

Solvmanifolds and noncommutative tori with real multiplication

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Based on

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Background:

- Hilbert modular surfaces

$$X = (\mathbb{H} \times \mathbb{H})/\Gamma$$

$\Gamma = \mathrm{SL}_2(\mathcal{O}_{\mathbb{K}})$, real quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{d})$

X has isolated singularities: link 3-dim solvmanifolds

- 1970s: Hirzebruch conjecture: signature defects

$$\sigma(X) = \int_X L - \eta(0)$$

$$\eta(0) = L(\Lambda', V, 0)$$

Shimizu L -function (for a lattice Λ with action of units V of \mathbb{K})

$$L(\Lambda, V, s) = \sum_{\mu \in (\Lambda \setminus \{0\})/V} \text{sign}(N(\mu)) |N(\mu)|^{-s}$$

norm $N(\mu) = \mu \mu'$ product with Galois conjugate

- 1983: proved Atiyah–Donnelly–Singer using Atiyah–Patodi–Singer index theorem for Dirac-type operator

$$\eta(s) = \sum_{\lambda \neq 0} \text{sign}(\lambda) |\lambda|^{-s}$$

(also higher rank totally real fields)

Atiyah's question: (2007)

- Eta function $\eta(s)$ splits as $L(\Lambda, V, s)$ plus “stuff” that does not matter in the $\eta(0)$ calculation
- Does the arithmetic part of the eta function come from an underlying *noncommutative geometry*?
- Is the 3-dimensional solvmanifold a “model up to homotopy” of this noncommutative geometry?

Answer: Yes!

Notation:

- $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ real quadratic field
- $\alpha_i : \mathbb{K} \hookrightarrow \mathbb{R}$, $i = 1, 2$ real embeddings
- $L \subset \mathbb{K}$ lattice \Rightarrow lattice $\Lambda \subset \mathbb{R}^2$

$$L \ni \ell \mapsto (\alpha_1(\ell), \alpha_2(\ell)) \subset \mathbb{R}^2$$

$$\Lambda = (\alpha_1, \alpha_2)(L)$$

- $V = \epsilon^{\mathbb{Z}}$ totally positive units preserving L

$$V = \{u \in O_{\mathbb{K}}^* \mid uL \subset L, \alpha_i(u) \in \mathbb{R}_+^*\}$$

($L = O_{\mathbb{K}} \Rightarrow \epsilon =$ fundamental unit)

- Action of V on Λ (tot. pos. units $N(\epsilon) = 1$)

$$\lambda = (\alpha_1(\ell), \alpha_2(\ell)) \mapsto (\epsilon\alpha_1(\ell), \epsilon'\alpha_2(\ell)) = (\epsilon\alpha_1(\ell), \epsilon^{-1}\alpha_2(\ell))$$

The 3-dimensional solvmanifold

$$S(\Lambda, V) = \Lambda \rtimes_{\epsilon} V$$

action of V on Λ by

$$A_{\epsilon} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

$$S(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}^2 \rtimes \mathbb{R}$$

action of \mathbb{R} on \mathbb{R}^2 by

$$\Theta_t(x, y) = (e^t x, e^{-t} y), \quad \Theta_t \in \mathrm{SL}_2(\mathbb{R})$$

$$X_{\epsilon} = S(\Lambda, V) \backslash S(\mathbb{R}^2, \mathbb{R})$$

$$\pi_1(X_{\epsilon}) = S(\Lambda, V)$$

Topology

$$H_1(X_\epsilon, \mathbb{Z}) = \Lambda / (1 - A_\epsilon)\Lambda \oplus \mathbb{Z}$$

abelianization of $\pi_1(X_\epsilon) = S(\Lambda, V)$

Surjective homomorphism $\pi(\lambda, n) = (\lambda \bmod (1 - A_\epsilon)\Lambda, n)$, Kernel: commutators

Poincaré duality

$$H^0(X_\epsilon, \mathbb{Z}) = \mathbb{Z} \quad H^1(X_\epsilon, \mathbb{Z}) = \mathbb{Z}$$

$$H^3(X_\epsilon, \mathbb{Z}) = \mathbb{Z} \quad H^2(X_\epsilon, \mathbb{Z}) = \mathbb{Z} \oplus \text{Coker}(1 - A_\epsilon)$$

$\text{Ker}(1 - A_\epsilon) = 0$ and $\text{Coker}(1 - A_\epsilon)$ torsion

Action: (equivalent formulation)

Basis $\{1, \theta\}$ of $\alpha_1(L) \subset \mathbb{R}$

$$\varphi_\epsilon = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

with $\epsilon = a + b\theta$, $\epsilon\theta = c + d\theta$

Fixed points $1/\theta$ and $1/\theta'$

Identifications

$$S(\Lambda, V) = \mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}$$

$$(\lambda, \epsilon^k) \mapsto (\lambda_1 = n + m\theta, k)$$

$$(\lambda, \epsilon^k) \mapsto (\lambda_2 = n + m\theta', -k)$$

Solvmanifold and universal family

\mathbb{H} = hyperbolic plane, \mathcal{U} = universal family

$$\mathcal{U} \xrightarrow{\pi} \mathbb{H}, \quad \pi^{-1}(\tau) = E_\tau$$

elliptic curves, up to equivalence descends to

$$\mathcal{U}/\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$$

Galois conjugate θ, θ' for lattice Λ

Geodesic $\ell_{\theta, \theta'} \Rightarrow$ closed geodesic in $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$

$$\mathrm{length}(\bar{\ell}_{\theta, \theta'}) = \log \epsilon$$

unit generating V acting on Λ

$X_\epsilon =$ restriction of $\mathcal{U}/\mathrm{SL}_2(\mathbb{Z})$ to geodesic $\bar{\ell}_{\theta, \theta'}$

Noncommutative tori with real multiplication

$$VU = e^{2\pi i\theta} UV$$

C^* -algebra \mathcal{A}_θ

Morita equivalences (Connes, Rieffel)

$$\mathcal{A}_{\theta_1} \simeq \mathcal{A}_{\theta_2} \Leftrightarrow g \in \mathrm{SL}_2(\mathbb{Z})$$

$$\theta_1 = g\theta_2$$

Quadratic irrationalities: $\mathbb{Q}(\theta)$ real quadratic

$$\exists g \in \mathrm{SL}_2(\mathbb{Z}), \quad g\theta = \theta$$

Self Morita equivalences

Manin: (2005) parallel to theory of elliptic curves with complex multiplication

Twisted group algebras

- Γ finitely generated discrete group

$$\sigma : \Gamma \times \Gamma \rightarrow U(1)$$

multiplier, 2-cocycle: $\sigma(\gamma, 1) = \sigma(1, \gamma) = 1$ and

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3)$$

- Hilbert space $\ell^2(\Gamma)$ left/right σ -regular representations

$$L_\gamma^\sigma f(\gamma') = f(\gamma^{-1}\gamma')\sigma(\gamma, \gamma^{-1}\gamma')$$

$$R_\gamma^\sigma f(\gamma') = f(\gamma'\gamma)\sigma(\gamma', \gamma)$$

$$L_\gamma^\sigma L_{\gamma'}^\sigma = \sigma(\gamma, \gamma')L_{\gamma\gamma'}^\sigma, \quad R_\gamma^\sigma R_{\gamma'}^\sigma = \sigma(\gamma, \gamma')R_{\gamma\gamma'}^\sigma$$

left σ -regular commutes with right $\bar{\sigma}$ -regular

- $\mathbb{C}(\Gamma, \sigma) =$ twisted group ring, gen by R_γ^σ
 $C_r^*(\Gamma, \sigma) =$ norm closure, (reduced) twisted group C^* -algebra

NC tori as twisted group algebras

$$\mathcal{A}_\theta = C_r^*(\mathbb{Z}^2, \sigma)$$

$$\sigma((n, m), (n', m')) = \exp(-2\pi i(\xi_1 nm' + \xi_2 mn'))$$

with $\theta = \xi_2 - \xi_1$

generators $U = R_{(0,1)}^\sigma$ and $V = R_{(1,0)}^\sigma$

$$Uf(n, m) = e^{-2\pi i\xi_2 n} f(n, m + 1)$$

$$Vf(n, m) = e^{-2\pi i\xi_1 m} f(n + 1, m)$$

$$UV = e^{2\pi i\theta} VU$$

Some freedom to choose ξ_1, ξ_2

Invariance under $SL_2(\mathbb{Z})$: cocycle $\sigma((n, m), (n', m'))$ satisfies

$$\sigma((n, m), (n', m')) = \sigma((n, m)\varphi, (n', m')\varphi)$$

for all $\varphi \in SL_2(\mathbb{Z})$ iff

$$\xi_2 = -\xi_1$$

\Rightarrow Set $\xi_2 = \theta/2 = -\xi_1$

$$\begin{aligned}\sigma_\theta((n, m), (k, r)) &= \exp(\pi i \theta (nr - mk)) \\ &= \exp(\pi i \theta (n, m) \wedge (k, r))\end{aligned}$$

for

$$(a, b) \wedge (c, d) := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

NC tori

$$\mathbb{T}_{\Lambda,1} = C^*(\mathbb{Z}^2, \sigma_\theta) = C^*(\Lambda, \sigma_{\theta(\theta' - \theta)^{-1}})$$

$$\mathbb{T}_{\Lambda,2} = C^*(\mathbb{Z}^2, \sigma_{\theta'}) = C^*(\Lambda, \sigma_{\theta'(\theta' - \theta)^{-1}})$$

For elliptic curves: natural object E/O_K^*

For NC tori: V acts on $\mathbb{T}_{\Lambda,i}$ (automorphisms)

$$v_\epsilon^k(R_{(n,m)}^\sigma) = R_{(n,m)\varphi_\epsilon^k}^\sigma$$

$$v_\epsilon^k(R_\lambda^\sigma) = R_{A_\epsilon^k(\lambda)}^\sigma$$

Crossed product $\mathbb{T}_{\Lambda,i} \rtimes_i V$

Note: in NCG crossed product replaces quotient $\mathbb{T}_\theta/\text{Aut}(\mathbb{T}_\theta)$

Twisted group algebra of $S(\Lambda, V)$

Given $\sigma((n, m), (k, r)) = \exp(\pi i \theta(nr - mk))$

$$\tilde{\sigma}((n, m, k), (n', m', k')) := \sigma((n, m), (n', m')) \varphi_\epsilon^k$$

is a multiplier for $S(\Lambda, V)$

$$C^*(\Lambda, \sigma) \rtimes V = C^*(S(\Lambda, V), \tilde{\sigma})$$

$$R_{(n,m,k)}^{\tilde{\sigma}} \mapsto R_{(n,m)}^\sigma v_\epsilon^k$$

$$\mathbb{T}_{\Lambda, V, i} = \mathbb{T}_{\Lambda, i} \rtimes_i V$$

$$\mathbb{T}_{\Lambda, V, 1} = C^*(\mathbb{Z}^2, \sigma_\theta) \rtimes_{v_1} \mathbb{Z} = C^*(\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}, \tilde{\sigma}_\theta) =$$

$$C^*(\Lambda, \sigma_{\theta(\theta' - \theta)^{-1}}) \rtimes_{v_1} V = C^*(S(\Lambda, V), \tilde{\sigma}_{\theta(\theta' - \theta)^{-1}})$$

$$\mathbb{T}_{\Lambda, V, 2} = C^*(\mathbb{Z}^2, \sigma_{\theta'}) \rtimes_{v_2} \mathbb{Z} = C^*(\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}, \tilde{\sigma}_{\theta'}) =$$

$$C^*(\Lambda, \sigma_{\theta'(\theta' - \theta)^{-1}}) \rtimes_{v_2} V = C^*(S(\Lambda, V), \tilde{\sigma}_{\theta'(\theta' - \theta)^{-1}})$$

Note: $S(\Lambda, V)$ amenable: $C_r^*(S(\Lambda, V), \tilde{\sigma}) = C_{\max}^*(S(\Lambda, V), \tilde{\sigma})$

Homotopy quotient (Baum–Connes)

- \mathcal{A} noncommutative C^* -algebra (NC space)
- X topological space: is a homotopy quotient model of \mathcal{A} if (analytic) K -theory of \mathcal{A} computed in terms of topological K -theory of X , through a natural map (Kasparov assembly map)

Claim: $X_\epsilon =$ homotopy quotient model for $C^*(S(\Lambda, V), \tilde{\sigma})$

To check: K-theory computation for $\mathcal{A} = C^*(\Lambda, \sigma)$

$$K_0(C^*(S(\Lambda, V), \tilde{\sigma})) \cong \Lambda$$

$$K_1(C^*(S(\Lambda, V), \tilde{\sigma})) \cong \Lambda \oplus \Lambda / (1 - A_\epsilon)\Lambda$$

(Pimsner–Voiculescu six terms exact sequence)

$$\begin{array}{ccccc} K_0(\mathcal{A}) & \xrightarrow{1-\alpha_*} & K_0(\mathcal{A}) & \longrightarrow & K_0(\mathcal{A} \rtimes \mathbb{Z}) \\ & & & & \downarrow \partial \\ & \uparrow \partial & & & \\ K_1(\mathcal{A} \rtimes \mathbb{Z}) & \longleftarrow & K_1(\mathcal{A}) & \xleftarrow{1-\beta_*} & K_1(\mathcal{A}) \end{array}$$

Twist: no effect on K-theory

$$K_i(C^*(S(\Lambda, V), \tilde{\sigma})) \cong K_i(C^*(S(\Lambda, V)))$$

use homotopy $\exp(-t\pi i\theta(mn' - nm'))$, $t \in [0, 1]$

Kasparov assembly map isomorphism

$$\mu : K^1(X_\epsilon) \xrightarrow{\cong} K_0(C^*(S(\Lambda, V)))$$

$$\mu : K^0(X_\epsilon) \xrightarrow{\cong} K_1(C^*(S(\Lambda, V)))$$

(mapping torus, Thom isomorphism)

$$K_{i+1}(C(X_\epsilon)) = K_i(C(T^2) \rtimes_{A_\epsilon} \mathbb{Z})$$

$\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}$ satisfies BC (Chabert-Echteroff)

Conclusion: Solvmanifold X_ϵ is a model up to homotopy of NC space $\mathbb{T}_\theta / \text{Aut}(\mathbb{T}_\theta)$ (analog of $E / \text{Aut}(E)$ for elliptic curves)

So far topology: what about geometry?

An arithmetic quantum Hall effect

- discrete group Γ acting on contractible space \tilde{X} w/ compact quotient $X = \tilde{X}/\Gamma$

- base point $x_0 \in \tilde{X}$, crystal charged ions Γx_0

\Rightarrow model electron-ion interactions

- Hamiltonian $\Delta + \mathcal{V}$ on $L^2(\tilde{X})$ with $T_\gamma \Delta = \Delta T_\gamma$ and \mathcal{V} invariant

- Magnetic field: closed 2-form ω w/ $\gamma^* \omega = \omega$

- global magnetic potential $\omega = d\chi$, hermitian connection $\nabla = d - i\chi$ with $\nabla^2 = i\omega$

$$\chi - \gamma^* \chi = d\phi_\gamma, \quad \phi_\gamma(x) = \int_{x_0}^x \chi - \gamma^* \chi$$

$$\phi_\gamma(x) - \phi_{\gamma'}(\gamma x) - \phi_{\gamma\gamma'}(x)$$

independent of $x \in \tilde{X}$ and $\phi_\gamma(x_0) = 0$

$$\sigma(\gamma, \gamma') = \exp(-i\phi_\gamma(\gamma' x_0))$$

multiplier $\sigma : \Gamma \times \Gamma \rightarrow U(1)$

After turning on magnetic field, ordinary Laplacian not invariant under magnetic translations: magnetic potential modifies connection $\nabla = d - i\chi$

Magnetic Laplacian:

$$\Delta^{\chi} = \nabla^* \nabla = (d - i\chi)^*(d - i\chi)$$

now invariant under magnetic translations

$$T_{\gamma}^{\phi} \Delta^{\chi} = \Delta^{\chi} T_{\gamma}^{\phi}$$

$$T_{\gamma}^{\phi} f(x) = \exp(i\phi_{\gamma}(x)) f(\gamma^{-1}x)$$

Independent electron approximation: \mathcal{V} effective potential invariant under T_{γ}^{ϕ}

Discrete model: Hilbert space $\ell^2(\Gamma)$

$$\mathcal{R} = \sum_{i=1}^r R_{\gamma_i} \quad \text{with} \quad R_{\gamma_i} f(\gamma) = f(\gamma\gamma_i)$$

Laplacian $\Rightarrow \Delta_{discr} = r - \mathcal{R}$ random walk operator
($r = \#$ symm set of generators)

Harper operator

$$\mathcal{H}_\sigma = \sum_{i=1}^r R_{\gamma_i}^\sigma, \quad R_{\gamma_i}^\sigma \in \mathbb{C}(\Gamma, \sigma)$$

$$\Delta_{discr}^\chi = r - \mathcal{H}_\sigma$$

magnetic Laplacian; potential $\mathcal{V} \in \mathbb{C}(\Gamma, \sigma)$

Harper operators and noncommutative tori

$$\mathcal{H}_\sigma = U + U^* + V + V^*$$

(spectral theory: Hofstadter butterfly, rational/irrational θ)

For X_ϵ Harper operator: $\mathcal{H}_{\tilde{\sigma}} = U + U^* + V + V^* + W + W^*$ with $U = R_{(0,1,0)}^{\tilde{\sigma}}$, $V = R_{(1,0,0)}^{\tilde{\sigma}}$ and $W = R_{(0,0,1)}^{\tilde{\sigma}}$

Problem: band structure, gaps in the spectrum (detected by range of the trace on K -theory)

$\mathcal{U}(\Gamma, \sigma) =$ von Neumann alg: weak closure $\mathbb{C}(\Gamma, \sigma)$

$$P_E = 1_{(-\infty, E]}(\mathcal{H}_{\sigma, V}) \in \mathcal{U}(\Gamma, \sigma)$$

spectral projections of $\mathcal{H}_{\sigma, V}$ satisfy $P_E \in C_r^*(\Gamma, \sigma)$ when E in a gap in the spectrum

\Rightarrow Counting of projections in $C_r^*(\Gamma, \sigma)$

\Leftrightarrow count proj's mod equiv

$$[\mathrm{tr}] : K_0(C_r^*(\Gamma, \sigma)) \rightarrow \mathbb{R}$$

where $\tau(a) = \langle a\delta_1, \delta_1 \rangle_{\ell^2(\Gamma)}$ on $C_r^*(\Gamma, \sigma)$ and

$$\mathrm{tr} = \tau \otimes \mathrm{Tr} : \{P \in C_r^*(\Gamma, \sigma) \otimes \mathcal{K} \mid P^* = P, P^2 = P\} \rightarrow \mathbb{R}$$

Estimate by: $[\mathrm{tr}](K_0(C_r^*(\Gamma, \sigma))) \cap [0, 1]$

Compute via a **(twisted) index theorem**:

$$[\mathrm{tr}](K_0(C^*(S(\Lambda, V), \tilde{\sigma}))) = \mathbb{Z} + \mathbb{Z}\theta(\theta' - \theta)^{-1}$$

Spectral flow: $K_1(C(X_\epsilon)) \ni [g], g : X_\epsilon \rightarrow GL_N(\mathbb{C})$

$$\beta(g) = g^{-1}dg \in \Omega^1(X_\epsilon, gl_N(\mathbb{C}))$$

connections $\nabla_u = d + u\beta(g)$ on $X_\epsilon \times \mathbb{C}^N \Rightarrow$ Chern–Simons form
(odd Chern character)

$$Ch(g) := cs(d, d + \beta(g)) = \int_0^1 \text{Tr}\left(\frac{d}{du}(\nabla_u)e^{\nabla_u^2}\right)du$$

$$Ch(g) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \text{Tr}(\beta(g)^{2k+1})$$

Odd Fredholm module pairing w/ $[g] \in K_1$

$$\langle D, [g] \rangle = SF(D, g^{-1}Dg)$$

Dirac operator \Rightarrow APS (mapping torus)

$$SF(\not{D}, g^{-1}\not{D}g) = -\frac{1}{(2\pi i)^2} \int_{X_\epsilon} \hat{A}(X_\epsilon) Ch(g)$$

Twisted index theorem:

range of trace on $K_0(C^*(S(\Lambda, V), \tilde{\sigma}))$

$$[\text{tr}](\mu_{\tilde{\sigma}}[g]) = \frac{-1}{(2\pi i)^2} \int_{X_\epsilon} \hat{A} e^{\omega_\epsilon} \text{Ch}(g)$$

$$\mu_{\tilde{\sigma}} : K^1(X_\epsilon) \rightarrow K_0(C^*(S(\Lambda, V), \tilde{\sigma}))$$

twisted Kasparov isomorphism $[g] \in K^1(X_\epsilon)$

ω_ϵ closed 2-form on X_ϵ of cocycle $\tilde{\sigma}$:

2 cocycle: $\tilde{\sigma} = \exp(2\pi i \zeta)$

$$\zeta((\lambda, k), (\eta, r)) = \frac{1}{4\pi i} \int_{\mathcal{R}} \omega$$

ω closed 2-form on $T^2 = \mathbb{R}^2/\Lambda$ flux

$$\int_{T^2} \omega = 2\pi i \theta (\theta' - \theta)^{-1}$$

parallelogram $\mathcal{R} = \{0, A_\epsilon^k(\eta), \lambda, \lambda + A_\epsilon^k(\eta)\}$

Sketch of proof:

$P^\pm =$ projs on L^2 -kernel of $\mathcal{D}_g \mathcal{D}_g^*$ and $\mathcal{D}_g^* \mathcal{D}_g$

$$\mathcal{D}_g P^+ = 0 \quad \mathcal{D}_g^* P^- = 0$$

$$\mathcal{D}_g = \frac{\partial}{\partial u} + D_u$$

on $\tilde{X}_\epsilon \times [0, 1]$ with $D_u = \tilde{\partial}_u \otimes \nabla$

$$\tilde{\partial}_u = (1-u)\tilde{\partial} + u g^{-1} \tilde{\partial} g, \quad [g] \in K^1(X_\epsilon)$$

P^\pm have smooth kernels $P^\pm(x, y)$

$$e^{-i\phi_\gamma(x)} P^\pm(\gamma x, \gamma y) e^{i\phi_\gamma(y)} = P^\pm(x, y)$$

$\Rightarrow P^\pm(x, x)$ is $\Gamma = S(\Lambda, V)$ -invariant

$$\mathrm{tr}(P^\pm) = \int_{X_\epsilon \times S^1} \mathrm{tr} P^\pm((x, t), (x, t)) dx dt,$$

von Neumann trace, $\mathrm{tr} P^\pm(x, x)$ ptwise trace

$$\mathrm{Ind}_{L^2}(\mathcal{D}_g) = \mathrm{tr}(P^+) - \mathrm{tr}(P^-)$$

Set $\bar{P}^\pm(x, y) = \int_{S^1} \text{tr} P^\pm((x, t), (y, t)) dt$

$$\int_{X_\epsilon \times S^1} \text{tr} P^\pm((x, t), (x, t)) dx dt = \int_{X_\epsilon} \text{tr} \bar{P}^\pm(x, x) dx$$

projections \bar{P}^\pm in von Neumann algebra $\mathcal{U}(\Gamma, \tilde{\sigma})$
(after compact perturbation)

$$\text{Ind}_{(\Gamma, \tilde{\sigma})}(\mathcal{D}_g) = [\bar{P}^+] - [\bar{P}^-] \in K_0(C_r^*(\Gamma, \tilde{\sigma}))$$

Twisted Kasparov map

$$\mu_{\tilde{\sigma}}[g] = \text{Ind}_{(\Gamma, \tilde{\sigma})}(\mathcal{D}_g)$$

L^2 -index:

$$\text{Ind}_{L^2}(\mathcal{D}_g) = \text{tr}(\bar{P}^+) - \text{tr}(\bar{P}^-) = \text{tr}(\text{Ind}_{(\Gamma, \tilde{\sigma})}(\mathcal{D}_g))$$

Heat kernel e^{-tD^2}

$$D = \begin{pmatrix} 0 & D_g^* \\ D_g & 0 \end{pmatrix} \quad \text{with} \quad D^2 = \begin{pmatrix} D_g^* D_g & 0 \\ 0 & D_g D_g^* \end{pmatrix}$$

$$\lim_{t \rightarrow \infty} \text{tr}_s(e^{-tD^2}) = \text{tr}(P^+) - \text{tr}(P^-)$$

$$\frac{\partial}{\partial t} \text{tr}_s(e^{-tD^2}) = -\text{tr}_s(D^2 e^{-tD^2}) = \text{tr}_s([D e^{-tD^2}, D]) = 0$$

$$\text{tr}(P^+) - \text{tr}(P^-) = \lim_{t \rightarrow \infty} \text{tr}_s(e^{-tD^2}) = \lim_{t \rightarrow 0} \text{tr}_s(e^{-tD^2})$$

$$= \frac{-1}{(2\pi i)^2} \int_{X_\epsilon \times S^1} \hat{A} \text{Ch}(\nabla_u) = \frac{-1}{(2\pi i)^2} \int_{X_\epsilon} \hat{A} e^{\omega_\epsilon} \text{Ch}(g),$$

$$\text{Ch}(\nabla_u) = \text{tr}(\beta e^{(d+u\beta)^2}), \quad \beta = g^{-1} dg,$$

$$\int_{S^1} \text{Ch}(\nabla_u) = \text{Ch}(g)$$

Range computation:

3-manifold X_ϵ

$$\hat{A}(X_\epsilon) = 1 - \frac{1}{24}p_1(X_\epsilon) + \dots \quad e^{\omega_\epsilon} = 1 + \omega_\epsilon + \frac{1}{2}\omega_\epsilon^2 + \dots$$

$$Ch(g) = -\frac{1}{6}\text{Tr}(\beta(g)) + \frac{1}{5!}\text{Tr}(\beta^3(g)) + \dots$$

terms up to $\dim=3$

$$\frac{1}{(2\pi)^2} \int_{X_\epsilon} \left(\frac{-1}{6}\text{Tr}(\beta(g)) \wedge \omega + \frac{1}{5!}\text{Tr}(\beta(g)^3) \right)$$

$$\frac{1}{(2\pi)^2} \int_{X_\epsilon} \frac{1}{5!}\text{Tr}(\beta(g)^3) = \frac{1}{(2\pi)^2} \int_{X_\epsilon} Ch(g)$$

(untwisted odd Chern character)

$$\frac{1}{(2\pi)^2} \frac{-1}{6} \int_{X_\epsilon} \text{Tr}(\beta(g)) \wedge \omega_\epsilon \in \mathbb{Z}R(\omega)$$

$R(\omega)$ = range of linear form

$$T_\omega : [g] \mapsto \frac{1}{(2\pi)^2} \frac{-1}{6} \int_{X_\epsilon} \text{Tr}(\beta(g)) \wedge \omega_\epsilon \in \mathbb{R}$$

$$\int_C Ch_1(g) = 2\pi i \deg(g|_C) \in 2\pi i\mathbb{Z}$$

with $Ch_1(g) = \frac{1}{6} \text{Tr}(\beta(g))$ and $C \in H_1(X_\epsilon, \mathbb{Z})$

$$\frac{1}{2\pi i} \int_{X_\epsilon} Ch_1(g) \wedge PD(C) \in \mathbb{Z}$$

$$PD(C) \in H^2(X_\epsilon, \mathbb{Z}) \hookrightarrow H^2(X_\epsilon, \mathbb{R})$$

$$\omega_\epsilon = 2\pi i \theta (\theta' - \theta)^{-1} \bar{\omega}_\epsilon, \quad \bar{\omega}_\epsilon \in H^2(X_\epsilon, \mathbb{Z})$$

$$\bar{\omega}_\epsilon(v, w) = A_\epsilon^k(\eta) \wedge \lambda,$$

for $v = ((0, 0), (\lambda, k))$ and $w = ((0, 0), (\eta, r))$

$$\begin{aligned} & \frac{1}{(2\pi)^2} \int_{X_\epsilon} Ch_1(g) \wedge \omega_\epsilon \\ &= -\frac{1}{2\pi i} \theta (\theta' - \theta)^{-1} \int_{PD(\bar{\omega}_\epsilon)} Ch_1(g) \\ &= \theta (\theta' - \theta)^{-1} \deg(g|_{PD(\bar{\omega}_\epsilon)}) \in \theta (\theta' - \theta)^{-1} \mathbb{Z} \end{aligned}$$

Dirac operator on X_ϵ

$\{dt, e^t dx, e^{-t} dy\}$ basis of cotangent bundle

of $S(\mathbb{R}^2, \mathbb{R}, \epsilon) = \mathbb{R}^2 \rtimes_\epsilon \mathbb{R}$

$$\not{D}_{X_\epsilon} = c(dt) \frac{\partial}{\partial t} + c(e^t dx) \frac{\partial}{\partial x} + c(e^{-t} dy) \frac{\partial}{\partial y}$$

$$\not{D}_{X_\epsilon} = \frac{\partial}{\partial t} \sigma_0 + e^t \frac{\partial}{\partial x} \sigma_1 + e^{-t} \frac{\partial}{\partial y} \sigma_2$$

$$\not{D}_{X_\epsilon} = \begin{pmatrix} \frac{\partial}{\partial t} & e^{-t} \frac{\partial}{\partial y} - ie^t \frac{\partial}{\partial x} \\ e^{-t} \frac{\partial}{\partial y} + ie^t \frac{\partial}{\partial x} & -\frac{\partial}{\partial t} \end{pmatrix}$$

σ_i , for $i = 0, 1, 2$ Pauli matrices

Commutative spectral triple

$$(C^\infty(X_\epsilon), L^2(X_\epsilon, S), \not{D}_{X_\epsilon})$$

Isospectral deformations (Connes–Landi)

Given $(C^\infty(X), L^2(X, S), \not{D}_X)$ with $T^2 \subset \text{Isom}(X)$

$$\pi(f) = \sum_{n,m \in \mathbb{Z}} \pi(f_{n,m})$$

$$\alpha_\tau(\pi(f_{n,m})) = e^{2\pi i(n\tau_1 + m\tau_2)} \pi(f_{n,m}), \quad \forall \tau = (\tau_1, \tau_2) \in T^2$$

$$\alpha_\tau(T) = U(\tau) T U(\tau)^*, \quad \text{for } T \in \mathcal{B}(\mathcal{H}), \tau \in T^2,$$

$$U(\tau)\psi(x) = \psi(\tau^{-1}(x)) \text{ on } \mathcal{H} = L^2(X, S)$$

$$U(\tau) = \exp(2\pi i \tau L) = \exp(2\pi i(\tau_1 L_1 + \tau_2 L_2))$$

w/ L_1 and L_2 the infinitesimal generators

Algebra in $\mathcal{B}(\mathcal{H})$ gen by

$$\pi_{\xi_1, \xi_2}(f) = \sum_{n,m} \pi(f_{n,m}) e^{-2\pi i(\xi_1 n L_2 + \xi_2 m L_1)}$$

$$\pi_{\xi_1, \xi_2}(f_{n,m}) \pi_{\xi_1, \xi_2}(h_{k,r}) = e^{-2\pi i(\xi_1 n r + \xi_2 m k)} \pi(f_{n,m}) \pi(h_{k,r})$$

$$\Rightarrow \text{cocycle } \sigma((n, m), (k, r)) = \exp(-2\pi i(\xi_1 n r + \xi_2 m k))$$

NC Spectral triple $(C^\infty(X)_{\xi_1, \xi_2}, L^2(X, S), \not{D}_X)$

Deforming X_ϵ isospectrally

Fibration $T^2 \rightarrow X_\epsilon \rightarrow S^1$ torus action

$$U(\tau)\psi((x, y), t) = \psi((x + e^t\tau_1, y + e^{-t}\tau_2), t)$$

preserves the metric $dt^2 + e^t dx^2 + e^{-t} dy^2 \Rightarrow U(\tau)\partial_{X_\epsilon} U(\tau)^* = \partial_{X_\epsilon}$
Infinitesimal generators

$$2\pi L_1 = e^t \frac{\partial}{\partial x}, \quad 2\pi L_2 = e^{-t} \frac{\partial}{\partial y}$$

$$U(\tau) = \exp(2\pi i(\tau_1 L_1 + \tau_2 L_2))$$

$$E_\lambda((x, y), t) := e^{2\pi i \langle \Theta_{-t}(x, y), \lambda \rangle}$$

$\Theta_{-t}(x, y) = (e^{-t}x, e^t y)$ and $\langle (a, b), \lambda \rangle = a\lambda_1 + b\lambda_2$

$$\Xi_u(\lambda, L_1, L_2) := \exp\left(i\pi \frac{u}{(\theta' - \theta)} \lambda \wedge (L_1, L_2)\right)$$

Connes-Landi deformation $C^\infty(X_\epsilon)_u$
 subalg of $\mathcal{B}(L^2(X_\epsilon, S))$ gen by

$$\pi_u(f) = E_\lambda \Xi_u(\lambda, L_1, L_2)$$

$\xi_2 = u/2 = -\xi_1$ twisted translations

$$\pi(R_\lambda^\sigma) = E_\lambda \Xi_u(\lambda, L_1, L_2)$$

$$\pi(R_\lambda^\sigma)\pi(R_\eta^\sigma) = \sigma(\lambda, \eta)\pi(R_{\lambda+\eta}^\sigma)$$

$$\sigma(\lambda, \eta) = \exp(2\pi i u_\theta \lambda \wedge \eta)$$

for $u = \theta$ and $u = \theta'$ get $\mathbb{T}_{\Lambda, i}$

$$\Rightarrow (C^*(\Lambda, \sigma), L^2(X_\epsilon, S), \not{D}_{X_\epsilon})$$

Conclusion: metric (Dirac) on solvmanifold induces metric
 (spectral triple) on NC space $C^*(\Lambda, \sigma)$

Modify Dirac by **unitary equivalences** (after ADS)

- Fourier modes on the fibers

$$\hat{\partial}_{X_\epsilon} \psi_\lambda = \left(\frac{\partial}{\partial t} \sigma_0 + 2\pi i \lambda_1 \sigma_1 + 2\pi i \lambda_2 \sigma_2 \right) \psi_\lambda$$

$$\pi(R_\eta^\sigma) \psi_\lambda = e^{2\pi i u \eta \wedge \lambda} \psi_{\eta+\lambda}$$

$$[\hat{\partial}_{X_\epsilon}, \pi(R_\eta^\sigma)] = (\eta_1 \sigma_1 + \eta_2 \sigma_2) R_\eta^\sigma$$

- $\mathcal{U} \psi_\lambda = \sigma_\lambda \psi_\lambda$: $\sigma_\lambda = \text{prod } \sigma_i, i = 1, 2, \lambda_i < 0$

$$\mathcal{U} \hat{\partial}_{X_\epsilon} \mathcal{U}^* = \text{sign}(N(\lambda)) \left(\frac{\partial}{\partial t} \sigma_0 + 2\pi i |\lambda_1| \sigma_1 + 2\pi i |\lambda_2| \sigma_2 \right)$$

$$\mathcal{U} \hat{\pi}(R_\eta^\sigma) \mathcal{U}^* : \sigma_\lambda \psi_\lambda \mapsto \sigma_{\lambda+\eta} \psi_{\lambda+\eta}$$

- For $\lambda = A_\epsilon^k(\mu) \neq 0, \mu \in \mathcal{F}_V$

$$\tilde{\mathcal{U}}(\sigma_\lambda \psi_\lambda)(t) = \sigma_\lambda \psi_\lambda \left(t - \log \frac{|\mu_1|}{|N(\mu)|^{1/2}} \right)$$

$$\tilde{\vartheta} = \tilde{\vartheta}^{(0)} + \sum_{\mu \in (\Lambda \setminus \{0\})/V} \tilde{\vartheta}^{(\mu)}$$

$$\tilde{\psi}_\lambda := \tilde{U}(\sigma_\lambda \psi_\lambda) = \sigma_\lambda \psi_{|N(\lambda)|^{1/2}(\text{sign}(\lambda_1)\epsilon^k, \text{sign}(\lambda_2)\epsilon^{-k})}$$

Unitarily equivalent: $\tilde{\pi}(R_\eta^\sigma) \tilde{\psi}_\lambda = \tilde{\psi}_{\lambda+\eta}$

$$\tilde{\vartheta}^{(\mu)} \tilde{\psi}_{A_\epsilon^k(\mu)} = \text{sign}(N(\mu)) |N(\mu)|^{1/2}.$$

$$\left(|N(\mu)|^{-1/2} \frac{\partial}{\partial t} \sigma_0 + 2\pi i \epsilon^k \sigma_1 + 2\pi i \epsilon^{-k} \sigma_2 \right) \tilde{\psi}_{A_\epsilon^k(\mu)}$$

$$\tilde{\vartheta}^{(\mu)} = D_\mu B_\mu$$

$$D_\mu \tilde{\psi}_{A_\epsilon^k(\mu)} = \text{sign}(N(\mu)) |N(\mu)|^{1/2} \tilde{\psi}_{A_\epsilon^k(\mu)}$$

$$B_\mu \tilde{\psi}_{A_\epsilon^k(\mu)} = \left(|N(\mu)|^{-1/2} \frac{\partial}{\partial t} \sigma_0 + 2\pi i \epsilon^k \sigma_1 + 2\pi i \epsilon^{-k} \sigma_2 \right) \tilde{\psi}_{A_\epsilon^k(\mu)}$$

Foliations, transverse measures, differential operators

$$\Theta_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

Fixed point $(0, 0)$, stable manifold axis $(0, y)$, unstable manifold axis $(x, 0)$

$$(x, y) = (s_1 + s_2\theta, s_1 + s_2\theta'), \quad (s_1, s_2) \in \mathbb{R}^2 / \mathbb{Z}^2$$

Kronecker foliations

$$L_\theta = \{s_1 + s_2\theta\}, \quad L_{\theta'} = \{s_1 + s_2\theta'\}$$

operators $e^t \frac{\partial}{\partial x}$ and $e^{-t} \frac{\partial}{\partial y}$ leaf-wise derivations

factors e^t and e^{-t} scaling of transverse measure

$$\delta_\theta : \psi_{n,m} \mapsto (n + m\theta) \psi_{n,m}$$

$$\delta_{\theta'} : \psi_{n,m} \mapsto (n + m\theta') \psi_{n,m}$$

$$\mathcal{D}_{\theta,\theta'} = \begin{pmatrix} 0 & \delta_{\theta'} - i\delta_\theta \\ \delta_{\theta'} + i\delta_\theta & 0 \end{pmatrix}$$

$\mathcal{D}_{\theta, \theta', 0}$ on complement of zero modes

$$\mathcal{D}_{\theta, \theta', 0} = \sum_{\mu \in (\Lambda \setminus \{0\})/V} \mathcal{D}_{\theta, \theta'}^{\mu}$$

$$\mathcal{D}_{\theta, \theta'}^{\mu} \psi_{A_{\epsilon}^k(\mu)} = (\lambda_1 \sigma_1 + \lambda_2 \sigma_2) \psi_{A_{\epsilon}^k(\mu)}$$

adiabatic limit of

$$\hat{\mathcal{D}}_{X_{\epsilon}} : \psi_{\lambda} \mapsto \left(\frac{\partial}{\partial t} \sigma_0 + 2\pi i \lambda_1 \sigma_1 + 2\pi i \lambda_2 \sigma_2 \right) \psi_{\lambda}$$

Unitary equivalence

$$\tilde{\mathcal{D}}_{\theta, \theta'}^{\mu} \tilde{\psi}_{A_{\epsilon}^k(\mu)} = \text{sign}(N(\mu)) |N(\mu)|^{1/2} (\epsilon^k \sigma_1 + \epsilon^{-k} \sigma_2) \tilde{\psi}_{A_{\epsilon}^k(\mu)}$$

$$\tilde{\mathcal{D}}_{\theta, \theta'}^{\mu} = D_{\theta}^{\mu} B_{\theta}$$

$$D_{\theta}^{\mu} \tilde{\psi}_{A_{\epsilon}^k(\mu)} = \text{sign}(N(\mu)) |N(\mu)|^{1/2} \tilde{\psi}_{A_{\epsilon}^k(\mu)}$$

$$B_{\theta} \tilde{\psi}_{A_{\epsilon}^k(\mu)} = (\epsilon^k \sigma_1 + \epsilon^{-k} \sigma_2) \tilde{\psi}_{A_{\epsilon}^k(\mu)}$$

Lorentzian geometry

$$N(\lambda) = \lambda_1 \lambda_2 = (n + m\theta)(n + m\theta') \Leftrightarrow \square = p_0^2 - p_1^2$$

wave operator $\mathcal{D}_\lambda^2 = \square_\lambda$

$$\mathcal{D}_\lambda = \begin{pmatrix} 0 & \mathcal{D}_\lambda^+ \\ \mathcal{D}_\lambda^- & 0 \end{pmatrix} := \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix}$$

$$\square_\lambda = \begin{pmatrix} N(\lambda) & 0 \\ 0 & N(\lambda) \end{pmatrix}$$

On $\mathcal{H} = \ell^2(\Lambda) \oplus \ell^2(\Lambda)$ with $C^*(\Lambda, \sigma)$ acting diagonally, operator

$$\mathcal{D}e_{\lambda, \pm} = \mathcal{D}_\lambda e_{\lambda, \pm}$$

$\mathbb{Z}/2\mathbb{Z}$ -grading

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Bounded commutators:

$$[\mathcal{D}, R_\eta^\sigma]e_{\lambda, \pm} = \sigma(\lambda, \eta) \eta_\mp e_{\eta+\lambda, \pm}$$

$\eta_+ = \eta_1$ and $\eta_- = \eta_2$

Problems with spectral triples in Lorentzian geometry

- \mathcal{D} not self-adjoint
- \mathcal{D} has infinite multiplicities in the spectrum: noncompact symmetry group $V = \epsilon^{\mathbb{Z}}$

Two issues:

- work over $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ instead of \mathbb{C}
- use indefinite inner product spaces (Krein)

Arithmetic Krein spaces

$c : \mathbb{K} \rightarrow \mathbb{K}$ Galois involution $c : x \mapsto x'$

\mathcal{V} a \mathbb{K} -vector space, $T : \mathcal{V} \rightarrow \mathcal{V}$ is c -linear if

$$T(av + bw) = c(a)T(v) + c(b)T(w)$$

- Lorentzian pairing:

$$(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{K}$$

non-degenerate, c -linear in first variable, linear in the second

- \mathbb{K} -Krein space: $(\mathcal{V}, (\cdot, \cdot))$ with c -linear involution $\kappa : \mathcal{V} \rightarrow \mathcal{V}$
 - $(\kappa \cdot, \cdot) = c(\cdot, \kappa \cdot)$
 - For all $v \neq 0$ in \mathcal{V} , the elements $(\kappa v, v) \in \mathbb{K}$ are totally positive
- Krein adjoint T^\dagger of \mathbb{K} -linear T

$$(v, Tw) = (T^\dagger v, w)$$

- Associated real Hilbert spaces

$$\mathcal{V}_{\mathbb{R},i} := \mathcal{V} \otimes_{\iota_i(\mathbb{K})} \mathbb{R}$$

two embeddings $\iota_i : \mathbb{K} \hookrightarrow \mathbb{R}$

$$\begin{aligned} \langle v, w \rangle &= \frac{1}{2} \iota_1((\kappa v, w) + (v, \kappa w)) \\ &= \frac{1}{2} \iota_2((\kappa v, w) + (v, \kappa w)) \end{aligned}$$

- For \mathbb{K} -linear operator T on \mathbb{K} -Krein space \mathcal{V}

$$\mathbb{M}_i(T) := \inf_{(v,v)=1} \iota_i(Tv, Tv), \quad i = 1, 2$$

Krein-bounded: $\mathbb{M}_i(T) > -\infty$

(Note: need not be bounded in associated Hilbert space)

Lorentzian \mathbb{K} -spectral triples: $(\mathcal{A}, \mathcal{V}, \mathcal{D})$

- \mathcal{A} involutive \mathbb{K} -algebra
- \mathcal{V} a \mathbb{K} -Krein space with action

$$\pi : \mathcal{A} \rightarrow \text{End}_{\mathbb{K}}(\mathcal{V})$$

by Krein-bounded

$$\mathbb{M}_i(\pi(a)) > -\infty$$

$$\pi(a^*) = \pi(a)^\dagger$$

- Densely defined \mathbb{K} -linear \mathcal{D} with $\mathcal{D}^\dagger = \mathcal{D}$
- Krein-bounded commutators $C_a := [\mathcal{D}, \pi(a)]$

$$\mathbb{M}_i(C_a) > -\infty, \quad \forall a \in \mathcal{A}$$

– \exists densely def \mathbb{K} -linear $U : \mathcal{V} \rightarrow \mathcal{V}$

$$(Uv, Uv) = (v, v), \quad v \in \text{Dom}(U)$$

$$U^\dagger = U^{-1} \text{ and } U^\dagger \mathcal{D} U = \mathcal{D}$$

– Commutators $C_{a,U} := [\mathcal{D}, \pi_U(a)]$ Krein-bounded

$$\mathbb{M}_i(C_{a,U}) > -\infty, \quad \forall a \in \mathcal{A}$$

$$\pi_U(a) = U^\dagger \pi(a) U$$

– U unbounded $U = U^*$ in assoc Hilbert space

– p -summability

$$\sum_n |\langle Ue_n, |\mathcal{D}^2| Ue_n \rangle|^{-s/2} < \infty, \quad \forall s \geq p$$

$e_n =$ o.n.basis of complement of zero modes of $|\mathcal{D}^2|$ on Hilbert space

Lorentzian \mathbb{K} -spectral triple for real multiplication NC tori

- \mathbb{K} -Krein space \mathcal{V}_Λ spanned by e_λ , $\lambda \in \Lambda$

$$(v, w) := \sum_{\lambda} c(a_\lambda) b_\lambda$$

$v = \sum_{\lambda} a_\lambda e_\lambda$ and $w = \sum_{\lambda} b_\lambda e_\lambda$

- Action of group ring $\mathbb{K}[\Lambda]$

$$R_\lambda e_\eta = e_{\lambda+\eta} \quad \text{with} \quad \mathbb{M}_i(R_\lambda) > -\infty$$

In this case: bounded in assoc Hilbert space

- Twisted group ring $\omega \in \mathbb{K}^*$ with $N(\omega) = \omega\omega' = 1$

$$\varpi(\lambda, \eta) = \omega^{(n,m) \wedge (r,k)}$$

$\lambda = (n + m\theta, n + m\theta')$ and $\eta = (r + k\theta, r + k\theta')$

\mathbb{K}^* -valued 2-cocycle ϖ on Λ

$$R_\lambda^\varpi R_\eta^\varpi = \varpi(\lambda, \eta) R_{\lambda+\eta}^\varpi$$

twisted $\mathbb{K}(\Lambda, \varpi)$

- Action of $\mathbb{K}(\Lambda, \varpi)$ on \mathcal{V}_Λ

$$R_\lambda^\varpi e_\eta = \varpi(\eta, \lambda) e_{\lambda+\eta}$$

Krein-bounded $M_i(R_\lambda^\varpi) > -\infty$ not-bounded in assoc Hilbert

- Preserves Lorentzian pairing

$$(R_\lambda^\varpi e_\eta, R_\lambda^\varpi e_\zeta) = c(\omega(\eta, \lambda)) \omega(\zeta, \lambda) \delta_{\eta, \zeta} =$$

$$N(\omega(\eta, \lambda)) \delta_{\eta, \zeta} = (e_\eta, e_\zeta)$$

- Lorentzian Dirac operator $\mathcal{D}_\mathbb{K}$

$$\mathcal{D}_{\mathbb{K}, \lambda} e_{\lambda, \pm} = \begin{pmatrix} 0 & \mathcal{D}_\lambda^+ \\ \mathcal{D}_\lambda^- & 0 \end{pmatrix} e_{\lambda, \pm} := \begin{pmatrix} 0 & \ell \\ c(\ell) & 0 \end{pmatrix} e_{\lambda, \pm}$$

Krein self-adjoint $\mathcal{D}_\mathbb{K}^\dagger = \mathcal{D}_\mathbb{K}$

- On associated real Hilbert spaces

$$\mathcal{D}_\lambda = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} \quad \text{and} \quad c(\mathcal{D}_\lambda) = \begin{pmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{pmatrix}$$

- Commutators Krein-bounded

$$[\mathcal{D}_\mathbb{K}, R_\lambda^\varpi] e_{\eta, \pm} = \varpi(\eta, \lambda) \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} e_{\lambda+\eta, \pm}$$

$$([\mathcal{D}_\mathbb{K}, R_\lambda^\varpi] v, [\mathcal{D}_\mathbb{K}, R_\lambda^\varpi] v) = N(\lambda)(v, v)$$

- Fundamental domain \mathcal{F}_V in Λ

$\rho(\lambda) \in \mathbb{Z}$ unique s.t. $\lambda = A_\epsilon^{\rho(\lambda)}(\mu)$, with $\mu \in \mathcal{F}_V$

$$T_\epsilon e_{\lambda, \pm} := \begin{pmatrix} \epsilon^{\rho(\lambda)} & 0 \\ 0 & \epsilon^{-\rho(\lambda)} \end{pmatrix} e_{\lambda, \pm}$$

$$J e_{\lambda, \pm} = e_{J(\lambda), \pm}, \quad J(A_\epsilon^k(\mu)) = A_\epsilon^{-k}(\mu), \quad U_\epsilon = T_\epsilon J$$

$$U_\epsilon e_{\lambda, \pm} = \begin{pmatrix} \epsilon^{-\rho(\lambda)} & 0 \\ 0 & \epsilon^{\rho(\lambda)} \end{pmatrix} e_{J(\lambda), \pm}$$

- $U_\epsilon^\dagger = U_\epsilon^{-1}$ and $(U_\epsilon v, U_\epsilon v) = (v, v)$. Symmetry

$$U_\epsilon^\dagger \mathcal{D}_\mathbb{K} U_\epsilon = \mathcal{D}_\mathbb{K}$$

- Krein-bounded commutators

$$\mathbb{M}_i([\mathcal{D}_\mathbb{K}, U_\epsilon^\dagger R_\lambda^\varpi U_\epsilon]) > -\infty$$

$$(U_\epsilon^\dagger [\mathcal{D}_\mathbb{K}, R_\lambda^\varpi] U_\epsilon v, U_\epsilon^\dagger [\mathcal{D}_\mathbb{K}, R_\lambda^\varpi] U_\epsilon v) = N(\lambda)(v, v)$$

- Hilbert space adjoint $U_\epsilon^* = U_\epsilon$

$$\begin{aligned} U_\epsilon^* e_{\lambda, \pm} &= (\kappa U_\epsilon^\dagger \kappa) e_{\lambda, \pm} = c(U_\epsilon^\dagger) e_{\lambda, \pm} \\ &= \begin{pmatrix} \epsilon^{-\rho(\lambda)} & 0 \\ 0 & \epsilon^{\rho(\lambda)} \end{pmatrix} e_{J(\lambda), \pm} = U_\epsilon e_{\lambda, \pm} \end{aligned}$$

unbounded

- Finite summability:

$$|\mathcal{D}_{\mathbb{K}}^2|e_{\lambda,\pm} = \begin{pmatrix} |N(\lambda)| & 0 \\ 0 & |N(\lambda)| \end{pmatrix} e_{\lambda,\pm}$$

- On complement of zero modes

$$\sum_{\lambda \neq 0} |\langle U_{\epsilon} e_{\lambda,\pm}, |\mathcal{D}_{\mathbb{K}}^2| U_{\epsilon} e_{\lambda,\pm} \rangle|^{-s/2} =$$

$$\sum_{\lambda \neq 0} (\epsilon^{2\rho(\lambda)} + \epsilon^{-2\rho(\lambda)})^{-s/2} |N(\lambda)|^{-s/2} =$$

$$\sum_{k \in \mathbb{Z}} (\epsilon^{2k} + \epsilon^{-2k})^{-s/2} \sum_{\mu \in (\Lambda \setminus \{0\})/V} |N(\mu)|^{-s/2}$$

Eta function

$$\eta_{\mathcal{D}}(s) := \sum_n \text{sign}(\langle Ue_n, \mathcal{D}^2 Ue_n \rangle) |\langle Ue_n, |\mathcal{D}^2| Ue_n \rangle|^{-s/2}$$

Shimizu L -function

$$\eta_{\mathcal{D}_{\mathbb{K}}}(s) = L\left(\Lambda, V, \frac{s}{2}\right) Z_{\epsilon}\left(\frac{s}{2}\right)$$

$L(\Lambda, V, s)$ Shimizu L -function and

$$Z_{\epsilon}(s/2) = \sum_{k \in \mathbb{Z}} (\epsilon^{2k} + \epsilon^{-2k})^{-s/2}$$

3-dim geometry adiabatic $\tilde{\mathcal{D}}_{\theta, \theta'}$

$$\zeta_{\tilde{\mathcal{D}}_{\theta, \theta'}}(s) = 2Z_{\epsilon}(s/2) \sum_{\mu \in (\Lambda \setminus \{0\})/V} |N(\mu)|^{-s/2}$$

Eta vanishes: symmetry

Restriction $\tilde{\mathcal{D}}_{\theta, \theta'}^+$ to positive modes of B_{θ}

$$\eta_{\tilde{\mathcal{D}}_{\theta, \theta'}^+}(s) = L(\Lambda, V, s/2) Z_{\epsilon}(s/2) = \eta_{\mathcal{D}_{\mathbb{K}}}(s)$$

Residue:

$$\operatorname{Res}_{s=0} \eta_{\tilde{\rho}_{\theta, \theta'}^+}(s) = \frac{L(\Lambda, V, 0)}{\log \epsilon}$$

$$Z_\epsilon(s) := \sum_{k \in \mathbb{Z}} (\epsilon^{2k} + \epsilon^{-2k})^{-s} \quad \text{with} \quad \Gamma(s)Z_\epsilon(s) = \int_0^\infty g_\epsilon(t) t^{s-1} dt$$

$$g_\epsilon(t) = \left(\sum_{k \in \mathbb{Z}} e^{-(\epsilon^{2k} + \epsilon^{-2k})t} \right)$$

$$g_\epsilon(t) = -e^{-2t} + 2h_\epsilon(t) - 2 \sum_{k=0}^{\infty} e^{-\epsilon^{2k}t} (1 - e^{-\epsilon^{-2k}t})$$

$$h_\epsilon(t) = \sum_{k=0}^{\infty} e^{-\epsilon^{2k}t} \quad \text{and} \quad h_\epsilon(t) - h_\epsilon(\epsilon^2 t) = e^{-t} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} t^r$$

$$h_\epsilon(t) = \frac{1}{2 \log \epsilon} \log(1/t) + C - \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(\epsilon^{2r} - 1)} t^r$$

$\Gamma(s)Z_\epsilon(s)$ double pole $s = 0$, simple poles $s \in \mathbb{Z}_{<0}$

$$\operatorname{Res}_{s=0} Z_\epsilon(s) = \frac{1}{\log \epsilon}$$

Conclusions

- The Shimizu L -function $L(\Lambda, V, s)$ arises as the η -function of a spectral triple on $C^*(\Lambda, V, \tilde{\sigma}) = C^*(\Lambda, \sigma) \rtimes V$ obtained by deforming the commutative spectral triple of the solvmanifold X_ϵ turning the fiber T^2 into NC tori \mathbb{T}_θ with real multiplication
- The induced spectral triple on \mathbb{T}_θ is the Wick rotation of a Lorentzian geometry, where the modes of the wave operator are norms in the real quadratic field
- the 3-dimensional solvmanifold is the homotopy quotient (in the sense of Baum–Connes) of the NC space $\mathbb{T}_\theta / \text{Aut}(\mathbb{T}_\theta) = C^*(\Lambda, V, \tilde{\sigma})$