

# Spectral Triples and Pati–Salam GUT

Matilde Marcolli

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## References

- A.H. Chamseddine, A. Connes, W. van Suijlekom, *Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification*, JHEP 1311 (2013) 132
- A.H. Chamseddine, A. Connes, W. van Suijlekom, *Grand Unification in the Spectral Pati-Salam Model*, arXiv:1507.08161

## Real even spectral triples

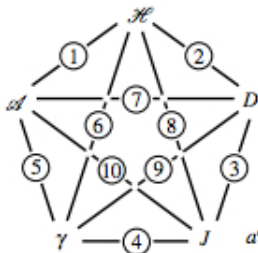
①  $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$

②  $D^* = D$ , comp. res.

③  $JD = \varepsilon' DJ$

④  $J\gamma = \varepsilon'' \gamma J$

⑤  $[\gamma, \mathcal{A}] = 0$



$\gamma^* = \gamma, \gamma^2 = \text{id}_{\mathcal{H}}$  ⑥

$[a, D] \in \mathcal{B}(\mathcal{H})$  ⑦

$J^2 = \varepsilon \text{id}_{\mathcal{H}}$  ⑧

$\gamma D = -D\gamma$  ⑨

$a^\sigma = Ja^*J^*, [a, b^\sigma] = 0$  ⑩

All previously discussed NCG particle physics models based on datum of a real even spectral triple

$$(\mathcal{A}, \mathcal{H}, D, J, \gamma)$$

possibly with additional  $R$ -symmetry for SUSY case

## Order one condition

- In particular very strong constraints on possible Dirac operators come from imposing the order one condition

$$[[D, a], b^0] = 0$$

with  $b^0 = Jb^*J^{-1}$

- natural question: **what kind of models arise without the order one condition?**
- NCG particle physics models without order one condition give **GUT models: Pati–Salam**
- there are other significant examples of noncommutative spaces without order one condition: quantum groups like  $SU_q(2)$

## Order one condition as a symmetry breaking mechanism

- A.H. Chamseddine, A. Connes, M. Marcolli, *Gravity and the Standard Model with Neutrino Mixing*, Adv. Theor. Math. Phys., Vol.11 (2007) 991–1090
  - shown that imposing  $[[D, a], b^0] = 0$  breaks down the L/R symmetry of the algebra  $\mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$  to the SM algebra  $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
- A.Chamseddine, A.Connes, *Why the Standard Model*, J.Gem.Phys. 58 (2008) 38–47
  - shown that same argument applies with initial choice of algebra

$$\mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4(\mathbb{C})$$

order one condition breaks it down to  $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

## Inner fluctuations without order one condition

- usual argument for conjugation of fluctuated Dirac operator  $D_A$  by unitary  $U = hJuJ^{-1}$  as gauge transformation

$$A \mapsto A^u = u[D, u^*] + uAu^*$$

only works if  $[JuJ^{-1}, A] = 0$  for  $A = \sum_j a_j[D, b_j]$  which requires order one condition

- without order one condition: general form of inner fluctuations

$$D' = D + A_{(1)} + \tilde{A}_{(1)} + A_{(2)}$$

$$A_{(1)} = \sum_j a_j[D, b_j]$$

$$\tilde{A}_{(1)} = \sum_j \hat{a}_j[D, \hat{b}_j], \quad \hat{a}_j = Ja_jJ^{-1}, \quad \hat{b}_j = Jb_jJ^{-1}$$

$$A_{(2)} = \sum_j \hat{a}_j[A_{(1)}, \hat{b}_j] = \sum_{j,k} \hat{a}_j a_k [[D, b_k], \hat{b}_j]$$

## Semigroup of inner perturbations

$$\text{Pert}(\mathcal{A}) = \left\{ \sum_j a_j \otimes b_j^{op} \in \mathcal{A} \otimes \mathcal{A}^{op} : \sum_j a_j b_j = 1, \sum_j a_j \otimes b_j^{op} = \sum_j b_j^* \otimes a_j^{op} \right\}$$

- acting on Dirac operator  $D$  by

$$\sum_j a_j \otimes b_j^{op} : D \mapsto \sum_j a_j D b_j$$

- semigroup structure implies that inner fluctuations of inner fluctuations are still inner fluctuations, even without order one condition

## Finite spectral triple without order one

- still assume order zero condition  $[a, b^0] = 0$  (bimodule)
- still assume KO-dimension = 6 (as for SM)
- these two requirements imply that center of the complexified algebra

$$Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$$

- dimension of the Hilbert space square of an integer  $\Rightarrow$   
 $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$
- imposing a symplectic symmetry on first algebra gives  $k = 2a$  and algebra  $M_a(\mathbb{H})$
- chirality operator on  $M_a(\mathbb{H})$  requires  $a$  even
- realistic physical assumption  $k = 4$

$$\mathcal{A} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$



## Gauge group

- inner automorphisms of algebra  $\mathcal{A}$ : Pati–Salam type left-right model

$$SU(2)_R \times SU(2)_L \times SU(4)$$

- $SU(4)$  color group has lepton number as 4-th color

## Fermions

- dimension of finite Hilbert space

$$384 = 2^7 \times 3$$

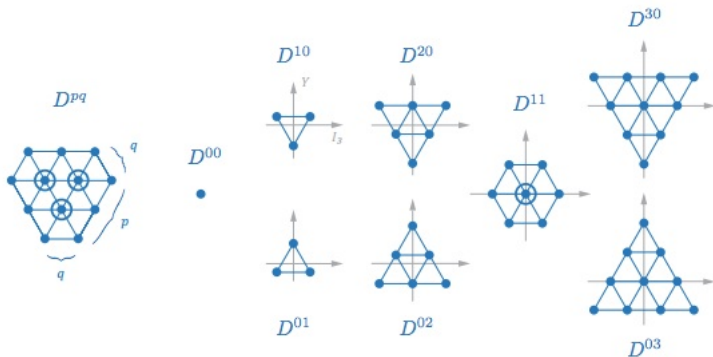
3 for generations,  $\mathbf{2}_L$  and  $\mathbf{2}_R$  for  $SU(2)_L$  and  $SU(2)_R$  and  $16 = 1 + 15$  for  $SU(4)$ , with further doubling for matter/antimatter

## $SU(N)$ representations

- $SU(N)$ : Lie algebra  $N^2 - 1$  dimensional, basis  $t_a$ ; there are  $N - 1$  Casimir operators (center of the universal enveloping algebra) that label irreducible representations
- $SU(2)$ : three dim Lie,  $\sigma_a$  (Pauli matrices), one Casimir operator  $J^2 = \sigma_a \sigma_a$ , eigenvalues  $j(j + 1)$  with  $j \in \frac{1}{2}\mathbb{Z}$ , irreducible representations labelled by  $p = 2j \in \mathbb{Z}_+$ , dimension  $D^p = p + 1$  (angular momentum)

- $SU(3)$ : eight dim Lie,  $\lambda_a$  (Gell-Mann matrices), two Casimir operators  $\lambda_a \lambda_a$  and  $f_{abc} \lambda_a \lambda_b \lambda_c$ : irreducible representations labelled by these with two quantum numbers  $p, q \in \mathbb{Z}_+$ , dimensions

$$D^{pq} = \frac{1}{2}(p+1)(q+1)(p+q+2)$$



- $SU(4)$ : three Casimir operators and irreducible representations parameterized by three quantum numbers  $p, q, r \in \mathbb{Z}_+$ , dimensions

$$D^{pqr} = \frac{1}{12}(p+1)(q+1)(p+q+2)(q+r+2)(p+q+r+3)$$

$$D^{000} = 1, D^{100} = D^{001} = 4, D^{010} = 8, D^{200} = D^{002} = 10, \\ D^{101} = 15$$

## Higgs fields

- assume that the unperturbed Dirac operator  $D$  satisfies order one condition when restricted to the SM subalgebra  $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$  (consistency with SM)
- then inner fluctuations in the vertical (NC) direction coming from terms  $A_{(2)}$  are composite, quadratic in those arising in the terms  $A_{(1)}$
- then (with order one on SM algebra) Higgs in representations:  $(2_R, 2_L, 1)$ ,  $(2_R, 1_L, 4)$  and  $(1_R, 1_L, 1 + 15)$  (Marshak–Mohapatra model)
- otherwise would have additional fundamental Higgs fields

## Spectral action computation

- Dirac operator  $D$  of the finite spectral triple is a  $384 \times 384$  matrix (written in an explicit tensor notation)
- Inner fluctuations computed (with order one on SM subalgebra)
- product geometry  $M \times F$  of 4-dim spacetime and finite geometry
- Spectral action  $\text{Tr}(f(D_A/\Lambda))$

$$\text{Tr}(f(D_A/\Lambda)) = \sum_{n=0}^{\infty} F_{4-n} \Lambda^{4-n} a_n$$

-  $a_n$  Seeley deWitt coefficients of heat kernel

-  $F_k(u) = f(v)$  with  $u = v^2$ : momenta  $F_4 = 2f_4$ ,  $F_2 = 2f_2$

$$F_4 = \int_0^{\infty} F(u) u du = 2 \int_0^{\infty} f(v) v^3 dv, \quad F_2 = \int_0^{\infty} F(u) du = 2 \int_0^{\infty} f(v) v dv$$

and  $F_0 = F(0) = f_0$  with remaining terms

$$F_{-2n} = (-1)^n F^{(n)}(0) = (-1)^n \left( \frac{1}{2v} \frac{d}{dv} \right)^n f|_{v=0}$$

## Seeley-deWitt coefficients

- $a_0$  coefficient (volume, cosmological term)

$$\begin{aligned}a_0 &= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} (1) \\ &= \frac{1}{16\pi^2} (4) (32) (3) \int d^4x \sqrt{g} \\ &= \frac{24}{\pi^2} \int d^4x \sqrt{g}\end{aligned}$$

- $a_2$  coefficient: Einstein–Hilbert and Higgs terms

$$a_2 = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left( E + \frac{1}{6} R \right)$$

$$\begin{aligned}a_2 &= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \left( (R(-96 + 64)) - 8 \left( H_{aIcK} H^{cK\dot{a}I} + 2 \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right) \right) \\ &= -\frac{2}{\pi^2} \int d^4x \sqrt{g} \left( R + \frac{1}{4} \left( H_{aIcK} H^{cK\dot{a}I} + 2 \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right) \right).\end{aligned}$$

- $a_4$  coefficient: modified gravity terms, Yang–Mills terms, and Higgs terms

$$a_4 = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left( \frac{1}{360} (5R^2 - 2R_{\mu\nu}^2 + 2R_{\mu\nu\rho\sigma}^2) 1 + \frac{1}{2} \left( E^2 + \frac{1}{3} RE + \frac{1}{6} \Omega_{\mu\nu}^2 \right) \right)$$

$$a_4 = \frac{1}{2\pi^2} \int d^4x \sqrt{g} \left[ -\frac{3}{5} C_{\mu\nu\rho\sigma}^2 + \frac{11}{30} R^* R^* + g_L^2 (W_{\mu\nu L}^\alpha)^2 + g_R^2 (W_{\mu\nu R}^\alpha)^2 + g^2 (V_{\mu\nu}^m)^2 \right. \\ \left. + \nabla_\mu \Sigma_{aI}^{\dot{c}K} \nabla^\mu \Sigma_{cK}^{aI} + \frac{1}{2} \nabla_\mu H_{\dot{a}I bJ} \nabla^\mu H^{\dot{a}I bJ} + \frac{1}{12} R \left( H_{\dot{a}I cK} H^{\dot{c}K aI} + 2\Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right) \right. \\ \left. + \frac{1}{2} \left| H_{\dot{a}I cK} H^{\dot{c}K bJ} \right|^2 + 2H_{\dot{a}I cK} \Sigma_{bJ}^{\dot{c}K} H^{\dot{a}I dL} \Sigma_{dL}^{bJ} + \Sigma_{aI}^{\dot{c}K} \Sigma_{cK}^{bJ} \Sigma_{bJ}^{\dot{d}L} \Sigma_{dL}^{aI} \right]$$



## Higgs potential

$$V = \frac{F_0}{2\pi^2} \left( \frac{1}{2} \left| H_{aIcK} H^{\dot{c}KbJ} \right|^2 + 2H_{aIcK} \Sigma_{bJ}^{\dot{c}K} H^{\dot{a}IdL} \Sigma_{\dot{d}L}^{bJ} + \Sigma_{aI}^{\dot{c}K} \Sigma_{\dot{c}K}^{bJ} \Sigma_{bJ}^{\dot{d}L} \Sigma_{\dot{d}L}^{aI} \right) - \frac{F_2}{2\pi^2} \left( H_{aIcK} H^{\dot{c}KaI} + 2\Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} \right).$$

- Quartic terms

$$\frac{1}{2} \left| H_{aIcK} H^{\dot{c}KbJ} \right|^2 = \frac{1}{2} |k^{\nu R}|^4 \left( \Delta_{\dot{a}K} \bar{\Delta}^{\dot{a}L} \Delta_{\dot{b}L} \bar{\Delta}^{\dot{b}K} \right)^2$$

$$\begin{aligned} \Sigma_{aI}^{\dot{c}K} \Sigma_{\dot{c}K}^{bJ} \Sigma_{bJ}^{\dot{d}L} \Sigma_{\dot{d}L}^{aI} &= \left( \left( (k^{\nu} - k^{\ast u}) \phi_a^{\dot{c}} + (k^{\ast e} - k^{\ast d}) \tilde{\phi}_a^{\dot{c}} \right) \Sigma_I^K + \left( k^{\ast u} \phi_a^{\dot{c}} + k^{\ast d} \tilde{\phi}_a^{\dot{c}} \right) \delta_I^K \right) \\ &\quad \left( \left( (k^{\nu} - k^u) \phi_c^{\dot{b}} + (k^e - k^d) \tilde{\phi}_c^{\dot{b}} \right) \Sigma_K^J + \left( k^u \phi_c^{\dot{b}} + k^d \tilde{\phi}_c^{\dot{b}} \right) \delta_K^J \right) \\ &\quad \left( \left( (k^{\ast \nu} - k^{\ast u}) \phi_b^{\dot{d}} + (k^{\ast e} - k^{\ast d}) \tilde{\phi}_b^{\dot{d}} \right) \Sigma_J^L + \left( k^{\ast u} \phi_b^{\dot{d}} + k^{\ast d} \tilde{\phi}_b^{\dot{d}} \right) \delta_J^L \right) \\ &\quad \left( \left( (k^{\nu} - k^u) \phi_d^{\dot{a}} + (k^e - k^d) \tilde{\phi}_d^{\dot{a}} \right) \Sigma_L^I + \left( k^u \phi_d^{\dot{a}} + k^d \tilde{\phi}_d^{\dot{a}} \right) \delta_L^I \right) \end{aligned}$$

$$\begin{aligned} 2H_{aIcK} \Sigma_{bJ}^{\dot{c}K} H^{\dot{a}IdL} \Sigma_{\dot{d}L}^{bJ} &= 2 |k^{\nu R}|^2 \left( \Delta_{\dot{a}K} \bar{\Delta}^{\dot{a}L} \Delta_{\dot{c}I} \bar{\Delta}^{\dot{c}I} \right) \\ &\quad \left( \left( (k^{\ast \nu} - k^{\ast u}) \phi_b^{\dot{c}} + (k^{\ast e} - k^{\ast d}) \tilde{\phi}_b^{\dot{c}} \right) \Sigma_J^K + \left( k^{\ast u} \phi_b^{\dot{c}} + k^{\ast d} \tilde{\phi}_b^{\dot{c}} \right) \delta_J^K \right) \\ &\quad \left( \left( (k^{\nu} - k^u) \phi_a^{\dot{b}} + (k^e - k^d) \tilde{\phi}_a^{\dot{b}} \right) \Sigma_L^J + \left( k^u \phi_a^{\dot{b}} + k^d \tilde{\phi}_a^{\dot{b}} \right) \delta_L^J \right). \end{aligned}$$

- also mass terms

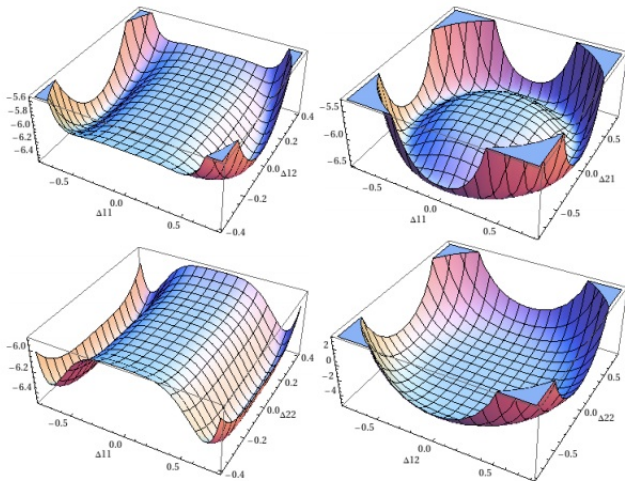


FIG. 1: The scalar potential in some of the  $\Delta_{\hat{a}I}$ -directions, with all other fields at their SM-vevs as in Equation (25). We have put  $k^\nu = k^e = 1$  and  $k^{\nu R} = k^u = k^d = 2$ . With these choices, the Standard Model vacuum corresponds to  $\Delta_{\hat{1}1} = \frac{1}{\sqrt{2}}$ ,  $\Sigma_1^1 = 2$ ,  $\phi_1^1 = \frac{1}{2}$  and all other fields are zero. At this point the Hessian in the  $\Delta$ -directions is nonnegative.

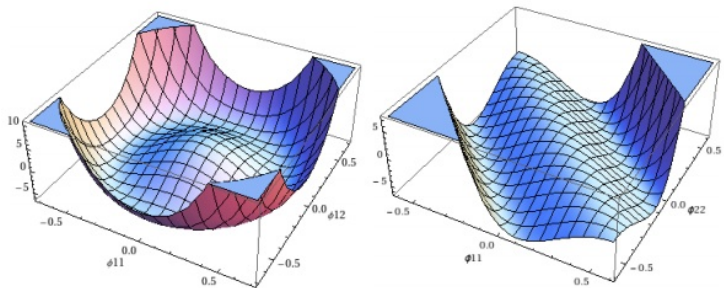


FIG. 2: The scalar potential in the  $\phi_a^b$ -directions, after the  $\Sigma$  and  $\Delta$ -fields have acquired their SM-vevs as in Equation (25). Again, we have put  $k^\nu = k^e = 1$  and  $k^{\nu R} = k^u = k^d = 2$ .