

Spectral Action Models of Gravity and Packed Swiss Cheese Cosmologies

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Based on:

- Adam Ball, Matilde Marcolli, *Spectral Action Models of Gravity on Packed Swiss Cheese Cosmology*, *Classical Quantum Gravity* 33 (2016), no. 11, 115018, 39 pp.



Homogeneity versus Isotropy in Cosmology

- Homogeneous and isotropic: Friedmann universe $\mathbb{R} \times S^3$

$$\pm dt^2 + a(t)^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

with round metric on S^3 with $SU(2)$ -invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations (i, j, k) of $(1, 2, 3)$

- Homogeneous **but not** isotropic:

Bianchi IX mixmaster models $\mathbb{R} \times S^3$

$$F(t) \left(\pm dt^2 + \frac{\sigma_1^2}{W_1^2(t)} + \frac{\sigma_2^2}{W_2^2(t)} + \frac{\sigma_3^2}{W_3^2(t)} \right)$$

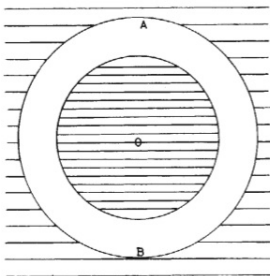
with a conformal factor $F(t) \sim W_1(t)W_2(t)W_3(t)$

- Isotropic **but not** homogeneous?

\Rightarrow Swiss Cheese Models

Main Idea:

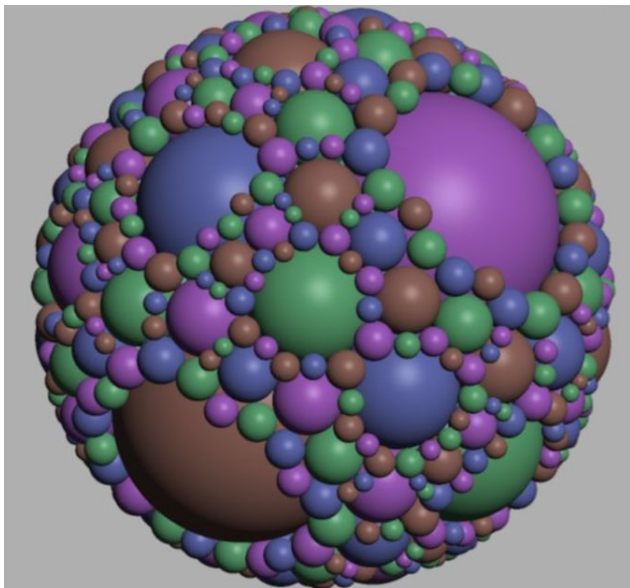
- M.J. Rees, D.W. Sciama, *Large-scale density inhomogeneities in the universe*, *Nature*, Vol.217 (1968) 511–516.



Cut off 4-balls from a FRW spacetime and replace with different density smaller region outside/inside patched across boundary with vanishing Weyl curvature tensor (isotropy preserved)

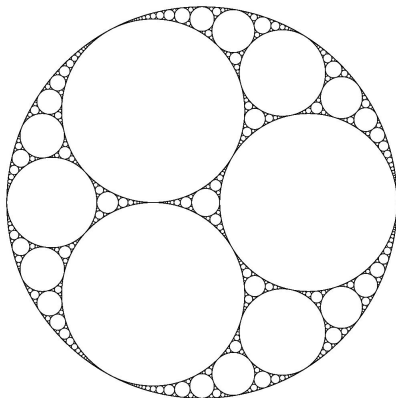
Packed Swiss Cheese Cosmology

- Iterate construction removing more and more balls \Rightarrow Apollonian sphere packing of 3-dimensional spheres
- Residual set of sphere packing is fractal
- Proposed as explanation for possible fractal distribution of matter in galaxies, clusters, and superclusters
 - F. Sylos Labini, M. Montuori, L. Pietroneo, *Scale-invariance of galaxy clustering*, Phys. Rep. Vol. 293 (1998) N. 2-4, 61–226.
 - J.R. Mureika, C.C. Dyer, *Multifractal analysis of Packed Swiss Cheese Cosmologies*, General Relativity and Gravitation, Vol.36 (2004) N.1, 151–184.



Apollonian sphere packings

- best known and understood case: Apollonian circle packing

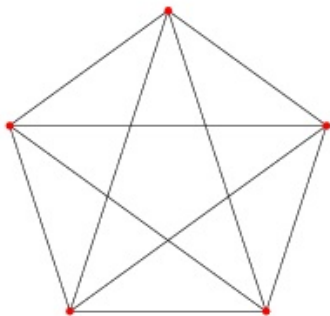


Configurations of mutually tangent circles in the plane, iterated on smaller scales filling a full volume region in the unit $2D$ ball:
residual set volume zero fractal of Hausdorff dimension $1.30568\dots$

- Many results (geometric, arithmetic, analytic) known about Apollonian circle packings: see for example
 - R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H. Yan, *Apollonian circle packings: number theory*, J. Number Theory 100 (2003) 1–45
 - A. Kontorovich, H. Oh, *Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds*, Journal of AMS, Vol 24 (2011) 603–648.
- **Higher dimensional** analogs of Apollonian packings: much more delicate and complicated geometry
 - R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H. Yan, *Apollonian Circle Packings: Geometry and Group Theory III. Higher Dimensions*, Discrete Comput. Geom. 35 (2006) 37–72.

Some known facts on Apollonian sphere packings

- **Descartes configuration** in D dimensions: $D + 2$ mutually tangent $(D - 1)$ -dimensional spheres
- Example: start with $D + 1$ equal size mutually tangent S^{D-1} centered at the vertices of D -simplex and one more smaller sphere in the center tangent to all



4-dimensional simplex

- **Quadratic Soddy–Gosset relation** between radii a_k

$$\left(\sum_{k=1}^{D+2} \frac{1}{a_k} \right)^2 = D \sum_{k=1}^{D+2} \left(\frac{1}{a_k} \right)^2$$

- **curvature-center coordinates:** $(D + 2)$ -vector

$$w = \left(\frac{\|x\|^2 - a^2}{a}, \frac{1}{a}, \frac{1}{a}x_1, \dots, \frac{1}{a}x_D \right)$$

(first coordinate curvature after inversion in the unit sphere)

- **Configuration space** \mathcal{M}_D of all Descartes configuration in D dimensions = all solutions \mathcal{W} to equation

$$\mathcal{W}^t Q_D \mathcal{W} = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2I_D \end{pmatrix}$$

with left and a right action of Lorentz group $O(D+1, 1)$

- **Dual Apollonian group** \mathcal{G}_D^\perp generated by reflections: inversion with respect to the j -th sphere

$$S_j^\perp = I_{D+2} + 2 \mathbf{1}_{D+2} e_j^t - 4 e_j e_j^t$$

$e_j = j$ -th unit coordinate vector

- $D \neq 3$: only relations in \mathcal{G}_D^\perp are $(S_j^\perp)^2 = 1$
- \mathcal{G}_D^\perp discrete subgroup of $GL(D+2, \mathbb{R})$
- Apollonian packing $\mathcal{P}_D =$ an orbit of \mathcal{G}_D^\perp on \mathcal{M}_D

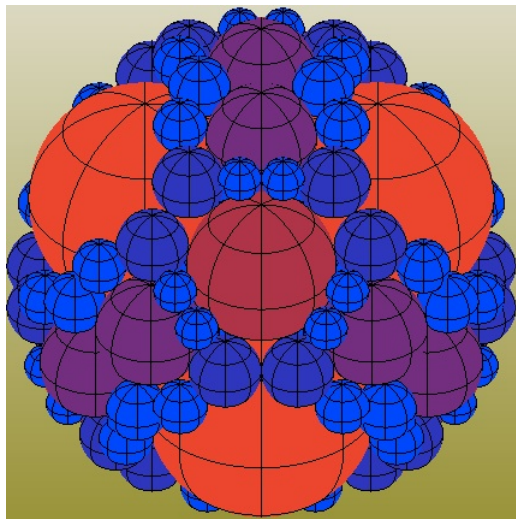
\Rightarrow **iterative construction**: at n -th step add spheres obtained from initial Descartes configuration via all possible

$$S_{j_1}^\perp S_{j_2}^\perp \cdots S_{j_n}^\perp, \quad j_k \neq j_{k+1}, \quad \forall k$$

there are N_n spheres in the n -th level

$$N_n = (D+2)(D+1)^{n-1}$$

iterative construction of sphere packings



- **Length spectrum:** radii of spheres in packing \mathcal{P}_D

$$\mathcal{L} = \mathcal{L}(\mathcal{P}_D) = \{a_{n,k} : n \in \mathbb{N}, 1 \leq k \leq (D+2)(D+1)^{n-1}\}$$

radii of spheres $S_{a_{n,k}}^{D-1}$

- **Melzak's packing constant** $\sigma_D(\mathcal{P}_D)$ exponent of convergence of series

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{(D+2)(D+1)^{n-1}} a_{n,k}^s$$

- **Residual set:** $\mathcal{R}(\mathcal{P}_D) = B^D \setminus \cup_{n,k} B_{a_{n,k}}^D$ with

$$\partial B_{a_{n,k}}^D = S_{a_{n,k}}^{D-1} \in \mathcal{P}_D$$

- Packing $\Rightarrow \text{Vol}_D(\mathcal{R}(\mathcal{P}_D)) = 0 \Rightarrow \sum_{\mathcal{L}} a_{n,k}^D < \infty \Rightarrow \sigma_D(\mathcal{P}_D) \leq D$
- **packing constant and Hausdorff dimension:**

$$\dim_H(\mathcal{R}(\mathcal{P}_D)) \leq \sigma_D(\mathcal{P}_D)$$

for Apollonian circles known to be same

- **Sphere counting function**: spheres with given curvature bound

$$\mathcal{N}_\alpha(\mathcal{P}_D) = \#\{S_{a_{n,k}}^{D-1} \in \mathcal{P}_D : a_{n,k} \geq \alpha\}$$

curvatures $c_{n,k} = a_{n,k}^{-1} \leq \alpha^{-1}$

- for Apollonian circles power law (Kontorovich–Oh)

$$\mathcal{N}_\alpha(\mathcal{P}_2) \sim_{\alpha \rightarrow 0} \alpha^{-\dim_H(\mathcal{R}(\mathcal{P}_2))}$$

- for higher dimensions (Boyd): packing constant

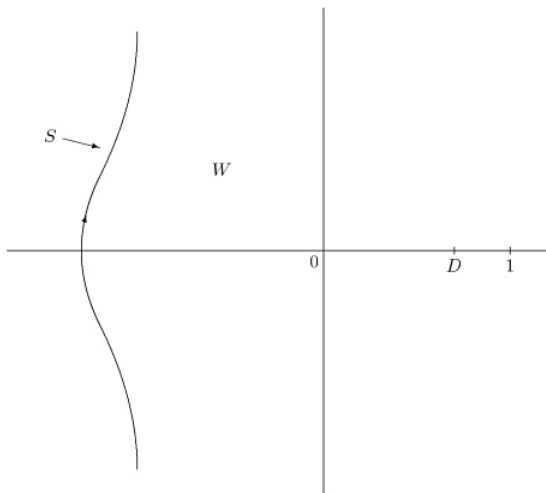
$$\limsup_{\alpha \rightarrow 0} -\frac{\log \mathcal{N}_\alpha(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}_D)$$

if limit exists $\mathcal{N}_\alpha(\mathcal{P}_D) \sim_{\alpha \rightarrow 0} \alpha^{-(\sigma_D(\mathcal{P}_D) + o(1))}$

Screens and Windows

- in general $\zeta_{\mathcal{L}_D}(s)$ need have analytic continuation to meromorphic on whole \mathbb{C}
 - \exists *screen* \mathcal{S} : curve $S(t) + it$ with $S : \mathbb{R} \rightarrow (-\infty, \sigma_D(\mathcal{P}_D)]$
 - *window* \mathcal{W} = region to the right of screen \mathcal{S} where analytic continuation
-
- M.L. Lapidus, M. van Frankenhuijsen, *Fractal geometry, complex dimensions and zeta functions. Geometry and spectra of fractal strings*, Second edition. Springer Monographs in Mathematics. Springer, 2013.

Screens and windows



Some additional assumptions

- **Definition:**

Apollonian packing \mathcal{P}_D of $(D - 1)$ -spheres is *analytic* if

- 1 $\zeta_{\mathcal{L}}(s)$ has analytic to meromorphic function on a region \mathcal{W} containing \mathbb{R}_+
 - 2 $\zeta_{\mathcal{L}}(s)$ has only one pole on \mathbb{R}_+ at $s = \sigma_D(\mathcal{P}_D)$.
 - 3 pole at $s = \sigma_D(\mathcal{P}_D)$ is simple
- **Also assume:** $\exists \lim_{\alpha \rightarrow 0} -\frac{\log \mathcal{N}_{\alpha}(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}_D)$
 - **Question:** in general when are these satisfied for packings \mathcal{P}_D ?
 - focus on $D = 4$ cases with these conditions

Rough estimate of the packing constant

- $\mathcal{P} = \mathcal{P}_4$ Apollonian packing of 3-spheres $S_{a_n,k}^3$
- at level n : average curvature

$$\frac{\gamma_n}{N_n} = \frac{1}{6 \cdot 5^{n-1}} \sum_{k=1}^{6 \cdot 5^{n-1}} \frac{1}{a_{n,k}}$$

- estimate $\sigma_4(\mathcal{P}_4)$ with averaged version: $\sum_n N_n \left(\frac{\gamma_n}{N_n}\right)^{-s}$

$$\sigma_{4,av}(\mathcal{P}) = \lim_{n \rightarrow \infty} \frac{\log(6 \cdot 5^{n-1})}{\log\left(\frac{\gamma_n}{6 \cdot 5^{n-1}}\right)}$$

- generating function of the γ_n known (Mallows)

$$G_{D=4} = \sum_{n=1}^{\infty} \gamma_n x^n = \frac{(1-x)(1-4x)u}{1 - \frac{22}{3}x - 5x^2}$$

u = sum of the curvatures of initial Descartes configuration

- obtain explicitly ($u = 1$ case)

$$\gamma_n = \frac{(11 + \sqrt{166})^n(-64 + 9\sqrt{166}) + (11 - \sqrt{166})^n(64 + 9\sqrt{166})}{3^n \cdot 10 \cdot \sqrt{166}}$$

- this gives a value

$$\sigma_{4,av}(\mathcal{P}) = 3.85193\dots$$

- in Apollonian circle case where $\sigma(\mathcal{P})$ known this method gives larger value, so expect $\sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P})$
- constraints on the packing constant:

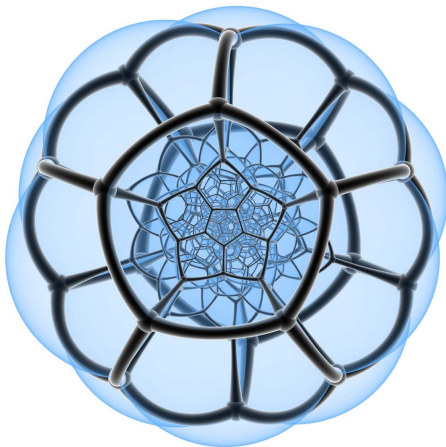
$$3 < \dim_H(\mathcal{R}(\mathcal{P})) \leq \sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P}) = 3.85193\dots$$

Models of (Euclidean, compactified) spacetimes

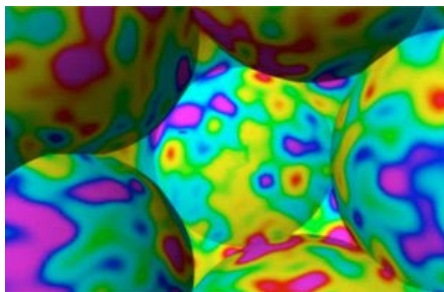
- 1 Homogeneous Isotropic cases: $S^1_\beta \times S^3_a$
- 2 Cosmic Topology cases: $S^1_\beta \times Y$ with Y a spherical space form S^3/Γ or a flat Bieberbach manifold T^3/Γ (modulo finite groups of isometries)
- 3 Packed Swiss Cheese: $S^1_\beta \times \mathcal{P}$ with Apollonian packing of 3-spheres $S^3_{a_{n,k}}$
- 4 Fractal arrangements with cosmic topology

Fractal arrangements with cosmic topology

- Example: Poincaré homology sphere, dodecahedral space S^3/\mathcal{I}_{120} , fundamental domain dodecahedron

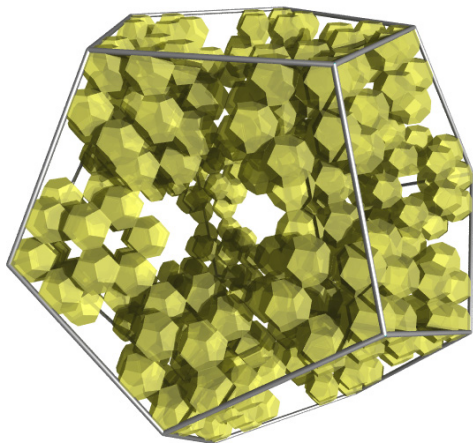


- considered a likely candidate for cosmic topology
 - S. Caillerie, M. Lachièze-Rey, J.P. Luminet, R. Lehoucq, A. Riazuelo, J. Weeks, *A new analysis of the Poincaré dodecahedral space model*, *Astron. and Astrophys.* 476 (2007) N.2, 691–696



- build a fractal model based on dodecahedral space

Fractal configurations of dodecahedra (Sierpinski dodecahedra)



- spherical dodecahedron has $\text{Vol}(Y) = \text{Vol}(S_a^3/\mathcal{I}_{120}) = \frac{\pi^2}{60} a^3$
- simpler than sphere packings because uniform scaling at each step: 20^n new dodecahedra, each scaled by a factor of $(2 + \phi)^{-n}$

$$\dim_H(\mathcal{P}_{\mathcal{I}_{120}}) = \frac{\log(20)}{\log(2 + \phi)} = 2.32958\dots$$

- close up all dodecahedra in the fractal identifying edges with \mathcal{I}_{120} : get fractal arrangement of Poincaré spheres $Y_{a(2+\phi)^{-n}}$
- zeta function has analytic continuation to all \mathbb{C}

$$\zeta_{\mathcal{L}}(s) = \sum_n 20^n (2 + \phi)^{-ns} = \frac{1}{1 - 20(2 + \phi)^{-s}}$$

exponent of convergence $\sigma = \dim_H(\mathcal{P}_{\mathcal{I}_{120}}) = \frac{\log(20)}{\log(2+\phi)}$ and poles

$$\sigma + \frac{2\pi im}{\log(2 + \phi)}, \quad m \in \mathbb{Z}$$

Spectral action models of gravity (modified gravity)

- **Spectral triple:** $(\mathcal{A}, \mathcal{H}, D)$
 - 1 unital associative algebra \mathcal{A}
 - 2 represented as bounded operators on a Hilbert space \mathcal{H}
 - 3 Dirac operator: self-adjoint $D^* = D$ with compact resolvent, with bounded commutators $[D, a]$
- prototype: $(C^\infty(M), L^2(M, S), \not{D}_M)$
- extends to non smooth objects (fractals) and noncommutative (NC tori, quantum groups, NC deformations, etc.)

Action functional

- Suppose *finitely summable* $ST = (\mathcal{A}, \mathcal{H}, D)$

$$\zeta_D(s) = \text{Tr}(|D|^{-s}) < \infty, \quad \Re(s) \gg 0$$

- **Spectral action** (Chamseddine–Connes)

$$\mathcal{S}_{ST}(\Lambda) = \text{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \text{Spec}(D)} \text{Mult}(\lambda) f(\lambda/\Lambda)$$

f = smooth approximation to (even) cutoff

Asymptotic expansion (Chamseddine–Connes) for
(almost) commutative geometries:

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\beta \in \Sigma_{ST}^+} f_\beta \Lambda^\beta \int |D|^{-\beta} + f(0) \zeta_D(0)$$

- Residues

$$\int |D|^{-\beta} = \frac{1}{2} \mathrm{Res}_{s=\beta} \zeta_D(s)$$

- Momenta $f_\beta = \int_0^\infty f(v) v^{\beta-1} dv$
- **Dimension Spectrum** Σ_{ST} poles of zeta functions
 $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$
- positive dimension spectrum $\Sigma_{ST}^+ = \Sigma_{ST} \cap \mathbb{R}_+^*$

Warning: for fractal spaces also oscillatory terms coming from part
of Σ_{ST} off the real line

Zeta function and heat kernel (manifolds)

- Mellin transform

$$|D|^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-tD^2} t^{\frac{s}{2}-1} dt$$

- heat kernel expansion

$$\mathrm{Tr}(e^{-tD^2}) = \sum_{\alpha} t^{\alpha} c_{\alpha} \quad \text{for } t \rightarrow 0$$

- zeta function expansion

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) = \sum_{\alpha} \frac{c_{\alpha}}{\Gamma(s/2)(\alpha + s/2)} + \text{holomorphic}$$

- taking residues

$$\mathrm{Res}_{s=-2\alpha} \zeta_D(s) = \frac{2c_{\alpha}}{\Gamma(-\alpha)}$$

Example spectral action of the round 3-sphere S^3

$$\mathcal{S}_{S^3}(\Lambda) = \text{Tr}(f(D_{S^3}/\Lambda)) = \sum_{n \in \mathbb{Z}} n(n+1) f\left(\left(n + \frac{1}{2}\right)/\Lambda\right)$$

- zeta function

$$\zeta_{D_{S^3}}(s) = 2\zeta\left(s - 2, \frac{3}{2}\right) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right)$$

$\zeta(s, q)$ = Hurwitz zeta function

- by asymptotic expansion

$$\mathcal{S}_{S^3}(\Lambda) \sim \Lambda^3 f_3 - \frac{1}{4}\Lambda f_1$$

- can also compute using Poisson summation formula (Chamseddine–Connes): estimate error term $O(\Lambda^{-\infty})$

Example: round 3-sphere S_a^3 radius a

$$\zeta_{D_{S_a^3}}(s) = a^s \left(2\zeta\left(s-2, \frac{3}{2}\right) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right) \right)$$

$$\mathcal{S}_{S_a^3}(\Lambda) \sim (\Lambda a)^3 f_3 - \frac{1}{4}(\Lambda a) f_1$$

Example: spherical space form $Y = S_a^3/\Gamma$ (Ćačić, Marcolli, Teh)

$$\mathcal{S}_Y(\Lambda) \sim \frac{1}{\#\Gamma} \mathcal{S}_{S_a^3}(\Lambda)$$

Why a model of (Euclidean) Gravity?

- M compact Riemannian 4-manifold

$$\text{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4$$

coefficients a_0 , a_2 and a_4 :

- cosmological term

$$f_4 \Lambda^4 \int |D|^{-4} = \frac{48 f_4 \Lambda^4}{\pi^2} \int \sqrt{g} d^4 x$$

- Einstein–Hilbert term

$$f_2 \Lambda^2 \int |D|^{-2} = \frac{96 f_2 \Lambda^2}{24 \pi^2} \int R \sqrt{g} d^4 x$$

- modified gravity terms (Weyl curvature and Gauss–Bonnet)

$$f(0) \zeta_D(0) = \frac{f_0}{10 \pi^2} \int \left(\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) \sqrt{g} d^4 x$$

$$C^{\mu\nu\rho\sigma} = \text{Weyl curvature and } R^* R^* = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R_{\mu\nu}^{\alpha\beta} R_{\rho\sigma}^{\gamma\delta}$$

momenta: (effective) gravitational and cosmological constant

Spectral action on a fractal spacetime:

- $S_{\beta}^1 \times \mathcal{P}$: Apollonian packing
 - $S_{\beta}^1 \times \mathcal{P}_{\Upsilon}$: fractal dodecahedral space
- 1 Construct a spectral triple for the geometries \mathcal{P} and \mathcal{P}_{Υ}
 - 2 Compute the zeta function
 - 3 Compute the asymptotic form of the spectral action
 - 4 Effect of product with S_{β}^1

\Rightarrow look for **new terms** in the spectral action (in addition to usual gravitational terms) that detect **presence of fractality**

The spectral triple of a fractal geometry

- case of Sierpinski gasket: Christensen, Ivan, Lapidus
- similar case for \mathcal{P} and \mathcal{P}_Y
- for D -dim packing

$$\mathcal{P}_D = \{S_{a_{n,k}}^{D-1} : n \in \mathbb{N}, 1 \leq k \leq (D+2)(D+1)^{n-1}\}$$

$$(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{\mathcal{P}_D}, \mathcal{D}_{\mathcal{P}_D}) = \bigoplus_{n,k} (\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{S_{a_{n,k}}^{D-1}}, \mathcal{D}_{S_{a_{n,k}}^{D-1}})$$

- for \mathcal{P}_Y with $Y_a = S^3/\mathcal{I}_{120}$:

$$(\mathcal{A}_{\mathcal{P}_Y}, \mathcal{H}_{\mathcal{P}_Y}, \mathcal{D}_{\mathcal{P}_Y}) = (\mathcal{A}_{\mathcal{P}_Y}, \bigoplus_n \mathcal{H}_{Y_{a_n}}, \bigoplus_n \mathcal{D}_{Y_{a_n}})$$

with $a_n = a(2 + \phi)^{-n}$

Zeta functions for Apollonian packing of 3-spheres:

- Lengths zeta function (fractal string)

$$\zeta_{\mathcal{L}}(s) := \sum_{n \in \mathbb{N}} \sum_{k=1}^{6 \cdot 5^{n-1}} a_{n,k}^s$$

with $\mathcal{L} = \mathcal{L}_4 = \{a_{n,k} \mid n \in \mathbb{N}, k \in \{1, \dots, 6 \cdot 5^{n-1}\}\}$

- zeta function of Dirac operator of the spectral triple

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n=1}^{\infty} \sum_{k=1}^{6 \cdot 5^{n-1}} \mathrm{Tr}(|D_{S_{a_{n,k}}^3}|^{-s})$$

each term $\mathrm{Tr}(|D_{S_{a_{n,k}}^3}|^{-s}) = a_{n,k}^s (2\zeta(s-2, \frac{3}{2}) - \frac{1}{2}\zeta(s, \frac{3}{2}))$ gives

$$\begin{aligned} \mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) &= \left(2\zeta\left(s-2, \frac{3}{2}\right) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right) \right) \sum_{n,k} a_{n,k}^s \\ &= \left(2\zeta\left(s-2, \frac{3}{2}\right) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right) \right) \zeta_{\mathcal{L}}(s) \end{aligned}$$

Spectral action for Apollonian packing of 3-spheres:
 (under good conditions on $\zeta_{\mathcal{L}}(s)$)

- Positive Dimension Spectrum: $\Sigma_{STPSC}^+ = \{1, 3, \sigma_4(\mathcal{P})\}$
- asymptotic spectral action

$$\begin{aligned} \text{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) &\sim \Lambda^3 \zeta_{\mathcal{L}}(3) f_3 - \Lambda \frac{1}{4} \zeta_{\mathcal{L}}(1) f_1 \\ &+ \Lambda^\sigma \left(\zeta\left(\sigma - 2, \frac{3}{2}\right) - \frac{1}{4} \zeta\left(\sigma, \frac{3}{2}\right) \right) \mathcal{R}_\sigma f_\sigma + \mathcal{S}_\Lambda^{\text{osc}} \end{aligned}$$

$\sigma = \sigma_4(\mathcal{P})$ packing constant; residue $\mathcal{R}_\sigma = \text{Res}_{s=\sigma} \zeta_{\mathcal{L}}(s)$, and momenta $f_\beta = \int_0^\infty v^{\beta-1} f(v) dv$

- additional term $\mathcal{S}_\Lambda^{\text{osc}}$ coming from series of contributions of poles of zeta function off the real line: **oscillatory terms**

Oscillatory terms (fractals)

- zeta function $\zeta_{\mathcal{L}}(s)$ on fractals in general has additional poles off the real line (position depends on Hausdorff and spectral dimension: depending on how homogeneous the fractal)
- best case exact self-similarity: $s = \sigma + \frac{2\pi im}{\log \ell}$, $m \in \mathbb{Z}$
- heat kernel on fractals has additional log-oscillatory terms in expansion

$$\frac{C}{t^\sigma} \left(1 + A \cos\left(\frac{2\pi}{\log \ell} \log t + \phi\right) \right) + \dots$$

for constants C, A, ϕ : series of terms for each complex pole

Log-oscillatory terms in expansion of the spectral action:

- G.V. Dunne, *Heat kernels and zeta functions on fractals*, J. Phys. A 45 (2012) 374016 [22p]
- M. Eckstein, B. Iochum, A. Sitarz, *Heat kernel and spectral action on the standard Podleś sphere*, Comm. Math. Phys. 332 (2014) 627–668
- M. Eckstein, A. Zającz, *Asymptotic and exact expansion of heat traces*, arXiv:1412.5100

effect of product with S_β^1 (leading term without oscillations)

- case of $S_\beta^1 \times S_a^3$ (Chamseddine–Connes)

$$D_{S_\beta^1 \times S_a^3} = \begin{pmatrix} 0 & D_{S_a^3} \otimes 1 + i \otimes D_{S_\beta^1} \\ D_{S_a^3} \otimes 1 - i \otimes D_{S_\beta^1} & 0 \end{pmatrix}$$

Spectral action

$$\mathrm{Tr}(h(D_{S_\beta^1 \times S_a^3}^2/\Lambda)) \sim 2\beta\Lambda \mathrm{Tr}(\kappa(D_{S_a^3}^2/\Lambda)),$$

test function $h(x)$, and test function

$$\kappa(x^2) = \int_{\mathbb{R}} h(x^2 + y^2) dy$$

- Case of $S_{\beta}^1 \times \mathcal{P}$:

$$\begin{aligned} \mathcal{S}_{S_{\beta}^1 \times \mathcal{P}}(\Lambda) &\sim 2\beta \left(\Lambda^4 \zeta_{\mathcal{L}}(3) \mathfrak{h}_3 - \Lambda^2 \frac{1}{4} \zeta_{\mathcal{L}}(1) \mathfrak{h}_1 \right) \\ &+ 2\beta \Lambda^{\sigma+1} \left(\zeta\left(\sigma - 2, \frac{3}{2}\right) - \frac{1}{4} \zeta\left(\sigma, \frac{3}{2}\right) \right) \mathcal{R}_{\sigma} \mathfrak{h}_{\sigma} \end{aligned}$$

with momenta

$$\begin{aligned} \mathfrak{h}_3 &:= \pi \int_0^{\infty} h(\rho^2) \rho^3 d\rho, & \mathfrak{h}_1 &:= 2\pi \int_0^{\infty} h(\rho^2) \rho d\rho \\ \mathfrak{h}_{\sigma} &= 2 \int_0^{\infty} h(\rho^2) \rho^{\sigma} d\rho \end{aligned}$$

Interpretation:

- Term $2\Lambda^4\beta a^3\mathfrak{h}_3 - \frac{1}{2}\Lambda^2\beta a\mathfrak{h}_1$, cosmological and Einstein–Hilbert terms, replaced by

$$2\Lambda^4\beta\zeta_{\mathcal{L}}(3)\mathfrak{h}_3 - \frac{1}{2}\Lambda^2\beta\zeta_{\mathcal{L}}(1)\mathfrak{h}_1$$

zeta regularization of divergent series of spectral actions of 3-spheres of packing

- Additional term in gravity action functional: corrections to gravity from fractality

$$2\beta\Lambda^{\sigma+1}\left(\zeta\left(\sigma-2, \frac{3}{2}\right) - \frac{1}{4}\zeta\left(\sigma, \frac{3}{2}\right)\right)\mathcal{R}_{\sigma}\mathfrak{h}_{\sigma}$$

Case of fractal dodecahedral space \mathcal{P}_Y

- Zeta functions

$$\zeta_{\mathcal{L}(\mathcal{P}_Y)}(s) = \sum_{n \geq 0} 20^n (2 + \phi)^{-ns}$$

$$\zeta_{\mathcal{D}_{\mathcal{P}_Y}}(s) = \frac{a^s}{120} \left(2\zeta\left(s-2, \frac{3}{2}\right) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right) \right) \zeta_{\mathcal{L}(\mathcal{P}_Y)}(s)$$

- Spectral action:

$$\begin{aligned} \mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}_Y}/\Lambda)) &\sim (\Lambda a)^3 \frac{\zeta_{\mathcal{L}(\mathcal{P}_Y)}(3)}{120} f_3 - \Lambda a \frac{\zeta_{\mathcal{L}(\mathcal{P}_Y)}(1)}{120} f_1 \\ &+ (\Lambda a)^\sigma \frac{\zeta\left(\sigma-2, \frac{3}{2}\right) - \frac{1}{4}\zeta\left(\sigma, \frac{3}{2}\right)}{120 \log(2+\phi)} f_\sigma + \mathcal{S}_{Y,\Lambda}^{\mathrm{osc}} \end{aligned}$$

$$\sigma = \dim_H(\mathcal{P}_Y) = \frac{\log(20)}{\log(2+\phi)} = 2.3296\dots$$

- on product geometry $S^1_\beta \times \mathcal{P}_Y$

$$\mathcal{S}_{S^1_\beta \times \mathcal{P}_Y}(\Lambda) \sim 2\beta \left(\Lambda^4 \frac{a^3 \zeta_{\mathcal{L}(\mathcal{P}_Y)}(3)}{120} \mathfrak{h}_3 - \Lambda^2 \frac{a \zeta_{\mathcal{L}(\mathcal{P}_Y)}(1)}{120} \mathfrak{h}_1 \right) \\ + 2\beta \Lambda^{\sigma+1} \frac{a^\sigma (\zeta(\sigma-2, \frac{3}{2}) - \frac{1}{4} \zeta(\sigma, \frac{3}{2}))}{120 \log(2+\phi)} \mathfrak{h}_\sigma + \mathcal{S}_{S^1_\beta \times Y, \Lambda}^{\text{osc}}$$

- **Note:** correction term now at different σ than Apollonian \mathcal{P}
- oscillatory terms $\mathcal{S}_{Y, \Lambda}^{\text{osc}}$ more explicit than in the Apollonian case

Oscillatory terms: dodecahedral case

- zeros of zeta function $\zeta_{\mathcal{L}}(s)$

$$s_m = \sigma + \frac{2\pi im}{\log(2 + \phi)}, \quad m \in \mathbb{Z}$$

with $\sigma = \log(20)/\log(2 + \phi)$

- contribution to heat kernel expansion of non-real zeros:

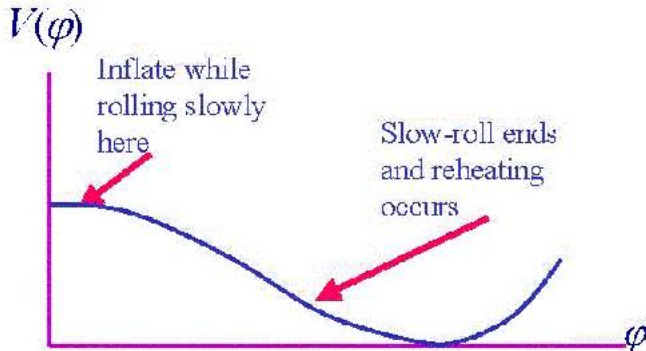
$$\frac{C}{t^\sigma} (a_0 + 2\Re(a_1 t^{-2\pi i/\log(2+\phi)}) + \dots)$$

with coefficients a_m proportional to $\Gamma(s_m)$: for fixed real part σ decays exponentially fast along vertical line

- oscillatory terms are small

Slow-roll inflation potential from the spectral action

- perturb the Dirac operator by a scalar field $D^2 + \phi^2 \Rightarrow$ spectral action gives potential $V(\phi)$



- shape of $V(\phi)$ distinguishes most cosmic topologies: spherical forms and Bieberbach manifolds (Marcolli, Pierpaoli, Teh)

Fractality corrections to potential $V(\phi)$

- additional term in potential

$$\mathcal{U}_\sigma(x) = \int_0^\infty u^{(\sigma-1)/2} (h(u+x) - h(u)) du$$

depends on σ fractal dimension

- size of correction depends on (leading term)

$$\left(\zeta\left(\sigma - 2, \frac{3}{2}\right) - \frac{1}{4}\zeta\left(\sigma, \frac{3}{2}\right)\right)\mathcal{R}_\sigma$$

- further corrections to \mathcal{U}_σ come from the oscillatory terms

⇒ presence of fractality (in this spectral action model of gravity)
can be read off the slow-roll potential (hence the slow-roll coefficients, which depend on V , V' , V'')