Spectral Action for Robertson–Walker metrics
Part II: Multifractal Cosmologies

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Reference

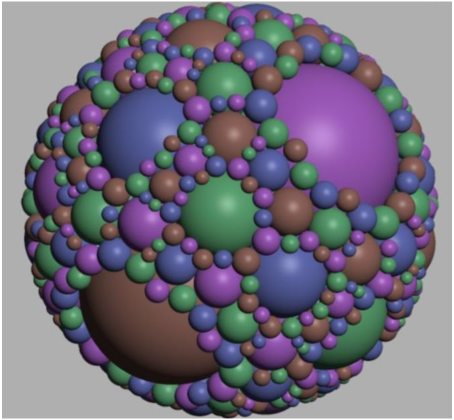
- Farzad Fathizadeh, Yeorgia Kafkoulis, Matilde Marcolli, *Bell polynomials and Brownian bridge in Spectral Gravity models on multifractal Robertson-Walker cosmologies*, arXiv:1811.02972

Other references

- A. Ball, M. Marcolli, *Spectral Action Models of Gravity on Packed Swiss Cheese Cosmology*, Classical and Quantum Gravity, 33 (2016), no. 11, 115018, 39 pp.

Packed Swiss Cheese Cosmology

- $\mathcal{P}$ Apollonian packing of 3-spheres radii $\{a_{n,k} : n \in \mathbb{N}, k = 1, \dots, 6 \cdot 5^{n-1}\}$
- iterative construction of packing: at n-th step $6 \cdot 5^{n-1}$ spheres $S^3_{a_{n,k}}$ are added
- spacetime that are isotropic but not homogeneous



- two possible choices of associated Robertson–Walker metrics
  1. round scaling (of full 4-dim spacetime)

$$ds^2_{n,k} = a^2_{n,k}\,(dt^2 + a(t)^2\,d\sigma^2), \qquad n \in \mathbb{N}, k = 1, \ldots, 6 \cdot 5^{n-1}$$

  2. non-round scaling (of spatial sections only)

$$ds^2_{n,k} = dt^2 + a(t)^2\,a^2_{n,k}\,d\sigma^2, \qquad n \in \mathbb{N}, k = 1, \ldots, 6 \cdot 5^{n-1}$$

- $D_{n,k}$ resulting Dirac operators on $\mathbb{R} \times S^3_{a_{n,k}}$
- entire (multifractal) spacetime $\mathbb{R} \times \mathcal{P}$
- spectral triple for $\mathbb{R} \times \mathcal{P}$: $\mathcal{A}$ subalgebra of $C_0(\mathbb{R} \times \mathcal{P})$, Hilbert space $\mathcal{H} = \oplus_{n,k}\mathcal{H}_{n,k}$ with $\mathcal{H}_{n,k} = L^2(S_{a_{n,k}}, \mathbb{S})$ and Dirac

$$D = D_{\mathbb{R}\times\mathcal{P}} := \bigoplus_{n\in\mathbb{N}} \bigoplus_{k=1}^{6\cdot 5^{n-1}} D_{n,k}$$

Mellin Transform and Zeta Functions

- meromorphic function $\phi(z)$ with poles at $\mathcal{S} \subset \mathbb{C}$, Laurent series expansion at a pole $z_0 \in \mathcal{S}$

$$\phi(z) = \sum_{-N \leq k} c_k (z - z_0)^k$$

- singular element at $z_0 \in \mathcal{S}$

$$S(\phi, z_0) := \sum_{-N \leq k \leq 0} c_k (z - z_0)^k$$

- singular expansion of ϕ

$$S_\phi(z) := \sum_{z \in \mathcal{S}} S(\phi, z)$$

- Example: for the Gamma function

$$\Gamma(z) \asymp \sum_{k \geq 0} \frac{(-1)^k}{k!} \frac{1}{z+k}$$

- Mellin transform

$$\phi(z) = \mathcal{M}(f)(z) = \int_0^\infty f(\tau)\tau^{z-1} d\tau$$

- relation between asymptotic expansion at $u \to 0$ of a function $f(u)$ and singular expansion of its Mellin transform $\phi(z) = \mathcal{M}(f)(z)$
- small time asymptotic expansion

$$f(u) \sim_{u \to 0^+} \sum_{\alpha \in \mathcal{S}, k_\alpha} c_{\alpha,k_\alpha} u^\alpha \log(u)^{k_\alpha}$$

- coefficients c_{α,k_α} determined by singular expansion of Mellin transform

$$\mathcal{M}(f)(z) \asymp S_{\mathcal{M}(f)}(z) = \sum_{\alpha \in \mathcal{S}, k_\alpha} c_{\alpha,k_\alpha} \frac{(-1)^{k_\alpha} k_\alpha!}{(s+\alpha)^{k_\alpha+1}}$$

- index k_α ranges over terms in singular element of $\phi(z) = \mathcal{M}(f)(z)$ at $z = \alpha$, up to order of pole at α

Example: Packing of 4-Spheres

- round S^4 is a Robertson–Walker metric $dt^2 + a(t)^2 d\sigma^2$ with $a(t) = \sin t$ $(0 \le t \le \pi)$ and $d\sigma^2$ round metric on S^3
- spectrum of Dirac operator on S_r^{D-1} radius $r > 0$

$$\mathrm{Spec}(D_{S_r^{D-1}}) = \left\{ \lambda_{\ell,\pm} = \pm r^{-1} \left(\frac{D-1}{2} + \ell \right) \mid l \in \mathbb{Z}_+ \right\}$$

multiplicities

$$\mathrm{m}_{\ell,\pm} = 2^{[\frac{D-1}{2}]} \binom{\ell + D}{\ell}.$$

- zeta function of Dirac operator

$$\zeta_D(s) = \mathrm{Tr}(|D_{S_r^4}|^{-s}) = \sum_{\ell,\pm} \mathrm{m}_{\ell,\pm} |\lambda_{\ell,\pm}|^{-s} = \frac{4}{3}\, r^s \,(\zeta(s-3) - \zeta(s-1))$$

$\zeta(s)$ Riemann zeta function

- fractal string zeta function $\zeta_{\mathcal{L}}(s) = \sum_{n,k} a_{n,k}^s$ of Apollonian packing $\mathcal{P}$ of $S^3_{a_{n,k}}$ with radii sequence $\mathcal{L} = \{a_{n,k}\}$
- resulting Dirac operator $\mathcal{D}_{\mathcal{P}}$ on associated packing of 4-spheres (each 3-sphere equator in a fixed hyperplane of a corresponding 4-sphere)
- zeta function of Dirac $\mathcal{D}_{\mathcal{P}}$ factors as product of zetas

$$\zeta_{\mathcal{D}_{\mathcal{P}}}(s) = \mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n,k} \frac{4}{3} a_{n,k}^s (\zeta(s-3) - \zeta(s-1))$$

$$= \zeta_{\mathcal{L}}(s) \zeta_{D_{S^4}}(s)$$

- Mellin transform relation between the zeta function of the Dirac operator and the heat-kernel of the Dirac Laplacian

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \frac{1}{\Gamma(s/2)} \int_0^\infty \mathrm{Tr}(e^{-t\mathcal{D}_{\mathcal{P}}^2})\, t^{s/2-1} dt$$

- use to compute spectral action leading terms from zeta function: $\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim$

$$f(0)\zeta_{\mathcal{D}_{\mathcal{P}}}(0) + f_2\Lambda^2 \frac{\zeta_{\mathcal{L}}(2)}{2} + f_4\Lambda^4 \frac{\zeta_{\mathcal{L}}(4)}{2} + \sum_{\sigma\in\mathcal{S}(\mathcal{L})} f_\sigma \Lambda^\sigma \frac{\zeta_{D_{S^4}}(\sigma)}{2} \mathcal{R}_\sigma$$

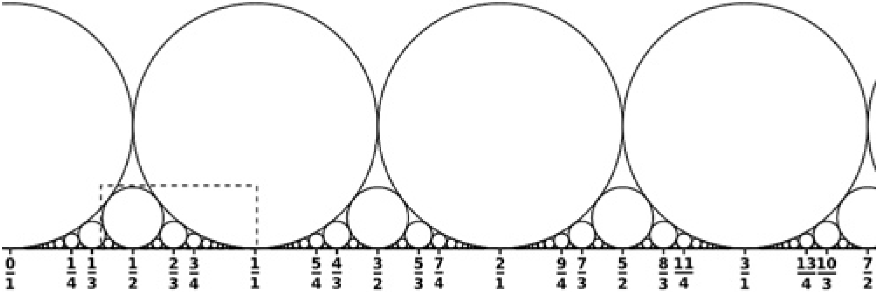
- $\mathcal{S}(\mathcal{L})$ set of poles of fractal string zeta $\zeta_{\mathcal{L}}(s)$ residues

$$\mathcal{R}_\sigma = \mathrm{Res}_{s=\sigma}\zeta_{\mathcal{L}}(s)$$

- $\zeta_{\mathcal{L}}(2)$ and $\zeta_{\mathcal{L}}(4)$ replace radii r^2 and r^4 for a single sphere S^4_r: zeta regularization of $\sum_{n,k} a^2_{n,k}$ and $\sum_{n,k} a^4_{n,k}$

Example: Lower Dimensional Apollonian Ford Circles

- Ford circles: tangent to the real line at points $(k/n, 0)$ with centers at points $(k/n, 1/(2n^2))$



- number of circles of radius $r_n = (2n^2)^{-1}$ is number of integers $1 \leq k \leq n$ coprime to n: multiplicity $m(r_n)$ given by Euler totient function

$$m(r_n) = \varphi(n),$$

$$\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

product over the distinct prime numbers dividing n

- Dirichlet series generating function of the Euler totient function

$$\mathcal{D}_\varphi(s) = \sum_{n \geq 1} \frac{\varphi(n)}{n^s}$$

- fractal string zeta function

$$\zeta_{\mathcal{L}}(s) = \sum_{n\geq 1} \varphi(n)\,(2n^2)^{-s} = 2^{-s}\sum_{n\geq 1}\varphi(n)\,n^{-2s} = 2^{-s}\mathcal{D}_\varphi(2s)$$

- using $\varphi(p^k) = p^k - p^{k-1}$

$$1 + \sum_k \varphi(p^k)p^{-sk} = \frac{1-p^{-s}}{1-p^{1-s}}$$

- using Euler product formula

$$\mathcal{D}_\varphi(s) = \frac{\zeta(s-1)}{\zeta(s)}$$

- so fractal string zeta function of Ford circles

$$\zeta_{\mathcal{L}}(s) = 2^{-s}\,\frac{\zeta(2s-1)}{\zeta(2s)}$$

- build a packing of 4-spheres over the Ford circles: collection of 2-spheres with Ford circles as equators in a given hyperplane, then same as equators of a collection of 3-spheres and then of 4-spheres
- for the resulting packing of 4-spheres spectral action

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim \frac{11}{140}\, f(0) + \frac{f_1}{\pi^2}\Lambda + \frac{45\,\zeta(3)}{4\pi^4}\, f_2\Lambda^2 + \frac{4725\,\zeta(7)}{16\pi^8}\, f_4\Lambda^4$$

$$+ \sum_{k\in\mathbb{N}} \frac{2^{k+1} f_{-k}}{3}\, \frac{\zeta(-k-3)-\zeta(-k-1)}{\zeta(-2k-1)}\Lambda^{-k}$$

$$+ \sum_{\sigma=a+ib} \frac{2^{-a}\cos(b\log 2)}{3}\Re(Z_\sigma)\, r(f)_\sigma\, \cos(b\log\Lambda)\Lambda^a$$

σ over nontrivial zeros of $\zeta(2s)$ with
$r(f)_\sigma = \int_0^\infty f(u)u^{a-1}\cos(bu)\,du$ and
$Z_\sigma = (\zeta(\sigma-3)-\zeta(\sigma-1))\zeta(2\sigma-1)$

General Case $\mathbb{R} \times \mathcal{P}$ with round scaling $a_{n,k}^2(dt^2 + a(t)^2 d\sigma^2)$

- $R = \{r_n\}$ sequence of $r_n \in \mathbb{R}_+^*$ so that $\zeta_R(z) = \sum_n r_n^{-z}$ converges for $\Re(z) > C$ for some $C > 0$
- function $f(\tau)$ with small time asymptotics

$$f(\tau) \sim \sum_N c_N \tau^N$$

- associated series

$$g_R(\tau) = \sum_n f(r_n \tau)$$

- then small time asymptotic expansion of $g_R(\tau)$

$$g_R(\tau) \sim_{\tau \to 0^+} \sum_N c_N \, \zeta_R(-N) \, \tau^N + \sum_{\sigma \in \mathcal{S}(\zeta_R)} \mathcal{R}_{R,\sigma} \, \mathcal{M}(f)(\sigma) \, \tau^{-\sigma}$$

with $\mathcal{S}(\zeta_R)$ poles of $\zeta_R(z)$

$$\mathcal{R}_{R,\sigma} := \mathrm{Res}_{z=\sigma} \zeta_R(z)$$

Sketch of proof:

- write associated series as

$$g_R(\tau) \sim \sum_{N,n} c_N r_n^N \tau^N = \sum_N \zeta_R(-N)\tau^N$$

- Mellin transform $\mathcal{M}(g)(z) = \int_0^\infty g(\tau)\tau^{z-1} d\tau$ gives

$$\mathcal{M}(g)(z) = (\sum_n r_n^{-z}) \int_0^\infty \sum_N c_N \, u^{N+z-1} du = \zeta_R(z) \cdot \mathcal{M}(f)(z)$$

- asymptotic expansion of $g_R(\tau)$ from Mellin transform $\mathcal{M}(g_R)(z)$ singular expansion

$$S_{\mathcal{M}(g_R)}(z) = \sum_{\sigma \in \mathcal{S}(\zeta_R)} \frac{\mathcal{R}_{R,\sigma} \, \mathcal{M}(f)(\sigma)}{z-\sigma} + \sum_{\sigma \in \mathcal{S}(\mathcal{M}(f))} \frac{\zeta_R(\sigma) c_\sigma}{z-\sigma}$$

- and from small time asymptotics of $f(\tau)$ know

$$S_{\mathcal{M}(f)}(z) = \sum_N \frac{c_N}{z+N}$$

Feynman–Kac formula on $\mathbb{R} \times \mathcal{P}$

- on each $\mathbb{R} \times S^3_{a_{n,k}}$ decompose Dirac $D_{a_{n,k}}$ using operators

$$H_{m,n,k} = -a_{n,k}^{-2} \frac{d^2}{dt^2} + V_{m,n,k}(t)$$

$$V_{m,n,k} = \frac{(m+\frac{3}{2})}{a_{n,k}^2 \cdot a(t)^2}((m+\frac{3}{2}) - a_{n,k} \cdot a'(t)$$

as in Chamseddine–Connes

- Feynman–Kac formula

$$e^{-\tau^2 H_{m,n,k}}(t,t) = e^{-\frac{\tau^2}{a_{n,k}^2}(\frac{d^2}{dt^2} + a_{n,k}^2 V_{m,n,k})}(t,t)$$

$$= \frac{a_{n,k}}{2\sqrt{\pi}\tau} \int \exp(-\tau^2 \int_0^1 V_{m,n,k}(t + \sqrt{2}\frac{\tau}{a_{n,k}}\alpha(u))du) D[\alpha]$$

- Poisson summation to replace sum

$$\sum_m \mu(m) e^{-\tau^2 H_{m,n,k}}(t,t)$$

with multiplicities $\mu(m)$ with the integral

$$\int_{-\infty}^{\infty} f_{\tau,\, n,k}(x)\, dx$$

$$f_{\tau,\, n,k}(x) = \left(x^2 - \frac{1}{4}\right) e^{-x^2 a_{n,k}^{-2} U - x a_{n,k}^{-1} V}$$

with U and V as in single sphere case

$$\sum_m \mu(m) e^{-\tau^2 H_{m,n,k}}(t,t) =$$

$$\int \frac{a_{n,k}}{\tau} \left(\frac{e^{\frac{V^2}{4U}}}{4} (-a_{n,k} U^{-1/2} + 2a_{n,k}^3 U^{-3/2} + a_{n,k}^3 V^2 U^{-5/2}) \right) D[\alpha]$$

$$= \int \frac{1}{\tau} \left(\frac{e^{\frac{V^2}{4U}}}{4} (-a_{n,k}^2 U^{-1/2} + a_{n,k}^4 (2U^{-3/2} + V^2 U^{-5/2})) \right) D[\alpha]$$

- same Taylor expansion method

$$e^{\frac{V^2}{4U}} U^r V^\ell = \tau^{2(r+\ell)} \sum_{M=0}^\infty a_{n,k}^{-M-2(r+\ell)} \, C_M^{(r,\ell)} \tau^M$$

with $C_M^{(r,\ell)}$ as in single sphere case $dt^2 + a(t)^2 d\sigma^2$

- resulting expansion

$$\frac{1}{\tau}\left(\frac{e^{\frac{V^2}{4U}}}{4}(-a_{n,k}^2 U^{-1/2} + a_{n,k}^4(2U^{-3/2} + V^2 U^{-5/2}))\right) =$$

$$\frac{1}{4}\sum_{M=0}^\infty \left(C_M^{(-5/2,2)} - C_M^{(-1/2,0)}\right)\zeta_{\mathcal{L}}(-M+2)\tau^{M-2}$$

$$+\frac{1}{2}\sum_{M=0}^\infty C_M^{(-3/2,0)}\zeta_{\mathcal{L}}(-M+4)\tau^{M-4}.$$

- Feynman–Kac formula for the whole $\mathbb{R} \times \mathcal{P}$

$$\sum_{n,k} \sum_{m} \mu(m) e^{-\tau^2 H_{m,n,k}}(t,t) =$$

$$\sum_{M=0}^{\infty} \tau^{2M-4} \zeta_{\mathcal{L}}(-2M+4) \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} (C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}) \right) D[\alpha]$$

  with only the term $\frac{1}{2} C_0^{(-3/2,0)}$ when $M = 0$

- obtained as a series

$$g_{\mathcal{L}}(\tau) = \sum_{n,k} f(a_{n,k}^{-1} \tau)$$

$$f(\tau) \sim \sum_{M} \tau^{2M-4} \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} (C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}) \right) D[\alpha]$$

Result: Spectral Action on Multifractal Robertson–Walker $\mathbb{R} \times \mathcal{P}$

$$\mathrm{Tr}(f(\mathcal{D}/\Lambda)) \sim$$

$$\sum_{M=0}^{\infty} \Lambda^{n_M} f_{n_M} \zeta_{\mathcal{L}}(n_M) \int \left(\frac{1}{2} C_{4-n_M}^{(-3/2,0)} + \frac{1}{4} (C_{2-n_M}^{(-5/2,2)} - C_{2-n_M}^{(-1/2,0)}) \right) D[\alpha]$$
$$+ \sum_{\sigma \in \mathcal{S}_{\mathcal{L}}} \tilde{f}(\sigma) \cdot f_\sigma \cdot \mathrm{Res}_{z=\sigma} \zeta_{\mathcal{L}} \cdot \Lambda^\sigma$$

$n_M = 4 - 2M$, set of poles $\mathcal{S}_{\mathcal{L}}$ of $\zeta_{\mathcal{L}}$ Mellin transform $\tilde{f}(z) = \mathcal{M}(f)(z)$ of $f(\tau) = \mathrm{Tr}(\exp(-\tau^2 D^2))$ with Dirac on $\mathbb{R} \times S^3$ with $dt^2 + a(t)^2 d\sigma^2$

Conclusion: presence of fractality detected by two types of effects

1. zeta regularization of coefficients $\zeta_{\mathcal{L}}(4 - 2M)$ in terms Λ^{4-2M} (including effective gravitational and cosmological constant in top terms)
2. additional terms from non-real poles of order $\Lambda^{\Re\sigma}$ (and log periodic) with $3 < \Re\sigma = \dim_H \mathcal{P} < 4$ between cosmological and Einstein–Hilbert term

Multifractal Robertson–Walker with non-round scaling

$dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

- rescaling $ds_a^2 = dt^2 + a^2 \cdot a(t)^2 \, d\sigma^2$ with $a > 0$ gives $U \mapsto a^{-2} \, U$ and $V \mapsto a^{-1} \, V$
- this gives rescaling

$$\frac{1}{4} \sum_{M=0}^{\infty} (a^3 \, C_M^{(-5/2,2)} - a \, C_M^{(-1/2,0)}) \, \tau^{M-2} + \frac{1}{2} \sum_{M=0}^{\infty} a^3 \, C_M^{(-3/2,0)} \, \tau^{M-4}$$

- expect presence of zeta regularized coefficients $\zeta_{\mathcal{L}}(3)$, $\zeta_{\mathcal{L}}(1)$
- to see this use a Mellin transform with respect to the "multiplicity variable" x in $f_s(x)$

Kummer confluent hypergeometric function

- notation: $a^{(n)} := a(a+1)\cdots(a+n-1)$ and $a^{(0)} := 1$
- Kummer confluent hypergeometric function defined by series

$$ {}_1F_1(a,b,t) = \sum_{n=0}^{\infty} \frac{a^{(n)} t^n}{b^{(n)} n!} $$

- solution of the Kummer equation

$$ t\frac{d^2 f}{dt^2} + (b-t)\frac{df}{dt} - af = 0. $$

Mellin transform and hypergeometric function

- Mellin transform in the x-variable of the function

$$f_{s,-}(x) := f_s(x) = (x^2 - \frac{1}{4})e^{-x^2U - xV}$$

given by

$$\mathcal{M}((x^2 - \frac{1}{4})e^{-x^2U-xV})(z) = \frac{1}{8}U^{-(z+3)/2}\times$$

$$\Big(U^{1/2}\,\Gamma(\frac{z}{2})(-U\,{}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}) + 2z\,{}_1F_1(\frac{z+2}{2},\frac{1}{2},\frac{V^2}{4U}))$$

$$+V\,\Gamma(\frac{z+1}{2})(U\,{}_1F_1(\frac{z+1}{2},\frac{3}{2},\frac{V^2}{4U}))-2(z+1)\,{}_1F_1(\frac{z+3}{2},\frac{3}{2},\frac{V^2}{4U}))\Big)$$

- similar expression for transform of $f_{s,+}(x) := (x^2 - \frac{1}{4})e^{-x^2U+xV}$

- multiplicity integral

$$\int_{-\infty}^{\infty} f_s(x)\, dx = \int_0^{\infty} f_{s,-}(x)\, dx + \int_0^{\infty} f_{s,+}(x)\, dx$$

$$f_{s,\pm}(x) = (x^2 - \frac{1}{4}) e^{-x^2 U \pm x V}$$

- multiplicity integral as special value at $z = 1$ of Mellin

$$\int_{-\infty}^{\infty} f_s(x)\, dx = \mathcal{M}(f_{s,-})(z)|_{z=1} + \mathcal{M}(f_{s,+})(z)|_{z=1}$$

- Mellin transform $\mathcal{M}(f_{s,-})(z) + \mathcal{M}(f_{s,+})(z)$

$$= -\frac{1}{4} U^{-1-\frac{z}{2}} \Gamma(\frac{z}{2})(U\, {}_1F_1(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}) - 2z\, {}_1F_1(1 + \frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}))$$

- value at $z = 1$

$$\left(-\frac{1}{4} U^{-(1+\frac{z}{2})} \Gamma(\frac{z}{2})(U\, {}_1F_1(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}) - 2z\, {}_1F_1(1 + \frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}))\right)|_{z=1} =$$

$$e^{\frac{V^2}{4U}} \frac{\sqrt{\pi}}{4}(-U^{-1/2} + 2U^{-3/2} + V^2 U^{-5/2})$$

Effect of scaling

- notation:

$$H_\lambda(\tau,z) := \quad U^{-z/2}\,\Gamma(z/2)\,{}_1F_1(\frac{z}{2},\lambda,\frac{V^2}{4U})$$

$$H(\tau,z) := \quad H_{1/2}(\tau,z) = U^{-z/2}\,\Gamma(z/2)\,{}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U})$$

$$H_{\mathcal{L}}(\tau,z) := \quad U^{-z/2}\,\zeta_{\mathcal{L}}(z)\,\Gamma(z/2)\,{}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}) = \zeta_{\mathcal{L}}(z)\,H(\tau,z)$$

- multiplicity integral with scaling $dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

$$\mathcal{M}(f_{s,n,k,-})(z) + \mathcal{M}(f_{s,n,k,+})(z) =$$

$$-\frac{1}{4}\,a_{n,k}^z\,U^{-\frac{z}{2}}\,\Gamma(\frac{z}{2}){}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U})$$

$$+a_{n,k}^{z+2}U^{1-\frac{z}{2}}\,\Gamma(1+\frac{z}{2})\,{}_1F_1(1+\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}))$$

$$-\frac{1}{4}a_{n,k}^z\,H(\tau,z) + a_{n,k}^{z+2}\,H(\tau,z+2).$$

- multiplicity integral over the full sphere packing $\mathbb{R} \times \mathcal{P}$

$$f_{\mathcal{P},s}(x) = \left(x^2 - \frac{1}{4}\right) \sum_{n,k} e^{-x^2 a_{n,k}^{-2} U - x a_{n,k}^{-1} V}$$

- as value of Mellin transform

$$\int_{-\infty}^{\infty} f_{\mathcal{P},s}(x)\, dx = \mathcal{M}(f_{\mathcal{P},s,-})(z)|_{z=1} + \mathcal{M}(f_{\mathcal{P},s,+})(z)|_{z=1}$$

$f_{\mathcal{P},s,\pm} = (x^2 - 1/4) \sum_{n,k} \exp(-x^2 a_{n,k}^{-2} U \pm x a_{n,k}^{-1} V)$

- Mellin transforms

$$\mathcal{M}(f_{\mathcal{P},s,-})(z) + \mathcal{M}(f_{\mathcal{P},s,+})(z) = -\frac{1}{4} H_{\mathcal{L}}(\tau, z) + H_{\mathcal{L}}(\tau, z+2)$$

- this shows one gets zeta regularized $\zeta_{\mathcal{L}}(3)$ and $\zeta_{\mathcal{L}}(1)$

Sketch of how to see the log periodic terms for non-round scaling $dt^2 + a^2_{n,k} a(t)^2 d\sigma^2$

- τ expansion

$$U = \tau^2 \sum_{n=0}^{\infty} \frac{u_n}{n!} \tau^n, \quad V = \tau^2 \sum_{n=0}^{\infty} \frac{v_n}{n!} \tau^n$$

- gives expansion of confluent hypergeometric function

$$H_{\mathcal{L}}(\tau, z) = \zeta_{\mathcal{L}}(z)\Gamma(z/2) U^{-z/2} \sum_{n=0}^{\infty} \frac{(z/2)_n}{4^n \, n! \, (1/2)_n} V^{2n} U^{-n}$$

  with $(a)_n = a(a+1)\cdots(a+n-1)$ the rising factorial
- the term $U^{-z/2}$ contributes a term with τ^z times a power series in τ, while confluent hypergeometric function contributes a power series in τ

- to see why τ^z gives rise to log periodic terms in the spectral action consider simplified case
- product of the Mellin transforms $\mathcal{M}(f_1)(z)\cdot\mathcal{M}(f_2)(z)$ is Mellin transform of convolution

$$\mathcal{M}(f_1)(z)\cdot\mathcal{M}(f_2)(z)=\mathcal{M}(f_1\star f_2)(z),$$
$$(f_1\star f_2)(x)=\int_0^\infty f_1(\frac{x}{u})f_2(u)\,\frac{du}{u}$$

- Mellin transform of a delta distribution

$$\tau^{z-1}=\mathcal{M}(\delta(x-\tau))$$

- Mellin transform of distribution $\Lambda_{\mathcal{P},\tau}:=\sum_{n,k}\tau\, a_{n,k}\ \delta(x-\tau\cdot a_{n,k})$

$$\langle\sum_{n,k}\tau\, a_{n,k}\ \delta(x-\tau\cdot a_{n,k}),\phi(x)\rangle=\sum_{n,k}\tau\, a_{n,k}\ \phi(\tau\, a_{n,k})$$

given by

$$\tau^z\,\zeta_{\mathcal{L}}(z)=\mathcal{M}(\sum_{n,k}\tau\, a_{n,k}\ \delta(x-\tau\cdot a_{n,k}))$$

- given function $g(x)$
  will want $g_\gamma(x) := \mathcal{M}^{-1}(\Gamma(z/2)\,{}_1F_1(z/2, 1/2, \gamma))$

$$\mathcal{M}(\Lambda_{\mathcal{P},\tau})(z) \cdot \mathcal{M}(g)(z) = \mathcal{M}(\Lambda_{\mathcal{P},\tau} \star g)(z)$$

$$= \mathcal{M}(\sum_{n,k} \tau a_{n,k} \int_0^\infty \delta(u - \tau a_{n,k})\, g(\frac{x}{u})\, \frac{du}{u}) = \sum_{n,k} \mathcal{M}(g(\frac{x}{\tau \cdot a_{n,k}}))$$

- take $h_z(\tau) := \mathcal{M}(g(\frac{x}{\tau}))$

$$L_z(\tau) := \sum_{n,k} h_z(\tau \cdot a_{n,k})$$

- asymptotic expansion for this function through singular expansion of Mellin transform in τ

$$\mathcal{M}_\tau(L_z(\tau))(\beta) = \zeta_{\mathcal{L}}(\beta) \cdot \mathcal{M}(h_z(\tau))(\beta)$$

- contributions from poles of $\zeta_{\mathcal{L}}(\beta)$ and of $\mathcal{M}(h_z(\tau))(\beta)$: log-periodic and zeta regularized terms as expected