \right\}_{n \in \mathbb{Z}}$$

Noncommutative Torus (interesting non-AF case)

irrational rotation algebra $\mathcal{A}_\theta$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$: universal C^*-algebra $C^*(U, V)$ generated by unitaries U, V (so $U^* U = U U^* = 1$ and $V^* V = V V^* = 1$) with relation

$$UV = e^{2\pi i \theta} VU$$

concrete realization as an algebra of bounded operators

$$U e_n = e_{n+1}, \qquad V e_n = e^{2\pi i n \theta} e_n$$

on $\{e_n\}$ Fourier basis of Hilbert space $\mathcal{H} = L^2(S^1)$

- If $\theta \in \mathbb{Q}$, algebra $\mathcal{A}_\theta$ Morita equivalent to commutative $C(T^2)$ (it is an almost-commutative geometry when θ rational)

Kronecker foliation on T^2 foliation $dx = \theta dy$, $x, y \in \mathbb{R}/\mathbb{Z}$.

- space of leaves: $X = \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}) \simeq S^1/\theta\mathbb{Z}$,

$$R_\theta x = x + \theta \mod 1$$

- transversal: $T = \{y = 0\}$, $T \cong S^1 \cong \mathbb{R}/\mathbb{Z}$

$$\mathcal{A}_\theta = \{(f_{ab}) \quad a, b \in T \quad \text{in the same leaf} \}$$

- (f_{ab}) as power series $b = \sum_{n\in\mathbb{Z}} b_n V^n$ where each b_n is an element of the algebra $C(S^1)$, with multiplication

$$VhV^{-1} = h \circ R_\theta^{-1}$$

- $C(S^1)$ generated by $U(t) = e^{2\pi i t}$ $\Rightarrow$ generating system (U, V) with relation

$$UV = e^{2\pi i\theta} VU$$

- crossed product algebra $\mathcal{A}_\theta = C(S^1) \rtimes_{R_\theta} \mathbb{Z}$

$$"S^1/\theta\mathbb{Z}" \sim C(S^1) \rtimes_{R_\theta} \mathbb{Z}$$

K-theory of C^*-algebras

- $K_0(\mathcal{A})$: idempotents (for C^*-algebras $\Leftrightarrow$ projections: $p \mapsto P = pp^*(1-(p-p^*)^2)^{-1})$, $P \sim Q$ iff $P = X^*X$ $Q = XX^*$, X = partial isometry ($X = XX^*X$); stable equivalence: $P \sim Q$, $P \in \mathcal{M}_n(\mathcal{A})$ $Q \in \mathcal{M}_m(\mathcal{A})$, $\exists R$ proj $P \oplus R \sim Q \oplus R$ $\Rightarrow K_0(\mathcal{A})^+$, Grothendieck group $= K_0(\mathcal{A})$
- $K_1(\mathcal{A})$:

$$\mathrm{GL}_n(\mathcal{A}) = \text{ invertible elements in } \mathcal{M}_n(\mathcal{A})$$

$$\mathrm{GL}_n^0(\mathcal{A}) = \text{ identity component}$$

$$\mathrm{GL}_n(\mathcal{A}) \to \mathrm{GL}_{n+1}(\mathcal{A}) \qquad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathrm{GL}_n(\mathcal{A})/\mathrm{GL}_n^0(\mathcal{A}) \to \mathrm{GL}_{n+1}(\mathcal{A})/\mathrm{GL}_{n+1}^0(\mathcal{A})$$

direct limit $= K_1(\mathcal{A})$

Pimsner–Voiculescu six terms exact sequence: crossed product $\mathcal{A} = \mathcal{B} \rtimes_R \mathbb{Z}$:

$$\begin{array}{ccccc} K_0(\mathcal{B}) & \xrightarrow{I-R_*} & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{A}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{A}) & \longleftarrow & K_1(\mathcal{B}) & \xleftarrow[I-R_*]{} & K_1(\mathcal{B}) \end{array}$$

For irrational rotation: $\mathcal{B} = C(S^1)$: $K_i(C(S^1)) = K^i_{top}(S^1) = \mathbb{Z}$
R_θ preserves rk of proj and winding number of det of invertible element $\Rightarrow R_{\theta *} = I$

$$K_0(\mathcal{A}_\theta) \cong \mathbb{Z}^2 \qquad K_1(\mathcal{A}_\theta) \cong \mathbb{Z}^2$$

Projections and the NC torus

• Rieffel projectors for $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\forall \alpha \in (\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$, $\exists$ projection P_α in $\mathcal{A}_\theta$, with $\mathrm{Tr}(P_\alpha) = \alpha$.

Rieffel's method

- if $\mu \in {}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ bimodule such that $\mu\langle \mu, \mu \rangle_{\mathcal{B}} = \mu$, then $P :=_{\mathcal{A}} \langle \mu, \mu \rangle$ is a projection
- if $\xi \in {}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ such that $\exists$ invertible $*$-invariant square root $\langle \xi, \xi \rangle_{\mathcal{B}}^{1/2}$, then can use $m := \xi \langle \xi, \xi \rangle_{\mathcal{B}}^{-1/2}$

Boca's projectors for $\mathcal{A}_\theta$ from $\xi =$ Gaussian element in some Heisenberg modules (Mumford's Tata lectures vol. III) $\Rightarrow$ the corresponding $\langle \xi, \xi \rangle_{\mathcal{B}}$ is a **quantum theta function** in the sense of Manin (Theta functions, quantum tori and Heisenberg groups)

Spectral triple for the NC torus

smooth subalgebra for $\mathcal{S}(\mathbb{Z}^2)$ = Schwartz space of sequences of rapid decay on $\mathbb{Z}^2$

$$\mathcal{A}_{\theta,0} = \{\sum_{\mathbb{Z}^2} b_{nm} U^n V^m \ , \ b \in \mathcal{S}(\mathbb{Z}^2)\}$$

Dirac operator derivations

$$\delta_1 = 2\pi i U \frac{\partial}{\partial U} \quad \delta_2 = 2\pi i V \frac{\partial}{\partial V}$$

$\partial = \delta_1 + \tau\delta_2$, $Im(\tau) > 0$: on $\mathcal{H} \oplus \mathcal{H}$

$$D = -i \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}$$

• $D^2 = -(\delta_1 + \tau\delta_2)(\delta_1 + \bar{\tau}\delta_2)$ eigenvalues (multiplicity two):

$$\mathrm{Spec}(D^2) = \{4\pi^2 |m + n\tau|^2\}_{m,n\in\mathbb{Z}}$$

• Zeta function= Eisenstein series

$$\mathrm{Tr}(|D|^{-s}) = 2^{1-s}\pi^{-s} E_s(\tau),$$

$$E_s(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{|m + n\tau|^s}$$

Morita equivalent noncommutative tori

• Morita equivalence $\Leftrightarrow$ changing choice of transversal

• Morita equivalence $\mathcal{A}_\theta \simeq \mathcal{A}_{-1/\theta}$: change of parameterization of space of leaves using $T = \{y = 0\}$ or $S = \{x = 0\}$ as transversal

Bimodules and Morita equivalence

$$\theta' = \frac{a\theta + b}{c\theta + d} = g\theta$$

$\mathcal{M}_{\theta,\theta'}$ = Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/c)$, right action of $\mathcal{A}_\theta$:

$$Uf\,(x,u) = f\left(x - \frac{c\theta + d}{c}, u - 1\right)$$

$$Vf\,(x,u) = \exp(2\pi i(x - ud/c))f(x,u)$$

left action of $\mathcal{A}_{\theta'}$:

$$U'f\,(x,u) = f\left(x - \frac{1}{c}, u - a\right)$$

$$V'f\,(x,u) = \exp\left(2\pi i\left(\frac{x}{c\theta + d} - \frac{u}{c}\right)\right) f(x,u)$$

Noncommutative modular curve

- isomorphisms: $\mathcal{A}_\theta \cong \mathcal{A}_{-\theta}$ and $\mathcal{A}_\theta \cong \mathcal{A}_{\theta\pm 1}$, Morita equivalence $\mathcal{A}_\theta \simeq \mathcal{A}_{-1/\theta}$
- Morita equivalence classes: $\theta \sim \theta'$, same $\mathrm{PGL}(2,\mathbb{Z})$ orbit
- Moduli space quotient

$$\mathrm{PGL}(2,\mathbb{Z})\backslash\mathbb{P}^1(\mathbb{R})$$

but this is a "bad quotient" classically, so treat also as a *noncommutative space*

$$C(\mathbb{P}^1(\mathbb{R})) \rtimes \mathrm{PGL}(2,\mathbb{Z})$$

- this noncommutative space can be seen as a "noncommutative boundary" of the classical modular curve $\mathrm{PGL}(2,\mathbb{Z})\backslash\mathbb{H}$ that extends the classical compactification by cusps $\mathrm{PGL}(2,\mathbb{Z})\backslash\mathbb{P}^1(\mathbb{Q})$

Degeneration of elliptic curves

Elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, $Im(\tau) > 0$

Jacobi uniformization $q \in \mathbb{C}^*$, $q = \exp(2\pi i \tau)$, $|q| < 1$

$$E_q = \mathbb{C}/q^{\mathbb{Z}}$$

fundamental domain: annulus radii 1 and q, identification via scaling and rotation

As $q \to \exp(2\pi i \theta) \in S^1$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$,

$$E_q \longrightarrow \text{noncomm. torus } \mathcal{A}_\theta$$

$\Rightarrow$ study **limiting behavior** (e.g. of arithmetic invariants defined on modular curves) when $\tau \to \theta \in \mathbb{R} \setminus \mathbb{Q}$

Lattices and Manin's pseudolattices

Elliptic curves and lattices Lattice Λ, embedding $j : \Lambda \hookrightarrow \mathbb{C}$, $j(\Lambda) = \mathbb{Z} + \tau\mathbb{Z}$, elliptic curve

$$E_\tau \cong \mathbb{C}/j(\Lambda)$$

$\Rightarrow$ Equivalence of categories

Pseudolattices $\Lambda =$ free abelian rk 2, $j : \Lambda \hookrightarrow \mathbb{C}$ image in a $\mathbb{R}$-line, orientation ϵ.
$j(0,1) = 1$, $j(1,0) = \theta$

- Morphism: $g \in \mathcal{M}_2(\mathbb{Z})$, $\theta' = \frac{\theta+b}{c\theta+d}$, $\epsilon' = sign(c\theta + d)\epsilon$.
Isomorphism: $g \in \mathrm{GL}(2,\mathbb{Z})$
- Equivalence of categories: (i) Pseudolattices with morphisms $g \in \mathrm{GL}(2,\mathbb{Z})$, (ii) noncommutative tori with morphisms Morita equivalences $\mathcal{M}_{\theta,\theta'}$
- if $End(\Lambda) \neq \mathbb{Z}$ then $\Lambda \subset \mathbb{K}$, some real quadratic field: **real multiplication** (quadratic irrationalities)

Noncommutative Riemann surfaces?

For tori: (1) cross product algebra, (2) foliation, (3) deformation of group algebra $\Rightarrow$ same result $\mathcal{A}_\theta$

Higher genus

- Cross product: $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$, $\Gamma =$ Fuchsian group (uniformization, as for modular curves)
- Foliations: interval exchange transformations
- Group algebra: $C^*(\Gamma, \sigma)$ (quantum Hall effect)

$\Rightarrow$ in classical case these points of view describe same object, but their noncommuative versions give *different* results unlike in the genus one case

Example: Algebras of directed graphs (extends AF algebras to larger class)

Directed graph

- $E = (E^0, E^1, E^1_+, r, s, \iota)$, with E^0 = vertices, E^1 = oriented edges $w = \{e, \epsilon\}$, E^1_+ = choice of orientation for each edge, $r, s : E^1 \to E^0$ range and source maps, $\iota(w) = \{e, -\epsilon\}$ orientation reversal.
- E finite, row finite (fin many exiting edges), locally finite
- Δ = universal covering tree, $E = \Delta/\Gamma$

- Admissible chain of edges $w_1 w_2$: if $w_2 \neq \iota(w_1)$ and $r(w_1) = s(w_2)$. Walks= chains of edges; Paths= chains of pos.oriented edges
- Boundary ∂E: *shift-tail equivalence* $\omega \sim \tilde{\omega}$ if $\exists N \geq 1, k \in \mathbb{Z}$ $\omega_i = \tilde{\omega}_{i-k}$, $\forall i \geq N$. Paths ending at sinks: $\omega \sim \tilde{\omega} \in \sigma^*$, iff $r(\omega) = r(\tilde{\omega})$

$$\partial\Delta = (\mathcal{P}^+ \cup \sigma^*)/\sim$$

Cuntz-Krieger family

$\{P_v\}_{v\in E^0}$ orthogonal projections and $\{S_w\}_{w\in E^1_+}$ partial isometries:

$$S_w^* S_w = P_{r(w)}$$

$$P_v = \sum_{w:s(w)=v} S_w S_w^*, \quad \forall v \in s(E^1_+)$$

$\Rightarrow$ universal C^*-algebra generated by $\{P_v, S_w\}$ with relations as above: $C^*(E)$

$U(1)$-gauge action: $\lambda : \{P_v, S_w\} \mapsto \{P_v, \lambda S_w\}$

Cuntz-Krieger algebras

Matrix A entries $\{0,1\} \Rightarrow$ algebra $\mathcal{O}_A$ generated by partial isometries S_i with $S_i S_i^*$ orthogonal proj and with relation:

$$S_i^* S_i = \sum_j A_{ij} S_j S_j^*$$

Introduced to study dynamics of subshifts of finite type

Edge matrix (directed graph): $A_+(w_i, w_j) = 1$ if $w_i w_j$ admissible path, $A_+(w_i, w_j) = 0$ otherwise; directed edge matrix: $A(w_i, w_j) = 1$ if $w_i w_j$ admissible walk, $A(w_i, w_j) = 0$ otherwise

Some cases of graph C^*-algebras

- $E = \Delta$, then $C^*(\Delta)$ AF-algebra Morita equivalent to $C_0(\partial\Delta)$
- $E = \Delta/\Gamma$, then $C^*(E)$ Morita equivalent to $C^*(\Delta) \rtimes \Gamma$
- If locally finite graph with no sinks, $C^*(E) \cong \mathcal{O}_{A_+}$

$\Rightarrow$ used to obtain NCG version of certain p-adic spaces (Mumford curves)

Groupoids

- $\mathcal{G}^0$ = units, $r, s : \mathcal{G} \to \mathcal{G}^0$ source/range maps
- Category: $\mathcal{G}^{(0)}$ = objects; $\mathcal{G}$ = morphisms, invertible:

$$s(\gamma^{-1}) = r(\gamma), \quad r(\gamma^{-1}) = s(\gamma)$$

with composition:

$$r(\gamma_1) = s(\gamma_2) \Rightarrow \gamma_1\gamma_2 \in \mathcal{G}$$

$$s(\gamma_1\gamma_2) = s(\gamma_1), \quad r(\gamma_1\gamma_2) = r(\gamma_2)$$

- Algebra: $\mathcal{A}_{\mathcal{G}} = \{f : \mathcal{G} \to \mathbb{C}\}$ finite support

$$(f_1 \star f_2)(\gamma) = \sum_{\gamma=\gamma_1\gamma_2} f_1(\gamma_1) f_2(\gamma_2)$$

$$f^{\vee}(\gamma) = \overline{f(\gamma^{-1})}$$

associative, noncommutative, involutive

- Representations of groupoid algebras

$$\mathcal{H}_x = \ell^2(\{\gamma \in \mathcal{G} \mid r(\gamma) = x\})$$

representation $\pi_x : \mathcal{A}_{\mathcal{G}} \to \mathcal{B}(\mathcal{H}_x)$

$$(\pi_x(f)\xi)(\gamma) = \sum_{\gamma=\gamma_1\gamma_2} f(\gamma_1)\xi(\gamma_2)$$

norm (when $\mathcal{G}^{(0)}$ compact)

$$\|f\| := \sup_{x\in\mathcal{G}^{(0)}} \|\pi_x(f)\|_{\mathcal{B}(\mathcal{H}_x)}$$

$\Rightarrow$ $C^*(\mathcal{G})$ completion of $\mathcal{A}_{\mathcal{G}}$

Algebras from categories

$\mathcal{C}$ = small category (semigroupoid)

$$\mathcal{A}_{\mathcal{C}} = \{f : Mor(\mathcal{C}) \to \mathbb{C} \,|\, \text{finite support}\}$$

$$(f_1 \star f_2)(\phi) = \sum_{\phi=\phi_1\circ\phi_2} f_1(\phi_1) f_2(\phi_2)$$

associative noncommutative (not involutive)

$$\mathcal{H}_Y = \ell^2(\{f \in Mor_{\mathcal{C}}(X, Y) \,|\, X \in Obj(\mathcal{C})\})$$

$$(\pi_Y(f)\xi)(\phi) = \sum_{\phi=\phi_1\phi_2} f(\phi_1)\xi(\phi_2)$$

$\pi_Y(f)^\vee$ = adjoint in $\mathcal{B}(\mathcal{H}_Y)$

like creation/annihilation operators

Group $G \Rightarrow$ group ring $\mathbb{C}[G] \Rightarrow$ representation on $\ell^2(G)$ by unitaries $U_\gamma^* = U_\gamma^{-1} = U_{\gamma^{-1}}$

Semigroup S with unit $\Rightarrow$ semigroup ring $\mathbb{C}[S] \Rightarrow$ representation by isometries $\mu_s^* \mu_s = 1$ with $\mu_s \mu_s^* = e_s$ idempotent (creation/annihilation operators)

Groupoid $\mathcal{G} \Rightarrow$ groupoid ring $\mathbb{C}[\mathcal{G}] \Rightarrow$ representations on $\ell^2(\mathcal{G}_x)$ by unitaries $U_\gamma^* = U_\gamma^{-1} = U_{\gamma^{-1}}$

Semigroupoid $\mathcal{S} \Rightarrow \mathbb{C}[\mathcal{S}]$ acting by isometries on $\ell^2(\mathcal{S}_x)$

Possible variant: algebras from 2-categories: two convolutions for horizontal and vertical compositions of 2-morphisms

Γ = directed graph (quiver) $\Rightarrow$
$\mathcal{C}(\Gamma)$ = category

- $Obj(\mathcal{C}(\Gamma)) = V(\Gamma)$
- $Mor_{\mathcal{C}(\Gamma)}(v, v')$ = paths in Γ from v to v'

Quiver representation of Γ in a category $\mathcal{C}$: Functor $F : \mathcal{C}(\Gamma) \to \mathcal{C}$

Graph algebras Γ = finite directed graph:
S_e, p_v Cuntz–Krieger

$$S_e^* S_e = p_{r(e)}, \qquad \sum_{s(e)=v} S_e S_e^* = p_v$$

Same as C^*-algebra completion of $\mathcal{A}_{\mathcal{C}(\Gamma)}$

Functoriality $F : \mathcal{C}_1 \to \mathcal{C}_2$ functor

Category $\mathcal{G}(F)$ (graph of F):

- $Obj(\mathcal{G}(F)) = \{\mathcal{X} = (X, F(X)) \,|\, X \in Obj(\mathcal{C}_1)\}$
- $Mor_{\mathcal{G}(F)}(\mathcal{X}, \mathcal{Y}) = \{\Phi = (\phi, F(\phi)) \,|\, \phi \in Mor_{\mathcal{C}_1}(X, Y)\}$

Bimodule $\mathcal{V}_{\mathcal{G}(F)} = \{\xi : Mor_{\mathcal{G}(F)} \to \mathbb{C} \,|\,$ finite support $\}$

Action of $\mathcal{A}_{\mathcal{C}_1}$:

$$(\pi_X(f)\xi)(\Phi) = \sum_{\phi=\phi_1\phi_2} f(\phi_1)\xi(\Phi_2)$$

Action of $\mathcal{A}_{\mathcal{C}_2}$:

$$(\pi_{F(X)}(h)\xi)(\Phi) = \sum_{F(\phi)=\psi F(\phi_2)} h(\psi)\xi(\Phi_2)$$

$$[\pi_X(f), \pi_{F(X)}(h)] = 0$$

Functor $F : \mathcal{C}_1 \to \mathcal{C}_2 \Rightarrow$ morphism of NC spaces $\mathcal{A}_{\mathcal{C}_1}$-$\mathcal{A}_{\mathcal{C}_2}$ bimodule