# Introduction: What is Noncommutative Geometry?

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**Noncommutative Geometry**:

- *Geometry adapted to quantum world*: physical observables are operators in Hilbert space, these do not commute (e.g. canonical commutation relation of position and momentum: $[x, p] = i\hbar$)

- A method to describe "bad quotients" of equivalence relations as if they were nice spaces (cf. other such methods, e.g. stacks)

- Generally a method for extending smooth geometries to objects that are not smooth manifolds (fractals, quantum groups, bad quotients, deformations, ...)

*Simplest example of a noncommutative geometry*: matrices $M_2(\mathbb{C})$

- Product is not commutative $AB \neq BA$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & v \\ x & y \end{pmatrix} = \begin{pmatrix} au + bx & av + by \\ cu + dx & cv + dy \end{pmatrix} \neq$$

$$\begin{pmatrix} au + cv & bu + dv \\ ax + cy & bx + dy \end{pmatrix} = \begin{pmatrix} u & v \\ x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- View product as a *convolution product*

$X = \{x_1, x_2\}$ space with two points

Equivalence relation $x_1 \sim x_2$ that identifies the two points: quotient (in classical sense) one point; graph of equivalence relation $R = \{(a, b) \in X \times X \,:\, a \sim b\} = X \times X$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

$A_{ij} = f(x_i, x_j) : R \to \mathbb{C}$ functions on $X \times X$

$$(f_1 \star f_2)(x_i, x_j) = \sum_{x_i \sim x_k \sim x_j} f_1(x_i, x_k) f_2(x_k, x_j)$$

- The algebra $M_2(\mathbb{C})$ is the algebra of functions on $X \times X$ with convolution product

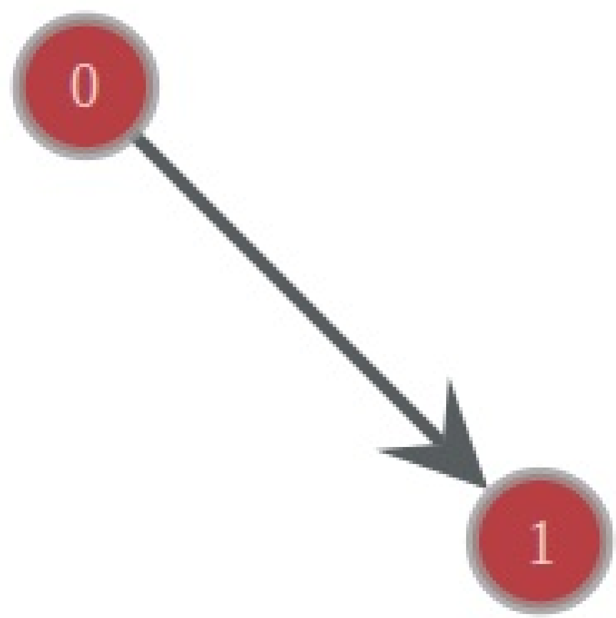
- Different description of the quotient $X/\sim$

- NCG space $M_2(\mathbb{C})$ is a point with internal degrees of freedom

- Intuition: useful to describe physical models with internal degrees of freedom

*Morita equivalence* (algebraic): rings $R$, $S$ that have equivalent categories $R - Mod \simeq S - Mod$ of (left)-modules

$R$ and $M_N(R)$ are Morita equivalent




The algebra $M_2(\mathbb{C})$ represents a two point space with an identification between points. Unlike the classical quotient with algebra $\mathbb{C}$, the non-commutative space $M_2(\mathbb{C})$ "remembers" how the quotient is obtained

## Noncommutative Geometry of Quotients

Equivalence relation $\mathcal{R}$ on $X$:
quotient $Y = X/\mathcal{R}$.

Even for very good $X \Rightarrow X/\mathcal{R}$ pathological!

Classical: functions on the quotient
$\mathcal{A}(Y) := \{f \in \mathcal{A}(X) \mid f \text{ is } \mathcal{R}-\text{invariant}\}$

$\Rightarrow$ often too few functions
$\mathcal{A}(Y) = \mathbb{C}$ only constants

NCG: $\mathcal{A}(Y)$ noncommutative algebra

$$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$$

functions on the graph $\Gamma_{\mathcal{R}} \subset X \times X$ of the equivalence relation
(compact support or rapid decay)

Convolution product

$$(f_1 * f_2)(x, y) = \sum_{x \sim u \sim y} f_1(x, u) f_2(u, y)$$

involution $f^*(x, y) = \overline{f(y, x)}$.

$\mathcal{A}(\Gamma_{\mathcal{R}})$ noncommutative algebra $\Rightarrow Y = X/\mathcal{R}$ *noncommutative space*

Recall: $C_0(X) \Leftrightarrow X$ Gelfand–Naimark equiv of categories abelian $C^*$-algebras, loc comp Hausdorff spaces

Result of NCG:

$Y = X/\mathcal{R}$ *noncommutative space* with NC algebra of functions $\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$ is

- as good as $X$ to do geometry
  (deRham forms, cohomology, vector bundles, connections, curvatures, integration, points and subvarieties)

- but with *new* phenomena
  (time evolution, thermodynamics, quantum phenomena)

## Tools needed for Physics Models

- Vector bundles and connections (gauge fields)

- Riemannian metrics (Euclidean gravity)

- Spinors (Fermions)

- Action Functional

*General idea*: reformulate usual geometry in algebraic terms (using the algebra of functions rather than the geometric space) and extend to case where algebra no longer commutative

**Remark**: Different forms of noncommutativity in physics

- Quantum mechanics: non-commuting operators

- Gauge theories: non-abelian gauge groups

- Gravity: hypothetical presence of noncommutativity in spacetime coordinates at high energy (some string compactifications with NC tori)

In the models we consider here the non-abelian nature of gauge groups is seen as an effect of an underlying non-commutativity of coordinates of "internal degrees of freedom" space (a kind of extra-dimensions model)

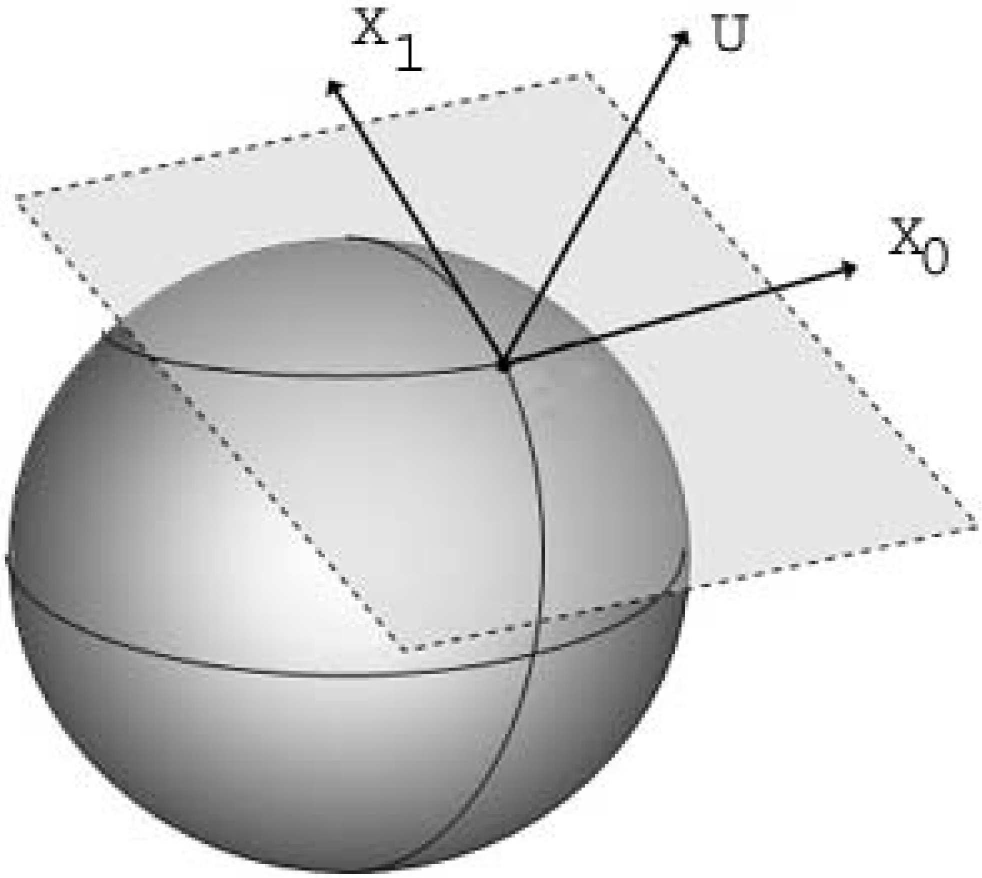
*Vector bundles* in the noncommutative world

- $M$ compact smooth manifold, $E$ vector bundle: space of smooth sections $\mathcal{C}^\infty(M, E)$ is a module over $\mathcal{C}^\infty(M)$

- The module $\mathcal{C}^\infty(M, E)$ over $\mathcal{C}^\infty(M)$ is finitely generated and projective (i.e. a vector bundle $E$ is a direct summand of some trivial bundle)

- Example: $TS^2 \oplus NS^2$ tangent and normal bundle give a trivial rk 3 bundle

- *Serre–Swan theorem*: any finitely generated projective module over $\mathcal{C}^\infty(M)$ is $\mathcal{C}^\infty(M, E)$ for some vector bundle $E$ over $M$




Tangent and normal bundle of $S^2$ add to trivial rank 3 bundle: more generally by Serre–Swan's theorem all vector bundles are summands of some trivial bundle

*Conclusion*: algebraic description of vector bundles as finite projective modules over the algebra of functions

See details (for smooth manifold case) in
Jet Nestruev, *Smooth manifolds and observables*, GTM Springer, Vol.220, 2003

*Vector bundles over a noncommutative space*:

- Only have the algebra $\mathcal{A}$ noncommutative, not the geometric space (usually not enough two-sided ideals to even have points of space in usual sense)

- Define vector bundles purely in terms of the algebra: $\mathcal{E}$ = finitely generated projective (left)-module over $\mathcal{A}$

*Connections on vector bundles*

- $\mathcal{E}$ finitely generated projective module over (noncommutative) algebra $\mathcal{A}$

- Suppose have differential graded algebra $(\Omega^\bullet, d)$, $d^2 = 0$ and
$$d(\alpha_1 \alpha_2) = d(\alpha_1)\alpha_2 + (-1)^{\deg(\alpha_1)} \alpha_1 d(\alpha_2)$$
with homomorphism $\mathcal{A} \to \Omega^0$
(hence bimodule)

- connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$ Leibniz rule
$$\nabla(\eta a) = \nabla(\eta) a + \eta \otimes da$$
for $a \in \mathcal{A}$ and $\eta \in \mathcal{E}$

**Spin Geometry**
(approach to Riemannian geometry in NCG)

*Spin manifold*

- Smooth $n$-dim manifold $M$ has tangent bundle $TM$

- Riemannian manifold (orientable): orthonormal frame bundle $FM$ on each fiber $E_x$ inner product space with oriented orthonormal basis

- $FM$ is a principal $SO(n)$-bundle

- Principal $G$-bundle: $\pi : P \to M$ with $G$-action $P \times G \to P$ preserving fibers $\pi^{-1}(x)$ on which free transitive (so each fiber $\pi^{-1}(x) \simeq G$ and base $M \simeq P/G$)

- Fundamental group $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$ so double cover universal cover:

$$Spin(n) \to SO(n)$$

- Manifold $M$ is *spin* if orthonormal frame bundle $FM$ lifts to a principal $Spin(n)$-bundle $PM$

- *Warning*: not all compact Riemannian manifolds are spin: there are topological obstruction

- In dimension $n = 4$ not all spin, but all at least $spin^{\mathbb{C}}$

- $spin^{\mathbb{C}}$ weaker form than spin: lift exists after tensoring $TM$ with a line bundle (or square root of a line bundle)

  $1 \to \mathbb{Z}/2\mathbb{Z} \to Spin^{\mathbb{C}}(n) \to SO(n) \times U(1) \to 1$

*Spinor bundle*

- Spin group $Spin(n)$ and Clifford algebra: vector space $V$ with quadratic form $q$

$$Cl(V,q) = T(V)/I(V,q)$$

tensor algebra mod ideal gen by $uv + vu = 2\langle u, v\rangle$ with $\langle u, v\rangle = (q(u+v) - q(u) - q(v))/2$

- Spin group is subgroup of group of units

$$Spin(V,q) \hookrightarrow \mathrm{GL}_1(Cl(V,q))$$

elements $v_1 \cdots v_{2k}$ prod of even number of $v_i \in V$ with $q(v_i) = 1$

- $Cl^{\mathbb{C}}(\mathbb{R}^n)$ complexification of Clifford alg of $\mathbb{R}^n$ with standard inn prod: unique min dim representation $\dim \Delta_n = 2^{[n/2]} \Rightarrow$ rep of $Spin(n)$ on $\Delta_n$ not factor through $SO(n)$

- Associated vector bundle of a principal $G$-bundle: $V$ linear representation $\rho : G \to GL(V)$ get vector bundle $E = P \times_G V$ (diagonal action of $G$)

- *Spinor bundle* $\mathbb{S} = P \times_\rho \Delta_n$ on spin manifold $M$

- *Spinors* sections $\psi \in \mathcal{C}^\infty(M, \mathbb{S})$

- Module over $\mathcal{C}^\infty(M)$ and also action by forms (Clifford multiplication) $c(\omega)$

- as vector space $Cl(V, q)$ same as $\Lambda^\bullet(V)$ not as algebra: under this vector space identification Clifford multiplication by a diff form

*Dirac operator*

- first order linear differential operator (elliptic on $M$ compact): "square root of Laplacian"

- $\gamma_a = c(e_a)$ Clifford action o.n.basis of $(V, q)$

- even dimension $n = 2m$: $\gamma = (-i)^m \gamma_1 \cdots \gamma_n$ with $\gamma^* = \gamma$ and $\gamma^2 = 1$ sign

$$\frac{1+\gamma}{2} \quad \text{and} \quad \frac{1-\gamma}{2}$$

orthogonal projections: $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$

- Spin connection $\nabla^S : \mathbb{S} \to \mathbb{S} \otimes \Omega^1(M)$

$$\nabla^S(c(\omega)\psi) = c(\nabla\omega)\psi + c(\omega)\nabla^S\psi$$

for $\omega \in \Omega^1(M)$ and $\psi \in \mathcal{C}^\infty(M, \mathbb{S})$ and $\nabla =$ Levi-Civita connection

- Dirac operator $\not{D} = -ic \circ \nabla^S$

$$\not{D} : \mathbb{S} \xrightarrow{\nabla^S} \mathbb{S} \otimes_{\mathcal{C}^\infty(M)} \Omega^1(M) \xrightarrow{-ic} \mathbb{S}$$

- $\not{D}\psi = -ic(dx^\mu)\nabla^S_{\partial_\mu}\psi = -i\gamma^\mu \nabla^S_\mu \psi$

- Hilbert space $\mathcal{H} = L^2(M, \mathbb{S})$ square integrable spinors

$$\langle \psi, \xi \rangle = \int_M \langle \psi(x), \xi(x) \rangle_x \sqrt{g}\, d^n x$$

- $\mathcal{C}^\infty(M)$ acting as bounded operators on $\mathcal{H}$ (Note: $M$ compact)

- Commutator: $[\not{D}, f]\psi = -ic(\nabla^S(f\psi)) + ifc(\nabla^S\psi)$

  $= -ic(\nabla^s(f\psi) - f\nabla^S\psi) = -ic(df \otimes \psi) = -ic(df)\psi$

  $[\not{D}, f] = -ic(df)$ bounded operator on $\mathcal{H}$ ($M$ compact)

*Analytic properties of Dirac on* $\mathcal{H} = L^2(M, \mathbb{S})$ on a compact Riemannian $M$

- Unbounded operator
- Self adjoint: $\not{D}^* = \not{D}$ with dense domain
- Compact resolvent: $(1+\not{D}^2)^{-1/2}$ is a compact operator (if no kernel $\not{D}^{-1}$ compact)
- Lichnerowicz formula: $\not{D}^2 = \Delta^S + \frac{1}{4} R$ with $R$ scalar curvature and Laplacian

$$\Delta^S = -g^{\mu\nu}(\nabla^S_\mu \nabla^S_\nu - \Gamma^\lambda_{\mu\nu} \nabla^S_\lambda)$$

*Main Idea:* abstract these properties into an algebraic definition of Dirac on NC spaces

*How to get metric $g_{\mu\nu}$ from Dirac $\not{D}$*

- Geodesic distance on $M$: length of curve $\ell(\gamma)$, piecewise smooth curves

$$d(x,y) = \inf_{\substack{\gamma:[0,1]\to M \\ \gamma(0)=x,\, \gamma(1)=y}} \{\ell(\gamma)\}$$

- *Myers–Steenrod theorem*: metric $g_{\mu\nu}$ uniquely determined from geodesic distance

- Show that geodesic distance can be computed using Dirac operator and algebra of functions

- $f \in \mathcal{C}(M)$ have

$$|f(x) - f(y)| \leq \int_0^1 |\nabla f(\gamma(t))|\, |\dot{\gamma}(t)|\, dt$$

$$\leq \|\nabla f\|_\infty \int_0^1 |\dot{\gamma}(t)|\, dt = \|\nabla f\|_\infty \ell(\gamma) = \|[\not{D}, f]\| \ell(\gamma)$$

- $|f(x) - f(y)| \leq \|[\not{D}, f]\| \ell(\gamma)$ gives

$$\sup_{f:\|[\not{D},f]\|\leq 1} \{|f(x) - f(y)|\} \leq \inf_{\gamma} \ell(\gamma) = d(x,y)$$

- Note: sup over $f \in \mathcal{C}^\infty(M)$ or over $f \in Lip(M)$ Lipschitz functions

$$|f(x) - f(y)| \leq C d(x,y)$$

- Take $f_x(y) = d(x,y)$ Lipschitz with

$$|f_x(y) - f_x(z)| \leq d(y,z)$$

(triangle inequality)

- $[\not{D}, f_x] = -ic(df_x)$ and $|\nabla f_x| = 1$, then $|f_x(y) - f_x(x)| = f_x(y) = d(x,y)$ realizes sup

- *Conclusion*: distance from Dirac

$$d(x,y) = \sup_{f:\|[\not{D},f]\|\leq 1} \{|f(x) - f(y)|\}$$

**Some references for Spin Geometry**:

- H. Blaine Lawson, Marie-Louise Michelsohn, *Spin Geometry*, Princeton 1989

- John Roe, *Elliptic Operators, Topology, and Asymptotic Methods*, CRC Press, 1999

*Spin Geometry and NCG, Dirac and distance*:

- Alain Connes, *Noncommutative Geometry*, Academic Press, 1995

- José M. Gracia-Bondia, Joseph C. Varilly, Hector Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, 2013

**Spectral triples**: abstracting Spin Geometry

- involutive algebra $\mathcal{A}$ with representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
- self adjoint operator $D$ on $\mathcal{H}$, dense domain
- compact resolvent $(1 + D^2)^{-1/2} \in \mathcal{K}$
- $[a, D]$ bounded $\forall a \in \mathcal{A}$
- *even* if $\mathbb{Z}/2$- grading $\gamma$ on $\mathcal{H}$

$$[\gamma, a] = 0, \ \forall a \in \mathcal{A}, \qquad D\gamma = -\gamma D$$

*Main example*: $(\mathcal{C}^\infty(M), L^2(M, \mathbb{S}), \not{D})$ with chirality $\gamma = (-i)^m \gamma_1 \cdots \gamma_n$ in even-dim $n = 2m$

Alain Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. 34 (1995), no. 3, 203–238.

**Real Structures** in Spin Geometry

- Clifford algebra $Cl(V, q)$ non-degenerate quadratic form of signature $(p, q)$, $p + q = n$

- $Cl_n^+ = Cl(\mathbb{R}^n, g_{n,0})$ and $Cl_n^- = Cl(\mathbb{R}^n, g_{0,n})$

- Periodicity: $Cl_{n+8}^{\pm} = Cl_n^{\pm} \otimes M_{16}(\mathbb{R})$

- Complexification: $Cl_n^{\pm} \subset \mathbb{C}l_n = Cl_n^{\pm} \otimes_{\mathbb{R}} \mathbb{C}$

| **n** | $Cl_n^+$ | $Cl_n^-$ | $\mathbb{C}l_n$ | $\Delta_n$ |
|---|---|---|---|---|
| 1 | $\mathbb{R} \oplus \mathbb{R}$ | $\mathbb{C}$ | $\mathbb{C} \oplus \mathbb{C}$ | $\mathbb{C}$ |
| 2 | $M_2(\mathbb{R})$ | $\mathbb{H}$ | $M_2(\mathbb{C})$ | $\mathbb{C}^2$ |
| 3 | $M_2(\mathbb{C})$ | $\mathbb{H} \oplus \mathbb{H}$ | $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ | $\mathbb{C}^2$ |
| 4 | $M_2(\mathbb{H})$ | $M_2(\mathbb{H})$ | $M_4(\mathbb{C})$ | $\mathbb{C}^4$ |
| 5 | $M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$ | $M_4(\mathbb{C})$ | $M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$ | $\mathbb{C}^4$ |
| 6 | $M_4(\mathbb{H})$ | $M_8(\mathbb{R})$ | $M_8(\mathbb{C})$ | $\mathbb{C}^8$ |
| 7 | $M_8(\mathbb{C})$ | $M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$ | $M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$ | $\mathbb{C}^8$ |
| 8 | $M_{16}(\mathbb{R})$ | $M_{16}(\mathbb{R})$ | $M_{16}(\mathbb{C})$ | $\mathbb{C}^{16}$ |

- Both real Clifford algebra and complexification act on spinor representation $\Delta_n$.

- $\exists$ antilinear $J : \Delta_n \to \Delta_n$ with $J^2 = 1$ and $[J, a] = 0$ for all $a$ in real algebra $\Rightarrow$ real subbundle $Jv = v$

- antilinear $J$ with $J^2 = -1$ and $[J, a] = 0$ $\Rightarrow$ quaternion structure

- real algebra: elements $a$ of complex algebra with $[J, a] = 0$, $JaJ^* = a$.

## Real Structures on Spectral Triples

$KO$-dimension $n \in \mathbb{Z}/8\mathbb{Z}$

antilinear isometry $J : \mathcal{H} \to \mathcal{H}$

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon'' \gamma J$$

| **n** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| $\varepsilon'$ | 1 | -1 | 1 | 1 | 1 | -1 | 1 | 1 |
| $\varepsilon''$ | 1 | | -1 | | 1 | | -1 | |

*Commutation*: $[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}$
where $b^0 = J b^* J^{-1} \qquad \forall b \in \mathcal{A}$

*Order one condition*:

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}$$

## **Finite Spectral Triples** $F = (\mathcal{A}_F, \mathcal{H}_F, D_F)$

- $\mathcal{A}$ finite dimensional (real) $C^*$-algebra

$$\mathcal{A} = \oplus_{i=1}^{N} M_{n_i}(\mathbb{K}_i)$$

$\mathbb{K}_i = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ quaternions (Wedderburn)

- Representation on finite dimensional Hilbert space $\mathcal{H}$, with bimodule structure given by $J$ (condition $[a, b^0] = 0$)

- $D_F^* = D_F$ with order one condition

$$[[D_F, a], b^0] = 0$$

- No analytic conditions: $D_F$ just a matrix

$\Rightarrow$ *Moduli spaces* (under unitary equivalence)

Branimir Ćaćić, *Moduli spaces of Dirac operators for finite spectral triples*, arXiv:0902.2068