Noncommutative spaces: geometry and dynamics

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Ma148b: Topics in Mathematical Physics, Caltech Winter 2021

Beyond the case of almost-commutative geometries

measure theory	von Neumann algebras
topology	C^*–algebras
smooth structures	smooth subalgebras
Riemannian geometry	spectral triples

Methods of constructing noncommutative spaces:

1. quotients
2. deformations
3. spaces defined by global properties

Von Neumann Algebras

• Hilbert space $\mathcal{H}$ (infinite dimensional, separable, over $\mathbb{C}$) algebra of bounded operators $\mathcal{B}(\mathcal{H})$ with operator norm

• Commutant of $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$:

$$\mathcal{M}' := \{T \in \mathcal{B}(\mathcal{H}) : \ TS = ST, \forall S \in \mathcal{M}\}.$$

• von Neumann algebra: $\mathcal{M} = \mathcal{M}''$ (double commutant) $\Leftrightarrow$ weakly closed

• Center: $Z(\mathcal{M}) = L^\infty(X, \mu)$.

• If $Z(\mathcal{M}) = \mathbb{C}$: factor

C^*-algebras

• involutive ($*$ anti-isomorphism) Banach algebra (complete in norm, $\|ab\| \leq \|a\| \cdot \|b\|$, $\|a^* a\| = \|a\|^2$)

• Gel'fand–Naimark correspondence: locally compact Hausdorff topological space $\Leftrightarrow$ commutative C^*-algebra:

$$X \Leftrightarrow C_0(X)$$

• representation of a C^*-algebra $\mathcal{A}$

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

C^*-algebra homomorphism

• state: continuous linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with positivity $\varphi(a^* a) \geq 0$ for all $a \in \mathcal{A}$ and $\varphi(1) = 1$

GNS representation

• cyclic vector ξ for a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a C^*-algebra if set $\{\pi(a)\xi \, : \, a \in \mathcal{A}\}$ dense in $\mathcal{H}$

• state from unit norm cyclic vector $\varphi(a) = \langle \pi(a)\xi, \xi \rangle$

• given a state $\varphi : \mathcal{A} \to \mathbb{C}$ construct a representation (GNS) where state is as above

• define $\langle a, b \rangle = \varphi(a^* b)$ for $a, b \in \mathcal{A}$

• $\mathcal{N} = \{a \in \mathcal{A} : \varphi(a^* a) = 0\}$ linear subspace but for C^*-algebras also a *left* ideal in $\mathcal{A}$

• $\mathcal{H} = \mathcal{A}/\mathcal{N}$ with $\langle a, b \rangle = \varphi(a^* b)$ Hilbert space

• the representation $\pi(a)b + \mathcal{N} = ab + \mathcal{N}$

• cyclic vector $\xi = 1 + \mathcal{N}$ unit of $\mathcal{A}$

Morita equivalence

C^*-algebras $\mathcal{A}_1 \sim \mathcal{A}_2$: $\exists$ bimodule $\mathcal{M}$, right Hilbert $\mathcal{A}_1$ module with $\langle \cdot, \cdot \rangle_{\mathcal{A}_1}$, left Hilbert $\mathcal{A}_2$-module with $\langle \cdot, \cdot \rangle_{\mathcal{A}_2}$:

- All $\mathcal{A}_i$ as closure of span
$$\{\langle \xi_1, \xi_2 \rangle_{\mathcal{A}_i} : \xi_1, \xi_2 \in \mathcal{M}\}$$
- $\forall \xi_1, \xi_2, \xi_3 \in \mathcal{M}$.
$$\langle \xi_1, \xi_2 \rangle_{\mathcal{A}_1} \xi_3 = \xi_1 \langle \xi_2, \xi_3 \rangle_{\mathcal{A}_2}$$
- $\mathcal{A}_1$ and $\mathcal{A}_2$ act on $\mathcal{M}$ by bounded operators

Isomorphism class of noncommutative spaces $\Leftrightarrow$ Morita equivalence class of C^*-algebras

Noncommutative Torus (interesting non-AF case)

irrational rotation algebra $\mathcal{A}_\theta$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$: universal C^*-algebra $C^*(U, V)$ generated by unitaries U, V (so $U^*U = UU^* = 1$ and $V^*V = VV^* = 1$) with relation

$$UV = e^{2\pi i\theta} VU$$

concrete realization as an algebra of bounded operators

$$Ue_n = e_{n+1}, \qquad Ve_n = e^{2\pi in\theta} e_n$$

on $\{e_n\}$ Fourier basis of Hilbert space $\mathcal{H} = L^2(S^1)$

• If $\theta \in \mathbb{Q}$, algebra $\mathcal{A}_\theta$ Morita equivalent to commutative $C(T^2)$ (it is an almost-commutative geometry when θ rational)

Kronecker foliation on T^2 foliation $dx = \theta dy$, $x, y \in \mathbb{R}/\mathbb{Z}$.

- space of leaves: $X = \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}) \simeq S^1/\theta\mathbb{Z}$,

$$R_\theta x = x + \theta \mod 1$$

- transversal: $T = \{y = 0\}$, $T \cong S^1 \cong \mathbb{R}/\mathbb{Z}$

$$\mathcal{A}_\theta = \{(f_{ab}) \quad a, b \in T \text{ in the same leaf } \}$$

- (f_{ab}) as power series $b = \sum_{n\in\mathbb{Z}} b_n V^n$ where each b_n is an element of the algebra $C(S^1)$, with multiplication

$$VhV^{-1} = h \circ R_\theta^{-1}$$

- $C(S^1)$ generated by $U(t) = e^{2\pi it} \Rightarrow$ generating system (U, V) with relation

$$UV = e^{2\pi i\theta} VU$$

- crossed product algebra $\mathcal{A}_\theta = C(S^1) \rtimes_{R_\theta} \mathbb{Z}$

$$"S^1/\theta\mathbb{Z}" \sim C(S^1) \rtimes_{R_\theta} \mathbb{Z}$$

K-theory of C^*-algebras

- $K_0(\mathcal{A})$: idempotents (for C^*-algebras $\Leftrightarrow$ projections: $p \mapsto P = pp^*(1-(p-p^*)^2)^{-1})$, $P \sim Q$ iff $P = X^*X$ $Q = XX^*$, $X =$ partial isometry ($X = XX^*X$); stable equivalence: $P \sim Q$, $P \in \mathcal{M}_n(\mathcal{A})$ $Q \in \mathcal{M}_m(\mathcal{A})$, $\exists R$ proj $P \oplus R \sim Q \oplus R$ $\Rightarrow K_0(\mathcal{A})^+$, Grothendieck group $= K_0(\mathcal{A})$
- $K_1(\mathcal{A})$:

$$\mathrm{GL}_n(\mathcal{A}) = \text{ invertible elements in } \mathcal{M}_n(\mathcal{A})$$

$$\mathrm{GL}_n^0(\mathcal{A}) = \text{ identity component}$$

$$\mathrm{GL}_n(\mathcal{A}) \to \mathrm{GL}_{n+1}(\mathcal{A}) \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathrm{GL}_n(\mathcal{A})/\mathrm{GL}_n^0(\mathcal{A}) \to \mathrm{GL}_{n+1}(\mathcal{A})/\mathrm{GL}_{n+1}^0(\mathcal{A})$$

direct limit $= K_1(\mathcal{A})$

Pimsner–Voiculescu six terms exact sequence: crossed product $\mathcal{A} = \mathcal{B} \rtimes_R \mathbb{Z}$:

$$\begin{array}{ccccc} K_0(\mathcal{B}) & \xrightarrow{I-R_*} & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{A}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{A}) & \longleftarrow & K_1(\mathcal{B}) & \xleftarrow[I-R_*]{} & K_1(\mathcal{B}) \end{array}$$

For irrational rotation: $\mathcal{B} = C(S^1)$: $K_i(C(S^1)) = K^i_{top}(S^1) = \mathbb{Z}$
R_θ preserves rk of proj and winding number of det of invertible element $\Rightarrow R_{\theta *} = I$

$$K_0(\mathcal{A}_\theta) \cong \mathbb{Z}^2 \qquad K_1(\mathcal{A}_\theta) \cong \mathbb{Z}^2$$

Projections and the NC torus

• Rieffel projectors for $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\forall \alpha \in (\mathbb{Z} + \mathbb{Z}\theta) \cap [0,1]$, $\exists$ projection P_α in $\mathcal{A}_\theta$, with $\mathrm{Tr}(P_\alpha) = \alpha$.

Rieffel's method

- if $\mu \in {}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ bimodule such that $\mu\langle\mu,\mu\rangle_{\mathcal{B}} = \mu$, then $P :=_{\mathcal{A}} \langle\mu,\mu\rangle$ is a projection
- if $\xi \in {}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ such that $\exists$ invertible $*$-invariant square root $\langle\xi,\xi\rangle_{\mathcal{B}}^{1/2}$, then can use $m := \xi\langle\xi,\xi\rangle_{\mathcal{B}}^{-1/2}$

Boca's projectors for $\mathcal{A}_\theta$ from $\xi =$ Gaussian element in some Heisenberg modules (Mumford's Tata lectures vol. III) $\Rightarrow$ the corresponding $\langle\xi,\xi\rangle_{\mathcal{B}}$ is a **quantum theta function** in the sense of Manin (Theta functions, quantum tori and Heisenberg groups)

Spectral triple for the NC torus

smooth subalgebra for $\mathcal{S}(\mathbb{Z}^2)$ = Schwartz space of sequences of rapid decay on $\mathbb{Z}^2$

$$\mathcal{A}_{\theta,0} = \{\sum_{\mathbb{Z}^2} b_{nm} U^n V^m \ , \ \ b \in \mathcal{S}(\mathbb{Z}^2)\}$$

Dirac operator derivations

$$\delta_1 = 2\pi i U \frac{\partial}{\partial U} \quad \delta_2 = 2\pi i V \frac{\partial}{\partial V}$$

$\partial = \delta_1 + \tau \delta_2$, $Im(\tau) > 0$: on $\mathcal{H} \oplus \mathcal{H}$

$$D = -i \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}$$

• $D^2 = -(\delta_1 + \tau\delta_2)(\delta_1 + \bar{\tau}\delta_2)$ eigenvalues (multiplicity two):

$$\mathrm{Spec}(D^2) = \{4\pi^2 |m + n\tau|^2\}_{m,n\in\mathbb{Z}}$$

• Zeta function= Eisenstein series

$$\mathrm{Tr}(|D|^{-s}) = 2^{1-s}\pi^{-s} E_s(\tau),$$

$$E_s(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{|m + n\tau|^s}$$

Morita equivalent noncommutative tori

• Morita equivalence $\Leftrightarrow$ changing choice of transversal

• Morita equivalence $\mathcal{A}_\theta \simeq \mathcal{A}_{-1/\theta}$: change of parameterization of space of leaves using $T = \{y = 0\}$ or $S = \{x = 0\}$ as transversal

Bimodules and Morita equivalence

$$\theta' = \frac{a\theta + b}{c\theta + d} = g\theta$$

$\mathcal{M}_{\theta,\theta'}$ = Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/c)$, right action of $\mathcal{A}_\theta$:

$$Uf\,(x,u) = f\left(x - \frac{c\theta + d}{c}, u - 1\right)$$

$$Vf\,(x,u) = \exp(2\pi i(x - ud/c))f(x,u)$$

left action of $\mathcal{A}_{\theta'}$:

$$U'f\,(x,u) = f\left(x - \frac{1}{c}, u - a\right)$$

$$V'f\,(x,u) = \exp\left(2\pi i\left(\frac{x}{c\theta + d} - \frac{u}{c}\right)\right)f(x,u)$$

Noncommutative modular curve

- isomorphisms: $\mathcal{A}_\theta \cong \mathcal{A}_{-\theta}$ and $\mathcal{A}_\theta \cong \mathcal{A}_{\theta\pm 1}$, Morita equivalence $\mathcal{A}_\theta \simeq \mathcal{A}_{-1/\theta}$
- Morita equivalence classes: $\theta \sim \theta'$, same $\mathrm{PGL}(2,\mathbb{Z})$ orbit
- Moduli space quotient

$$\mathrm{PGL}(2,\mathbb{Z})\backslash\mathbb{P}^1(\mathbb{R})$$

but this is a "bad quotient" classically, so treat also as a *noncommutative space*

$$C(\mathbb{P}^1(\mathbb{R})) \rtimes \mathrm{PGL}(2,\mathbb{Z})$$

- this noncommutative space can be seen as a "noncommutative boundary" of the classical modular curve $\mathrm{PGL}(2,\mathbb{Z})\backslash\mathbb{H}$ that extends the classical compactification by cusps $\mathrm{PGL}(2,\mathbb{Z})\backslash\mathbb{P}^1(\mathbb{Q})$

Degeneration of elliptic curves

Elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, $Im(\tau) > 0$

Jacobi uniformization $q \in \mathbb{C}^*$, $q = \exp(2\pi i \tau)$, $|q| < 1$

$$E_q = \mathbb{C}/q^{\mathbb{Z}}$$

fundamental domain: annulus radii 1 and q, identification via scaling and rotation

As $q \to \exp(2\pi i\theta) \in S^1$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$,

$$E_q \longrightarrow \text{noncomm. torus } \mathcal{A}_\theta$$

$\Rightarrow$ study **limiting behavior** (e.g. of arithmetic invariants defined on modular curves) when $\tau \to \theta \in \mathbb{R} \setminus \mathbb{Q}$

Lattices and Manin's pseudolattices

Elliptic curves and lattices Lattice Λ, embedding $j : \Lambda \hookrightarrow \mathbb{C}$, $j(\Lambda) = \mathbb{Z} + \tau\mathbb{Z}$, elliptic curve

$$E_\tau \cong \mathbb{C}/j(\Lambda)$$

$\Rightarrow$ Equivalence of categories

Pseudolattices $\Lambda =$ free abelian rk 2, $j : \Lambda \hookrightarrow \mathbb{C}$ image in a $\mathbb{R}$-line, orientation ϵ.
$j(0,1) = 1$, $j(1,0) = \theta$

- Morphism: $g \in \mathcal{M}_2(\mathbb{Z})$, $\theta' = \frac{\theta+b}{c\theta+d}$, $\epsilon' = sign(c\theta + d)\epsilon$.
Isomorphism: $g \in \mathrm{GL}(2,\mathbb{Z})$

- Equivalence of categories: (i) Pseudolattices with morphisms $g \in \mathrm{GL}(2,\mathbb{Z})$, (ii) noncommutative tori with morphisms Morita equivalences $\mathcal{M}_{\theta,\theta'}$

- if $End(\Lambda) \neq \mathbb{Z}$ then $\Lambda \subset \mathbb{K}$, some real quadratic field: **real multiplication** (quadratic irrationalities)

Noncommutative Riemann surfaces?

For tori: (1) cross product algebra, (2) foliation, (3) deformation of group algebra $\Rightarrow$ same result $\mathcal{A}_\theta$

Higher genus

- Cross product: $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$, $\Gamma =$ Fuchsian group (uniformization, as for modular curves)
- Foliations: interval exchange transformations
- Group algebra: $C^*(\Gamma, \sigma)$ (quantum Hall effect)

$\Rightarrow$ in classical case these points of view describe same object, but their noncommuative versions give *different* results unlike in the genus one case

Example: Algebras of directed graphs (extends AF algebras to larger class)

Directed graph

- $E = (E^0, E^1, E^1_+, r, s, \iota)$, with $E^0 =$ vertices, $E^1 =$ oriented edges $w = \{e, \epsilon\}$, $E^1_+ =$ choice of orientation for each edge, $r, s : E^1 \to E^0$ range and source maps, $\iota(w) = \{e, -\epsilon\}$ orientation reversal.
- E finite, row finite (fin many exiting edges), locally finite
- $\Delta =$ universal covering tree, $E = \Delta/\Gamma$
- Admissible chain of edges $w_1 w_2$: if $w_2 \neq \iota(w_1)$ and $r(w_1) = s(w_2)$. Walks= chains of edges; Paths= chains of pos.oriented edges
- Boundary ∂E: *shift-tail equivalence* $\omega \sim \tilde{\omega}$ if $\exists N \geq 1, k \in \mathbb{Z}$ $\omega_i = \tilde{\omega}_{i-k}$, $\forall i \geq N$. Paths ending at sinks: $\omega \sim \tilde{\omega} \in \sigma^*$, iff $r(\omega) = r(\tilde{\omega})$

$$\partial\Delta = (\mathcal{P}^+ \cup \sigma^*)/\sim$$

Cuntz-Krieger family

$\{P_v\}_{v\in E^0}$ orthogonal projections and $\{S_w\}_{w\in E^1_+}$ partial isometries:

$$S_w^* S_w = P_{r(w)}$$

$$P_v = \sum_{w:s(w)=v} S_w S_w^*, \quad \forall v \in s(E^1_+)$$

$\Rightarrow$ universal C^*-algebra generated by $\{P_v, S_w\}$ with relations as above: $C^*(E)$

$U(1)$-gauge action: $\lambda : \{P_v, S_w\} \mapsto \{P_v, \lambda S_w\}$

Cuntz-Krieger algebras

Matrix A entries $\{0,1\} \Rightarrow$ algebra $\mathcal{O}_A$ generated by partial isometries S_i with $S_i S_i^*$ orthogonal proj and with relation:

$$S_i^* S_i = \sum_j A_{ij} S_j S_j^*$$

Introduced to study dynamics of subshifts of finite type

Edge matrix (directed graph): $A_+(w_i, w_j) = 1$ if $w_i w_j$ admissible path, $A_+(w_i, w_j) = 0$ otherwise; directed edge matrix: $A(w_i, w_j) = 1$ if $w_i w_j$ admissible walk, $A(w_i, w_j) = 0$ otherwise

Some cases of graph C^*-algebras

- $E = \Delta$, then $C^*(\Delta)$ AF-algebra Morita equivalent to $C_0(\partial\Delta)$
- $E = \Delta/\Gamma$, then $C^*(E)$ Morita equivalent to $C^*(\Delta) \rtimes \Gamma$
- If locally finite graph with no sinks, $C^*(E) \cong \mathcal{O}_{A_+}$

$\Rightarrow$ used to obtain NCG version of certain p-adic spaces (Mumford curves)

Groupoids

- $\mathcal{G}^0$ = units, $r, s : \mathcal{G} \to \mathcal{G}^0$ source/range maps
- Category: $\mathcal{G}^{(0)}$ = objects; $\mathcal{G}$ = morphisms, invertible:

$$s(\gamma^{-1}) = r(\gamma), \quad r(\gamma^{-1}) = s(\gamma)$$

with composition:

$$r(\gamma_1) = s(\gamma_2) \Rightarrow \gamma_1\gamma_2 \in \mathcal{G}$$

$$s(\gamma_1\gamma_2) = s(\gamma_1), \quad r(\gamma_1\gamma_2) = r(\gamma_2)$$

- Algebra: $\mathcal{A}_{\mathcal{G}} = \{f : \mathcal{G} \to \mathbb{C}\}$ finite support

$$(f_1 \star f_2)(\gamma) = \sum_{\gamma=\gamma_1\gamma_2} f_1(\gamma_1) f_2(\gamma_2)$$

$$f^\vee(\gamma) = \overline{f(\gamma^{-1})}$$

associative, noncommutative, involutive

- Representations of groupoid algebras

$$\mathcal{H}_x = \ell^2(\{\gamma \in \mathcal{G} \,|\, r(\gamma) = x\})$$

representation $\pi_x : \mathcal{A}_{\mathcal{G}} \to \mathcal{B}(\mathcal{H}_x)$

$$(\pi_x(f)\xi)(\gamma) = \sum_{\gamma=\gamma_1\gamma_2} f(\gamma_1)\xi(\gamma_2)$$

norm (when $\mathcal{G}^{(0)}$ compact)

$$\|f\| := \sup_{x\in\mathcal{G}^{(0)}} \|\pi_x(f)\|_{\mathcal{B}(\mathcal{H}_x)}$$

$\Rightarrow$ $C^*(\mathcal{G})$ completion of $\mathcal{A}_{\mathcal{G}}$

Algebras from categories

$\mathcal{C}$ = small category (semigroupoid)

$$\mathcal{A}_{\mathcal{C}} = \{ f : Mor(\mathcal{C}) \to \mathbb{C} \,|\, \text{finite support} \}$$

$$(f_1 \star f_2)(\phi) = \sum_{\phi = \phi_1 \circ \phi_2} f_1(\phi_1) f_2(\phi_2)$$

associative noncommutative (not involutive)

$$\mathcal{H}_Y = \ell^2(\{ f \in Mor_{\mathcal{C}}(X, Y) \,|\, X \in Obj(\mathcal{C}) \})$$

$$(\pi_Y(f)\xi)(\phi) = \sum_{\phi = \phi_1 \phi_2} f(\phi_1)\xi(\phi_2)$$

$\pi_Y(f)^\vee$ = adjoint in $\mathcal{B}(\mathcal{H}_Y)$
like creation/annihilation operators

Group $G \Rightarrow$ group ring $\mathbb{C}[G] \Rightarrow$ representation on $\ell^2(G)$ by unitaries $U_\gamma^* = U_\gamma^{-1} = U_{\gamma^{-1}}$

Semigroup S with unit $\Rightarrow$ semigroup ring $\mathbb{C}[S] \Rightarrow$ representation by isometries $\mu_s^* \mu_s = 1$ with $\mu_s \mu_s^* = e_s$ idempotent (creation/annihilation operators)

Groupoid $\mathcal{G} \Rightarrow$ groupoid ring $\mathbb{C}[\mathcal{G}] \Rightarrow$ representations on $\ell^2(\mathcal{G}_x)$ by unitaries $U_\gamma^* = U_\gamma^{-1} = U_{\gamma^{-1}}$

Semigroupoid $\mathcal{S} \Rightarrow \mathbb{C}[\mathcal{S}]$ acting by isometries on $\ell^2(\mathcal{S}_x)$

Possible variant: algebras from 2-categories: two convolutions for horizontal and vertical compositions of 2-morphisms

Γ = directed graph (quiver) $\Rightarrow$
$\mathcal{C}(\Gamma)$ = category

- $Obj(\mathcal{C}(\Gamma)) = V(\Gamma)$
- $Mor_{\mathcal{C}(\Gamma)}(v, v')$ = paths in Γ from v to v'

Quiver representation of Γ in a category $\mathcal{C}$: Functor $F : \mathcal{C}(\Gamma) \to \mathcal{C}$

Graph algebras Γ = finite directed graph:
S_e, p_v Cuntz–Krieger

$$S_e^* S_e = p_{r(e)}, \qquad \sum_{s(e)=v} S_e S_e^* = p_v$$

Same as C^*-algebra completion of $\mathcal{A}_{\mathcal{C}(\Gamma)}$

Functoriality $F : \mathcal{C}_1 \to \mathcal{C}_2$ functor

Category $\mathcal{G}(F)$ (graph of F):

- $Obj(\mathcal{G}(F)) = \{\mathcal{X} = (X, F(X)) \,|\, X \in Obj(\mathcal{C}_1)\}$
- $Mor_{\mathcal{G}(F)}(\mathcal{X}, \mathcal{Y}) = \{\Phi = (\phi, F(\phi)) \,|\, \phi \in Mor_{\mathcal{C}_1}(X, Y)\}$

Bimodule $\mathcal{V}_{\mathcal{G}(F)} = \{\xi : Mor_{\mathcal{G}(F)} \to \mathbb{C} \,|\,$ finite support $\}$
Action of $\mathcal{A}_{\mathcal{C}_1}$:

$$(\pi_X(f)\xi)(\Phi) = \sum_{\phi = \phi_1 \phi_2} f(\phi_1)\xi(\Phi_2)$$

Action of $\mathcal{A}_{\mathcal{C}_2}$:

$$(\pi_{F(X)}(h)\xi)(\Phi) = \sum_{F(\phi) = \psi F(\phi_2)} h(\psi)\xi(\Phi_2)$$

$$[\pi_X(f), \pi_{F(X)}(h)] = 0$$

Functor $F : \mathcal{C}_1 \to \mathcal{C}_2 \Rightarrow$ morphism of NC spaces $\mathcal{A}_{\mathcal{C}_1}$-$\mathcal{A}_{\mathcal{C}_2}$ bimodule