Spectral Action of Bianchi IX Gravitational Instantons

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References:

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• Wentao Fan, Farzad Fathizadeh, Matilde Marcolli, *Modular forms in the spectral action of Bianchi IX gravitational instantons*, Journal of High Energy Physics (2019) 234, 38 pp.

• Wentao Fan, Farzad Fathizadeh, Matilde Marcolli, *Motives and periods in Bianchi IX gravity models*, Lett. Math. Phys. 108 (2018), no. 12, 2729–2747.

spacetime geometries (Euclidean)

- homogeneous and isotropic: Robertson–Walker
- homogeneous and non-isotropic: Bianchi IX, Kasner, mixmaster...
- non-homogeneous and isotropic: Rees-Sciama, swiss-cheese...

$SU(2)$-Bianchi IX cosmologies (Euclidean, compactified)

• another version of Bianchi IX mixmaster cosmologies, with $SU(2)$ symmetry (Euclidean version)

$$g = w_1 w_2 w_3 \, dt^2 + \frac{w_2 w_3}{w_1} \, \sigma_1^2 + \frac{w_1 w_3}{w_2} \, \sigma_2^2 + \frac{w_1 w_2}{w_3} \, \sigma_3^2$$

with $w_i = w_i(t)$, or more generally

$$g = F \, (d\mu^2 + \frac{\sigma_1^2}{w_1^2} + \frac{\sigma_2^2}{w_2^2} + \frac{\sigma_3^2}{w_3^3})$$

with a conformal factor $F \sim w_1 w_2 w_3$

• $SU(2)$–invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations (i, j, k) of $(1, 2, 3)$

$SU(2)$–invariant 1-forms

$$\sigma_1 = x_1\, dx_2 - x_2\, dx_1 + x_3\, dx_0 - x_0\, dx_3 = \frac{1}{2}(d\psi + \cos\theta\, d\phi),$$

$$\sigma_2 = x_2\, dx_3 - x_3 dx_2 + x_1\, dx_0 - x_0\, dx_1 = \frac{1}{2}(\sin\psi\, d\theta - \sin\theta \cos\psi\, d\phi),$$

$$\sigma_3 = x_3\, dx_1 - x_1\, dx_3 + x_2\, dx_0 - x_0\, dx_2 = \frac{1}{2}(-\cos\psi\, d\theta - \sin\theta\, \sin\psi\, d\phi),$$

Euler angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$ ($SU(2)$ case)

Explicit form of the metric

- more explicitly ds^2 is

$$w_1 w_2 w_3\, dt\, dt + \frac{w_1 w_2 \cos(\eta)}{w_3} d\phi\, d\psi + \frac{w_1 w_2 \cos(\eta)}{w_3} d\psi\, d\phi$$
$$+ \left(\frac{w_2 w_3 \sin^2(\eta) \cos^2(\psi)}{w_1} + w_1 \left(\frac{w_3 \sin^2(\eta) \sin^2(\psi)}{w_2} + \frac{w_2 \cos^2(\eta)}{w_3} \right) \right) d\phi\, d\phi$$
$$+ \frac{\left(w_1^2 - w_2^2\right) w_3 \sin(\eta) \sin(\psi) \cos(\psi)}{w_1 w_2} d\eta\, d\phi + \frac{\left(w_1^2 - w_2^2\right) w_3 \sin(\eta) \sin(\psi) \cos(\psi)}{w_1 w_2} d\phi\, d\eta$$
$$+ \left(\frac{w_2 w_3 \sin^2(\psi)}{w_1} + \frac{w_1 w_3 \cos^2(\psi)}{w_2} \right) d\eta\, d\eta + \frac{w_1 w_2}{w_3} d\psi\, d\psi$$

- identifying S^3 with unit quaternions $SU(2)$
- The metrics on S^3

$$\frac{w_2 w_3}{w_1}\, \sigma_1^2 + \frac{w_1 w_3}{w_2}\, \sigma_2^2 + \frac{w_1 w_2}{w_3}\, \sigma_3^2$$

are left-invariants under the action of $SU(2)$ but *not* right-invariant (unlike the round metric on S^3)

Dirac operator

• orthonormal coframe $\{\theta^a\}$

$$D = \sum_a \theta^a \nabla^S_{\theta_a}$$

• spin connection ∇^S with matrix of 1-forms $\omega = (\omega^a_b)$ with

$$\nabla \theta^a = \sum_b \omega^a_b \otimes \theta^b$$

• metric-compatibility and torsion-freeness (Levi–Civita connection)

$$\omega^a_b = -\omega^b_a, \qquad d\theta^a = \sum_b \omega^a_b \wedge \theta^b$$

Dirac operator

$$D = \sum_{a,\mu} \gamma^a dx^\mu(\theta_a) \frac{\partial}{\partial x^\mu} + \frac{1}{4} \sum_{a,b,c} \gamma^c \omega^b_{ac} \gamma^a \gamma^b$$

with $\omega^b_a = \sum_c \omega^b_{ac} \theta^c$

- matrices γ^a Clifford action of θ^a on spin bundle:

$(\gamma^a)^2 = -I$

$\gamma^a\gamma^b + \gamma^b\gamma^a = 0$ for $a \neq b$

Dirac operator on Bianchi IX metrics

- local coordinates $(x^\mu) = (t, \eta, \phi, \psi)$ with $\mathbb{S}^3$ parametrized by

$$(\eta, \phi, \psi) \mapsto \left(\cos(\eta/2)e^{i(\phi+\psi)/2}, \sin(\eta/2)e^{i(\phi-\psi)/2}\right)$$

with $0 \leq \eta \leq \pi, 0 \leq \phi < 2\pi, 0 \leq \psi < 4\pi$.

- orthonormal frame

$$\begin{aligned}
\theta^0 &= \sqrt{w_1 w_2 w_3}\, dt, \\
\theta^1 &= \sin(\eta)\cos(\psi)\sqrt{\frac{w_2 w_3}{w_1}}\, d\phi - \sin(\psi)\sqrt{\frac{w_2 w_3}{w_1}}\, d\eta, \\
\theta^2 &= \sin(\eta)\sin(\psi)\sqrt{\frac{w_1 w_3}{w_2}}\, d\phi + \cos(\psi)\sqrt{\frac{w_1 w_3}{w_2}}\, d\eta, \\
\theta^3 &= \cos(\eta)\sqrt{\frac{w_1 w_2}{w_3}}\, d\phi + \sqrt{\frac{w_1 w_2}{w_3}}\, d\psi.
\end{aligned}$$

- non-vanishing ω_{ac}^{b}

$$\omega_{11}^{0} = -\frac{w_2\left(w_1 w_3' - w_3 w_1'\right) + w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}}, \quad \omega_{22}^{0} = -\frac{w_2\left(w_3 w_1' + w_1 w_3'\right) - w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}},$$

$$\omega_{33}^{0} = -\frac{w_2\left(w_3 w_1' - w_1 w_3'\right) + w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}}, \quad \omega_{23}^{1} = -\frac{w_1^2 w_2^2 - w_3^2\left(w_1^2 + w_2^2\right)}{2(w_1 w_2 w_3)^{3/2}},$$

$$\omega_{32}^{1} = -\frac{w_1^2\left(w_2^2 - w_3^2\right) + w_2^2 w_3^2}{2(w_1 w_2 w_3)^{3/2}}, \quad \omega_{31}^{2} = -\frac{w_2^2 w_3^2 - w_1^2\left(w_2^2 + w_3^2\right)}{2(w_1 w_2 w_3)^{3/2}}.$$

pseudo-differential symbol of Dirac

$$\begin{aligned}
\sigma(D)(x,\xi) &= \sum_{a,\mu} i\gamma^a e_a^\mu \xi_{\mu+1} + \frac{1}{4\sqrt{w_1 w_2 w_3}}\left(\frac{w_1'}{w_1}+\frac{w_2'}{w_2}+\frac{w_3'}{w_3}\right)\gamma^1 \\
&\quad - \frac{\sqrt{w_1 w_2 w_3}}{4}\left(\frac{1}{w_1^2}+\frac{1}{w_2^2}+\frac{1}{w_3^2}\right)\gamma^2\gamma^3\gamma^4 \\
&= -\frac{i\gamma^2\sqrt{w_1}\left(\csc(\eta)\cos(\psi)\left(\xi_4\cos(\eta)-\xi_3\right)+\xi_2\sin(\psi)\right)}{\sqrt{w_2}\sqrt{w_3}} \\
&\quad + \frac{i\gamma^3\sqrt{w_2}\left(\sin(\psi)\left(\xi_3\csc(\eta)-\xi_4\cot(\eta)\right)+\xi_2\cos(\psi)\right)}{\sqrt{w_1}\sqrt{w_3}} \\
&\quad + \frac{i\gamma^1\xi_1}{\sqrt{w_1}\sqrt{w_2}\sqrt{w_3}} + \frac{i\gamma^4\xi_4\sqrt{w_3}}{\sqrt{w_1}\sqrt{w_2}} \\
&\quad + \frac{1}{4\sqrt{w_1 w_2 w_3}}\left(\frac{w_1'}{w_1}+\frac{w_2'}{w_2}+\frac{w_3'}{w_3}\right)\gamma^1 \\
&\quad - \frac{\sqrt{w_1 w_2 w_3}}{4}\left(\frac{1}{w_1^2}+\frac{1}{w_2^2}+\frac{1}{w_3^2}\right)\gamma^2\gamma^3\gamma^4
\end{aligned}$$

- with non-vanishing e_a^μ:

$$e_0^0 = \frac{1}{\sqrt{w_1 w_2 w_3}}, \qquad e_1^1 = -\frac{\sqrt{w_1}\sin(\psi)}{\sqrt{w_2 w_3}},$$

$$e_2^1 = \frac{\sqrt{w_2}\cos(\psi)}{\sqrt{w_1 w_3}}, \qquad e_1^2 = \frac{\sqrt{w_1}\csc(\eta)\cos(\psi)}{\sqrt{w_2 w_3}},$$

$$e_2^2 = \frac{\sqrt{w_2}\csc(\eta)\sin(\psi)}{\sqrt{w_1 w_3}}, \qquad e_1^3 = -\frac{\sqrt{w_1}\cot(\eta)\cos(\psi)}{\sqrt{w_2 w_3}},$$

$$e_2^3 = -\frac{\sqrt{w_2}\cot(\eta)\sin(\psi)}{\sqrt{w_1 w_3}}, \qquad e_3^3 = \frac{\sqrt{w_3}}{\sqrt{w_1 w_2}}$$

- get from the symbol the homogeneous components $p_k(x,\xi)$ with

$$\sigma(D^2)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$$

- Example: for $p_0(x,\xi)$ get

$$\left(-\frac{w_1'}{8w_1w_2^2} - \frac{w_1'}{8w_1w_3^2} + \frac{3w_1'}{8w_1^3} - \frac{w_2'}{8w_1^2w_2} - \frac{w_3'}{8w_1^2w_3} - \frac{w_2'}{8w_2w_3^2} + \frac{3w_2'}{8w_2^3} - \frac{w_3'}{8w_2^2w_3} + \frac{3w_3'}{8w_3^3}\right)\gamma^1\gamma^2\gamma^3\gamma^4 +$$
$$\left(-\frac{w_1''}{4w_1^2w_2w_3} + \frac{w_1'w_2'}{8w_1^2w_2^2w_3} + \frac{w_1'w_3'}{8w_1^2w_2w_3^2} + \frac{5w_1'^2}{16w_1^3w_2w_3} - \frac{w_2''}{4w_1w_2^2w_3} + \frac{w_2'w_3'}{8w_1w_2^2w_3^2} + \frac{5w_2'^2}{16w_1w_2^3w_3} - \frac{w_3''}{4w_1w_2w_3^2} + \frac{5w_3'^2}{16w_1w_2w_3^3} + \frac{w_2w_3}{16w_1^3} + \frac{w_3}{8w_1w_2} + \frac{w_1w_3}{16w_2^3} + \frac{w_2}{8w_1w_3} + \frac{w_1}{8w_2w_3} + \frac{w_1w_2}{16w_3^3}\right)I.$$

- also manageable expression for $p_2(x,\xi)$, longer one for $p_1(x,\xi)$

Applying Parametrix Method to this Dirac operator

$$a_{2n}(x,D^2) = \frac{(2\pi)^{-m}}{2\pi i} \int \int_\gamma e^{-\lambda}\, \mathrm{tr}\,(r_{2n}(x,\xi,\lambda))\, d\lambda\, d^m\xi$$

- Find a_0, a_2, a_4 explicitly

$$a_0(D^2) = 4 w_1 w_2 w_3$$

$$\begin{aligned} a_2(D^2) \;=\; & -\frac{w_1^2}{3} - \frac{w_2^2}{3} - \frac{w_3^2}{3} + \frac{w_1^2 w_2^2}{6 w_3^2} + \frac{w_1^2 w_3^2}{6 w_2^2} + \frac{w_2^2 w_3^2}{6 w_1^2} - \frac{(w_1')^2}{6 w_1^2} - \frac{(w_2')^2}{6 w_2^2} \\ & - \frac{(w_3')^2}{6 w_3^2} - \frac{w_1' w_2'}{3 w_1 w_2} - \frac{w_1' w_3'}{3 w_1 w_3} - \frac{w_2' w_3'}{3 w_2 w_3} + \frac{w_1''}{3 w_1} + \frac{w_2''}{3 w_2} + \frac{w_3''}{3 w_3}. \end{aligned}$$

and a much longer and more complicated expression for $a_4(D^2)$

Observation: all coefficients in these expressions (also for a_4) are rational numbers ... what about other terms in expansion?

Wodzicki Residue Method for $SU(2)$-Bianchi IX metrics

- setting $\zeta_{\mu+1} = \sum_\nu e^\nu_\mu \xi_{\nu+1}$ find inductively for $n \geq 2$

$$\sigma_{-2-n}(x,\xi)|_{S^*(M\times\mathbb{T}^{n-2})} = \sigma_{-2-n}(x,\xi(\zeta))|_{\zeta\in\mathbb{S}^{n+1}} = (w_1 w_2 w_3)^{-\frac{3}{2}n} P_n(\zeta)$$

polynomials $P_n(\zeta)$ coefficients functions of w_i and derivatives

- these explicitly give

$$a_{2n}(D^2) = (w_1 w_2 w_3)^{1-3n} Q_{2n}\left(w_1, w_2, w_3, w_1', w_2', w_3', \ldots, w_1^{(2n)}, w_2^{(2n)}, w_3^{(2n)}\right)$$

with Q_{2n} polynomials with rational coefficients

$$Q_{2n} = \frac{1}{2\pi^{n+1}} \int_{\mathbb{S}^{2n+1}} \mathrm{Tr}(P_{2n}(\zeta)(\Delta^{-1}))\, d^{2n+1}\zeta$$

Question: is this rationality a sign of an arithmetic structure of Bianchi IX metrics that persists in the Spectral Action?

Motives and periods for Bianchi IX metrics

- Wodzicki Residue Method (products with auxiliary flat tori)

$$a_{2n} = \frac{1}{32\,\pi^{n+3}} \mathrm{Res}(\Delta_{2n}^{-1}),$$

$$\Delta_{2n} = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^{2n-2}},$$

$\Delta_{\mathbb{T}^{2n-2}}$ flat Laplacian on $\mathbb{T}^{2n-2} = (\mathbb{R}/\mathbb{Z})^{2n-2}$ with symbol

$$\sigma(\Delta_{2n}^{-1})(x,\xi) \sim_{\xi\to\infty} \sum_{m=-\infty}^{-2} \sigma_m(\Delta_{2n}^{-1})(x,\xi)$$

each $\sigma_m(\Delta_{2n}^{-1})$ homogeneous order m in ξ

$$\mathrm{Res}\left(\Delta_{2n}^{-1}\right) = \int_{M\times\mathbb{T}^{2n-2}} \left(\int_{|\xi|=1} \mathrm{tr}\left(\sigma_{-2n-2}(x,\xi)\right) \sigma_{\xi,\,2n+1} \right) dx^1 \wedge \cdots \wedge dx^{2n+2}$$

volume form on the unit sphere in the cotangent bundle

$$\sigma_{\xi,\,2n+1} = \sum_{j=1}^{2n+2} (-1)^{j-1} \xi_j \, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_{2n+2}$$

- recursive relations (from parametrix method)

$$\sigma_{-2}(\Delta_{2n}^{-1})\,(x,\xi) = \left(p_2(x,\xi_1,\ldots,\xi_4) + \left(\xi_5^2 + \cdots + \xi_{2n+2}^2\right) I\right)^{-1}$$

$$\sigma_m(\Delta_{2n}^{-1})\,(x,\xi) =$$

$$-\left(\sum_{\substack{\alpha_1,\alpha_2,\alpha_4 \in \mathbb{Z}_{\geq 0} \\ m<j\leq -2, \quad 0\leq k\leq 2 \\ j-\alpha_1-\alpha_2-\alpha_4+k=m+2}} \frac{(-i)^{\alpha_1+\alpha_2+\alpha_4}}{\alpha_1!\,\alpha_2!\,\alpha_4!} \left(\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_4}^{\alpha_4}\sigma_j(\Delta_{2n}^{-1})\right)\left(\partial_t^{\alpha_1}\partial_\eta^{\alpha_2}\partial_\psi^{\alpha_4} p_k\right) \right) \cdot$$

$$\sigma_{-2}(\Delta_{2n}^{-1})$$

- change of variables

$$W_1 = \frac{1}{\sqrt{w_1(t)}\sqrt{w_2(t)}\sqrt{w_3(t)}}, \quad W_2 = -\frac{\sqrt{w_1(t)}}{\sqrt{w_2(t)}\sqrt{w_3(t)}},$$

$$W_3 = \frac{\sqrt{w_2(t)}}{\sqrt{w_1(t)}\sqrt{w_3(t)}}, \qquad W_4 = \frac{\sqrt{w_3(t)}}{\sqrt{w_1(t)}\sqrt{w_2(t)}}$$

$$\zeta_1 = \xi_1$$

$$\zeta_2 = \xi_4 \cot(\eta)\cos(\psi) - \xi_3 \csc(\eta)\cos(\psi) + \xi_2 \sin(\psi)$$

$$\zeta_3 = -\xi_4 \cot(\eta)\sin(\psi) + \xi_3 \csc(\eta)\sin(\psi) + \xi_2 \cos(\psi)$$

$$\zeta_4 = \xi_4, \qquad \zeta_5 = \xi_5, \qquad \cdots \qquad \zeta_{2n+2} = \xi_{2n+2}$$

- rewrite integrand

$$\mathrm{tr}\,(\sigma_{-2n-2}) = \sum_{j=1}^{M_n} \Big\{ c_{j,2n}\,(\sin\eta)^{\beta_{0,1,j}}(\cos\eta)^{\beta_{0,2,j}}\,(\sin\psi)^{\beta_{1,1,j}}(\cos\psi)^{\beta_{1,2,j}}$$

$$\frac{\zeta_1^{\beta_{1,j}}\zeta_2^{\beta_{2,j}}\cdots\zeta_{2n+2}^{\beta_{2n+2,j}}}{Q_{W,2n}^{\rho_{j,2n}}}\prod_{i=1}^{3}\omega_{i,0}^{k_{i,0,j}}\omega_{i,1}^{k_{i,1,j}}\cdots\omega_{i,2n}^{k_{i,2n,j}}\Big\}$$

$$Q_{W,2n}(\zeta_1,\ldots,\zeta_{2n+2}) = W_1^2\zeta_1^2 + W_2^2\zeta_2^2 + W_3^2\zeta_3^2 + W_4^2\zeta_4^2 + \zeta_5^2 + \cdots + \zeta_{2n+2}^2$$

with $c_{j,2n} \in \mathbb{Q}$ and

$$\beta_{0,1,j}, \beta_{0,2,j}, \beta_{1,1,j}, \beta_{1,2,j}, k_{i,0,j} \in \mathbb{Z}$$

$$\beta_{1,j}, \ldots, \beta_{2n+2,j}, \rho_{j,2n}, k_{i,1,j}, \ldots, k_{i,2n,j} \in \mathbb{Z}_{\geq 0}$$

and parameters

$$\omega_{i,0} = w_i(t), \qquad \omega_{i,1} = w_i'(t), \qquad \ldots \qquad \omega_{i,2n} = w_i^{(2n)}(t)$$

• integration on the cosphere bundle and the 3-manifold
$|\xi|_g = \sum_{\mu,\nu=1}^4 g^{\mu\nu}\xi_\mu\xi_\nu + \xi_5^2 + \cdots + \xi_{2n+2}^2$

$$a_{2n}(t) = \frac{1}{32\,\pi^{n+3}} \int_0^\pi d\eta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi \int_{|\xi|_g=1} \mathrm{tr}\,(\sigma_{-2n-2})\,\sigma_{\xi,\,2n+1}$$

• use one-side $SU(2)$ symmetry of Bianchi IX to show that

$$\frac{1}{\sin(\eta)\,w_1(t)\,w_2(t)\,w_3(t)} \int_{|\xi|_g=1} \mathrm{tr}\,(\sigma_{-2n-2})\;\sigma_{\xi,\,2n+1}$$

is independent of the variables η, ϕ, ψ
• volume form in the ζ_i coordinates

$$\sigma_{\xi,\,2n+1} = \sum_{j=1}^{2n+2} (-1)^{j-1}\xi_j\, d\xi_1 \wedge \cdots \wedge \widehat{d}\xi_j \wedge \cdots \wedge d\xi_{2n+2} =$$

$$= \sin(\eta) \sum_{j=1}^{2n+2} (-1)^{j-1}\zeta_j\, d\zeta_1 \wedge \cdots \wedge \widehat{d}\zeta_j \wedge \cdots \wedge d\zeta_{2n+2}$$

$$= \sin(\eta)\;\sigma_{\zeta,\,2n+1}$$

- also replace domain $|\xi|_g = 1$ with homologous cycle given by unit sphere in the ζ_i coords

$$\sum_{i=1}^{2n+2} \zeta_i^2 = \xi_1^2 + \xi_2^2 + \csc^2(\eta)\xi_3^2 + \csc^2(\eta)\xi_4^2 - 2\cot(\eta)\csc(\eta)\xi_3\xi_4 + \xi_5^2 + \cdots + \xi_{2n+2}^2 = 1$$

- all terms with some $\beta_{0,1,j}, \beta_{0,2,j}, \beta_{1,1,j}, \beta_{1,2,j}$ odd integrate to zero
- denote by b_{-2n-2} remaining terms in $\mathrm{tr}(\sigma_{-2n-2})$
- coordinates μ_1 and μ_2 defined by

$$\mu_1 = -\cos(\eta)\cos(\psi), \qquad \mu_2 = \sin(\psi)$$

with

$$\sin^2(\psi) = \mu_2^2, \qquad \cos^2(\psi) = 1 - \mu_2^2$$

$$\sin^2(\eta) = \frac{1 - \mu_1^2 - \mu_2^2}{1 - \mu_2^2}, \qquad \cos^2(\eta) = \frac{\mu_1^2}{1 - \mu_2^2}$$

Period form of the integrals $\alpha_{2n} = a_{2n}(t)$

$$\alpha_{2n} = \frac{1}{\pi^{n+2}} \int_{A_{2n}} \frac{b_{-2n-2}}{1-\mu_2^2} \, d\mu_1 \wedge d\mu_2 \wedge \sigma_{\zeta,\, 2n+1}$$

- algebraic differential form

$$\frac{b_{-2n-2}}{1-\mu_2^2} \, d\mu_1 \wedge d\mu_2 \wedge \sigma_{\zeta,\, 2n+1}$$

defined on the complement in $\mathbb{A}^{2n+4}$ of the union of two hyperplanes

$$H_{\pm} = \{(\mu_1, \mu_2, \zeta_1, \dots, \zeta_{2n}) \in \mathbb{A}^{2n+4} \, : \, \mu_2 = \pm 1\}$$

and quadric $Q_{W,2n}(\zeta_1, \dots, \zeta_{2n}) = 0$

- integration over semi-algebraic set

$$A_{2n} = \Big\{ (\mu_1, \mu_2, \zeta_1, \zeta_2, \dots, \zeta_{2n+2}) \in \mathbb{A}^{2n+4}(\mathbb{R}) :$$

$$0 < \mu_1, \mu_2 < 1 \quad \text{and} \quad \sum_{i=1}^{2n+2} \zeta_i^2 = 1 \Big\}$$

Motive underlying the period

- number field $\mathbb{K}$, assume $W = (W_1, \dots, W_4) \in \mathbb{G}_m(\mathbb{K})^4$
- $Z_{W,2n} \subset \mathbb{P}^{2n+1}$ projective quadric defined by

$$Q_{W,2n}(\zeta_1, \dots, \zeta_{2n+2}) = \sum_{i=1}^{4} W_i^2 \zeta_i^2 + \sum_{i=5}^{2n+2} \zeta_i^2$$

$$Z_{W,2n} = \{(\zeta_1 : \dots : \zeta_{2n+2}) \in \mathbb{P}^{2n+1} \, : \, Q_{W,2n}(\zeta_1, \dots, \zeta_{2n+2}) = 0\}$$

- $\widehat{Z}_{W,2n}$ the affine cone in $\mathbb{A}^{2n+2}$
- $C^2 Z_{W,2n}$ the projective cone of $Z_{W,2n}$ in $\mathbb{P}^{2n+3}$
- $\widehat{C^2 Z}_{W,2n}$ the affine cone of $C^2 Z_{W,2n}$ in $\mathbb{A}^{2n+4}$

Motive

$$\mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus (H_+ \cup H_- \cup \widehat{C^2 Z}_{W,2n}), \Sigma)$$

$H_\pm$ hyperplanes

$$H_\pm = \{(\mu_1, \mu_2, \zeta_1, \dots, \zeta_{2n+2}) \in \mathbb{A}^{2n+4} \,:\, \mu_2 = \pm 1\}$$

Σ divisor in $\mathbb{A}^{2n+4}$ given by

$$\Sigma = \cup_{i=1}^2 \cup_{j=0}^1 H_{i,j}$$

$H_{i,j}$ hyperplanes

$$H_{i,j} = \{(\mu_1, \mu_2, \zeta_1, \dots, \zeta_{2n+2}) \in \mathbb{A}^{2n+4} \,:\, \mu_i = j\}$$

Grothendieck class

$$[\mathbb{A}^{2n+4} \smallsetminus (H_+ \cup H_- \cup \widehat{C^2 Z}_{W,2n})]$$

- Grothendieck classes and cones
  - $\mathbb{L} = [\mathbb{A}^1]$ Lefschetz motive, class of the affine line
  - $[\hat{Z}] = (\mathbb{L} - 1)[Z] + 1$
  - $[CZ] = \mathbb{L}[Z] + 1$ (projective cone union of a copy of Z and a copy of affine cone $\hat{Z}$)
  - $[C^2 Z] = \mathbb{L}[CZ] + 1 = \mathbb{L}^2[Z] + \mathbb{L} + 1$
  - $[\widehat{C^2 Z}] = (\mathbb{L} - 1)[C^2 Z] + 1 = \mathbb{L}^3[Z] + \mathbb{L}^2 - \mathbb{L}^2[Z] = \mathbb{L}^3[Z] - \mathbb{L}^2([Z] - 1)$
  - $[\widehat{CZ}] = (\mathbb{L} - 1)[CZ] + 1 = \mathbb{L}^2[Z] - \mathbb{L}([Z] - 1)$
- so have $[\mathbb{A}^{2n+2} \smallsetminus \hat{Z}] = (\mathbb{L} - 1)[\mathbb{P}^{2n+1} \smallsetminus Z]$

$$[\mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}] = (\mathbb{L}-1)[\mathbb{P}^{2n+3} \smallsetminus C^2 Z] = \mathbb{L}^{2n+4} - 1 - (\mathbb{L}-1)[C^2 Z] =$$

$$\mathbb{L}^{2n+4} - 1 - (\mathbb{L} - 1)(\mathbb{L}^2[Z] + \mathbb{L} + 1) = \mathbb{L}^{2n+4} - \mathbb{L}^3[Z] + \mathbb{L}^2([Z] - 1)$$

- by inclusion-exclusion

$$[\widehat{C^2 Z} \cup H_+ \cup H_-] = [\widehat{C^2 Z}] + [H_- \cup H_+] - [\widehat{C^2 Z} \cap (H_+ \cup H_-)]$$

$$= [\widehat{C^2 Z}] + 2\mathbb{L}^{2n+3} - 2[\widehat{CZ}]$$

$$[\widehat{C^2 Z} \cup H_+ \cup H_-] = \mathbb{L}^3[Z] - \mathbb{L}^2([Z]-1) + 2\mathbb{L}^{2n+3} - 2(\mathbb{L}^2[Z] - \mathbb{L}([Z]-1))$$

$$= 2\mathbb{L}^{2n+3} + \mathbb{L}^3[Z] - 3\mathbb{L}^2[Z] + 2\mathbb{L}[Z] + \mathbb{L}^2 - 2\mathbb{L}$$

- so get
  - $[\mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}] = \mathbb{L}^{2n+4} - \mathbb{L}^3[Z] + \mathbb{L}^2([Z]-1)$
  - $[\mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z} \cup H_+ \cup H_-)] =$
    $\mathbb{L}^{2n+4} - 2\mathbb{L}^{2n+3} - \mathbb{L}^3[Z] + 3\mathbb{L}^2[Z] - 2\mathbb{L}[Z] - \mathbb{L}^2 + 2\mathbb{L}$

Quadratic forms and field extensions

• assume number field $\mathbb{K}$ contains $\mathbb{Q}(\sqrt{-1})$ then change of variables

$$X_1 = W_1\zeta_1 + iW_2\zeta_2, \qquad Y_1 = W_1\zeta_1 - iW_2\zeta_2$$
$$X_2 = i(W_3\zeta_3 + iW_4\zeta_4), \quad Y_2 = i(W_3\zeta_3 - iW_4\zeta_4).$$

quadratic form $Q_{W,2}$ becomes

$$X_1 Y_1 - X_2 Y_2$$

projective quadric $Z_{W,2} \subset \mathbb{P}^3$ is Segre embedding

$$Z_{W,2} = \{X_1 Y_1 - X_2 Y_2 = 0\} \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

then changes of coordinates

$$X_n = \zeta_{2n-1} + i\zeta_{2n}, \quad Y_n = \zeta_{2n-1} - i\zeta_{2n}$$

quadratic form $Q_{W,2n}$ becomes

$$Q_{W,2n-2}(\zeta_1, \ldots, \zeta_{2n}) + X_n Y_n$$

Grothendieck class of the quadric and its complement

- compute inductively $C_{2n} = [\mathbb{A}^{2n+2} \smallsetminus \hat{Z}_{W,2n}]$:
  - $C_{2n} = \mathbb{L}^{2n+2} - \mathbb{L}^{2n+1} - \mathbb{L}^{n+1} + \mathbb{L}^n$
  - $[Z_{W,2n}] = 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1} + 2\mathbb{L}^n + \mathbb{L}^{n+1} + \cdots + \mathbb{L}^{2n}$
- after change of variables $Q_{W,2}$ becomes quadric $X_1 Y_1 - X_2 Y_2$

$$[Z_{W,2}] = [\mathbb{P}^1 \times \mathbb{P}^1] = \mathbb{L}^2 + 2\mathbb{L} + 1$$

$$[\hat{Z}_{W,2}] = (\mathbb{L} - 1)[Z_{W,2}] + 1 = (\mathbb{L} - 1)(\mathbb{L}^2 + 2\mathbb{L} + 1) + 1$$

$$= \mathbb{L}^3 + 2\mathbb{L}^2 + \mathbb{L} - \mathbb{L}^2 - 2\mathbb{L} - 1 + 1 = \mathbb{L}^3 + \mathbb{L}^2 - \mathbb{L}$$

$$C_2 = \mathbb{L}^4 - \mathbb{L}^3 - \mathbb{L}^2 + \mathbb{L}$$

- recursion relation: complement where

$$Q_{W,2n-2}(\zeta_1,\dots\zeta_{2n})+X_nY_n\neq 0$$

  - if $X_n=0$ then $Y_n\in\mathbb{A}^1$ and $Q_{W,2n-2}(\zeta_1,\dots\zeta_{2n})\neq 0$ so contribute $\mathbb{L}\cdot C_{2n-2}$ to class C_{2n}
  - if $X_n\neq 0$ then $Y_n\neq\frac{Q_{W,2n-2}(\zeta_1,\dots\zeta_{2n})}{X_n}$ with $(\zeta_1,\dots\zeta_{2n})\in\mathbb{A}^{2n}$ and $Y_n\in\mathbb{G}_m$ and $X_n\in\mathbb{G}_m$ contributes $[\mathbb{G}_m]^2\mathbb{L}^{2n}=\mathbb{L}^{2n}(\mathbb{L}-1)^2$

$$C_{2n}=\mathbb{L}^{2n+2}-2\mathbb{L}^{2n+1}+\mathbb{L}^{2n}+\mathbb{L}\cdot C_{2n-2}$$

- assume $C_{2n-2}=\mathbb{L}^{2n}-\mathbb{L}^{2n-1}-\mathbb{L}^n+\mathbb{L}^{n-1}$ then get

$$C_{2n}=\mathbb{L}^{2n+2}-2\mathbb{L}^{2n+1}+\mathbb{L}^{2n}+\mathbb{L}(\mathbb{L}^{2n}-\mathbb{L}^{2n-1}-\mathbb{L}^n+\mathbb{L}^{n-1})$$

$$=\mathbb{L}^{2n+2}-\mathbb{L}^{2n+1}-\mathbb{L}^{n+1}+\mathbb{L}^n$$

$$[Z_{W,2n}]=([\hat{Z}_{W,2n}]-1)(\mathbb{L}-1)^{-1}=(\mathbb{L}^{2n+1}+\mathbb{L}^{n+1}-\mathbb{L}^n-1)(\mathbb{L}-1)^{-1}$$

$$=1+\mathbb{L}+\cdots+\mathbb{L}^{2n}+\mathbb{L}^n$$

Grothendieck class

$$[\mathbb{A}^{2n+4} \smallsetminus (H_+ \cup H_- \cup \widehat{C^2 Z}_{W,2n})] =$$
$$\mathbb{L}^{2n+4} - 3\mathbb{L}^{2n+3} + 2\mathbb{L}^{2n+2} - \mathbb{L}^{n+3} + 3\mathbb{L}^{n+2} - 2\mathbb{L}^{n+1}$$

Motive:

- motive of a quandric $\mathfrak{m}(Z_{W,2n})$ is a Tate motive
- $\mathfrak{m}(\mathbb{P}^{2n+1} \smallsetminus Z_{W,2n})$ is mixed Tate because triangle

$$\mathfrak{m}(\mathbb{P}^{2n+1} \smallsetminus Z_{W,2n}) \to \mathfrak{m}(\mathbb{P}^{2n+1}) \to \mathfrak{m}(Z_{W,2n})(1)[2] \to \mathfrak{m}(\mathbb{P}^{2n+1} \smallsetminus Z_{W,2n})[1]$$

- $\mathbb{A}^1$-fibrations $\mathbb{P}^{2n+2} \smallsetminus CZ_{W,2n} \to \mathbb{P}^{2n+1} \smallsetminus Z_{W,2n}$ and $\mathbb{P}^{2n+3} \smallsetminus C^2 Z_{W,2n} \to \mathbb{P}^{2n+2} \smallsetminus CZ_{W,2n}$ so mixed Tate $\mathfrak{m}(\mathbb{P}^{2n+3} \smallsetminus C^2 Z_{W,2n})$
- $\mathbb{G}_m$-bundle $\mathcal{T} = \mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}_{W,2n} \to \mathbb{P}^{2n+3} \smallsetminus C^2 Z_{W,2n}$ and associated $\mathbb{P}^1$-bundle $\mathcal{P}$ with Gysin triangle

$$\mathfrak{m}(\mathcal{T}) \to \mathfrak{m}(\mathcal{P}) \to \mathfrak{m}(\mathcal{P} \smallsetminus \mathcal{T})^*(1)[2] \to \mathfrak{m}(\mathcal{T})[1]$$

and $\mathfrak{m}(\mathcal{P} \smallsetminus \mathcal{T})$ mixed Tate (two copies of base) so $\mathfrak{m}(\mathcal{T})$ mixed Tate

- $\mathfrak{m}(\mathcal{T}) = \mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}_{W,2n})$ mixed Tate
- Mayer-Vietoris distinguished triangle

$$\mathfrak{m}(U \cap V) \to \mathfrak{m}(U) \oplus \mathfrak{m}(V) \to \mathfrak{m}(U \cup V) \to \mathfrak{m}(U \cap V)[1]$$

with $U = \mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}_{W,2n}$ and $V = \mathbb{A}^{2n+4} \smallsetminus (H_+ \cup H_-)$, with $U \cup V = \mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z}_{W,2n} \cap (H_+ \cup H_-))$ and $U \cap V = \mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z}_{W,2n} \cup H_+ \cup H_-)$

- know $\mathfrak{m}(U)$ mixed Tate by previous
- form $\mathfrak{m}(V)$ know $\mathfrak{m}(H_+ \cup H_-)$ mixed Tate then Gysin triangle to get $\mathfrak{m}(V)$ mixed Tate
- intersection $\widehat{C^2 Z}_{W,2n} \cap (H_+ \cup H_-)$ is two sections of the cone isomorphic to $\widehat{CZ}_{W,2n}$

$$\mathfrak{m}(\widehat{C^2 Z}_{W,2n} \cap (H_+ \cup H_-)) = \mathfrak{m}(\widehat{CZ}_{W,2n}) \oplus \mathfrak{m}(\widehat{CZ}_{W,2n})$$

- $\mathfrak{m}(\widehat{CZ}_{W,2n})$ mixted Tate because complement is (Gysin triangle)

- so motive of union $\mathfrak{m}(U \cup V) = \mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z}_{W,2n} \cap (H_+ \cup H_-))$ also mixed Tate
- now in Mayer-Vietoris triangle $\mathfrak{m}(U)$, $\mathfrak{m}(V)$, $\mathfrak{m}(U \cup V)$ mixed Tate, so also $\mathfrak{m}(U \cap V)$
- get $\mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z}_{W,2n} \cup H_+ \cup H_-))$ mixed Tate
- for $\mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus (H_+ \cup H_- \cup \widehat{C^2 Z}_{W,2n}), \Sigma)$ also distinguished triangle for relative cohomology with $\mathfrak{m}(\Sigma)$ and previous mixed Tate so motive underlying period integral is mixed Tate

Gravitational instantons: these results hold for arbitrary Bianchi IX metrics, specially interesting case of Bianchi IX gravitational instantons has additional arithmetic structure given by modular forms

Blanchi IX gravitational instantons and Painlevé VI

• Euclidean Bianchi IX metrics with $SU(2)$-symmetry that are
- self-dual (Weyl curvature tensor W self-dual)
- Einstein metrics (Ricci tensor proportional to the metric)

• Self-dual equations for a Riemannian 4-manifold are PDEs; with $SU(2)$-symmetry reduce to ODEs

• This ODE is a Painlevé VI equation with

$$(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$$

- N.J. Hitchin. *Twistor spaces, Einstein metrics and isomonodromic deformations*, J.Diff.Geom., Vol. 42, No. 1 (1995), 30–112.
- K.P. Tod. *Self–dual Einstein metrics from the Painlevé VI equation*, Phys. Lett. A 190 (1994), 221–224.
- S. Okumura. *The self–dual Einstein–Weyl metric and classical solutions of Painlevé VI*, Lett. in Math. Phys., 46 (1998), 219–232.
- M.V. Babich, D.A. Korotkin, *Self–dual $SU(2)$–Invariant Einstein Metrics and Modular Dependence of Theta–Functions.* Lett. Math. Phys. 46 (1998), 323–337

Painlevé VI equations

• *Painlevé transcendents*: solutions of nonlinear second-order ODEs in the plane with *Painlevé property* (the only movable singularities are poles) not solvable in terms of elementary functions; classification in types

• *Painlevé VI*: 4-parameter family $(\alpha, \beta, \gamma, \delta)$

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2$$

$$-\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right).$$

Painlevé VI and elliptic curves

• Painlevé VI rewritten as (Fuchs)

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}\right]\int_{\infty}^{(X,Y)}\frac{dx}{\sqrt{x(x-1)(x-t)}}=$$

$$=\alpha Y+\beta\frac{tY}{X^2}+\gamma\frac{(t-1)Y}{(X-1)^2}+(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2}$$

where $(X,Y):=(X(t),Y(t))$ is a section (local and/or multivalued) $P:=(X(t),Y(t))$ of the generic elliptic curve $E=E(t):\ Y^2=X(X-1)(X-t)$

• left-hand-side $\mu(P)$ satisfies $\mu(P+Q)=\mu(P)+\mu(Q)$ for $P+Q$ addition on the elliptic curve E (in particular $\mu(Q)=0$ for points of finite order)

• analytic description of the elliptic curve $E_\tau = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, with $\tau \in \mathbb{H}$

• then Painlevé VI rewritten as (Manin)

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp_z\left(z + \frac{T_j}{2}, \tau\right)$$

with $(\alpha_0, \ldots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$ and $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1 + \tau)$, and

$$\wp(z, \tau) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2} \right)$$

• also have, for $e_i(\tau) = \wp(\frac{T_i}{2}, \tau)$

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau))$$

so $e_1 + e_2 + e_3 = 0$

• a multivalued solution $z = z(\tau)$ defines a multi–section of the family, which is a covering of $\mathbb{H}$

• is know ramification and monodromy can study behavior over geodesics in $\mathbb{H}$

- Yu.I. Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbb{P}^2$, in "Geometry of Differential Equations", Amer. Math. Soc. Transl. (2) Vol. 186 (1998) 131–151

Theta characteristics

• explicit paremeterization of solutions for coefficients W_i of the Bianchi IX gravitational instantons (from solutions of Painlevé VI)

• theta–characteristics with parameters (p, q):

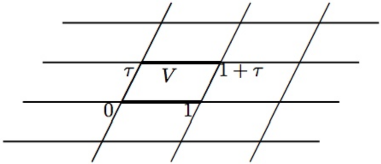
$$\vartheta[p,q](z,i\mu) := \sum_{m\in\mathbb{Z}} \exp\left(-\pi(m+p)^2\mu + 2\pi i(m+p)(z+q)\right)$$

• theta-characteristics and theta functions with vanishing characteristics

$$\vartheta[p,q](z,i\mu) = \exp\left(-\pi p^2\mu + 2\pi i pq\right) \cdot \vartheta[0,0](z + pi\mu + q, i\mu)$$

What's nice about theta characteristics?

• lattice in $\mathbb{C}$: entire doubly-periodic complex functions are constant, but quasi-periodic functions are interesting



• basic theta function $\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau)\exp(2\pi i n z)$

$$\vartheta(z+1,\tau) = \vartheta(z,\tau), \qquad \vartheta(z+\tau,\tau) = \exp(-\pi i \tau - 2\pi i z)\vartheta(z,\tau)$$

$$\vartheta(z+a\tau+b,\tau) = \exp(-\pi i a^2 \tau - 2\pi i a z)\vartheta(z,\tau)$$

automorphy factor $e_\tau(\lambda, z) = \exp(-\pi i a^2 \tau - 2\pi i a z)$

• more generally quasi-periodic theta characteristics
$\vartheta[p,q](z,\tau) = \sum_n \exp(\pi i(n+p)^2\tau)\exp(2\pi i(n+p)(z+q))$

• geometrically: no non-constant holomorphic functions on $E = \mathbb{C}/\Lambda$ but holomorphic sections of line bundles (from quasi-periodic functions)

• Abel theorem: meromorphic functions on $E = \mathbb{C}/\Lambda$ with zeros at a_i of order n_i and poles at b_j of order m_j

$$z \mapsto \frac{\prod_i \vartheta_\sigma(z-a_i,\tau)^{n_i}}{\prod_j \vartheta_\sigma(z-b_j,\tau)^{m_j}}$$

using $\vartheta_\sigma = \vartheta[\frac{1}{2},\frac{1}{2}]$ because simple zeros at $z \in \Lambda$

• theta characteristics ... function theory on elliptic curves

$$\wp(z) = -(\frac{\vartheta'_\sigma}{\vartheta_\sigma})'(z) + c$$

Weierstrass $\wp$-function

Gravitational instantons and theta characteristics

- use notation $\vartheta[p,q] := \vartheta[p,q](0, i\mu)$, and

$$\vartheta_2 := \vartheta[1/2, 0], \qquad \vartheta_3 := \vartheta[0,0], \qquad \vartheta_4 := \vartheta[0, 1/2]$$

- self-dual metrics

$$g = F\,(d\mu^2 + \frac{\sigma_1^2}{w_1^2} + \frac{\sigma_2^2}{w_2^2} + \frac{\sigma_3^2}{w_3^3})$$

with

$$w_1 = -\frac{i}{2}\vartheta_3\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p, q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]}, \quad w_2 = \frac{i}{2}\vartheta_2\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2}, q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]},$$

$$w_3 = -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2}, q]}{\vartheta[p,q]},$$

- with non-zero cosmological constant Λ:

$$F = \frac{2}{\pi\Lambda}\frac{w_1 w_2 w_3}{(\frac{\partial}{\partial q}\log\vartheta[p,q])^2}$$

- these metrics also satisfy Einstein equation if either

1. $\Lambda < 0$ with $p \in \mathbb{R}$ and $q \in \frac{1}{2} + i\mathbb{R}$
2. $\Lambda > 0$ with $q \in \mathbb{R}$ and $p \in \frac{1}{2} + i\mathbb{R}$

- also case with vanishing cosmological constant:

$$w_1 = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_2, \quad w_2 = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_3,$$

$$w_3 = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_4, \qquad F = C(\mu + q_0)^2\, w_1 w_2 w_3$$

with $q_0, C \in \mathbb{R}$, $C > 0$.

- M.V. Babich, D.A. Korotkin, *Self–dual $SU(2)$–Invariant Einstein Metrics and Modular Dependence of Theta–Functions.* Lett. Math. Phys. 46 (1998), 323–337
- Yuri Manin, Matilde Marcolli, *Symbolic Dynamics, Modular Curves, and Bianchi IX Cosmologies*, arXiv:1504.04005 [gr-qc]

Bianchi IX: time-dependent conformal perturbations

- original triaxial Bianchi IX:

$$ds^2 = w_1 w_2 w_3 \, d\mu^2 + \frac{w_2 w_3}{w_1} \sigma_1^2 + \frac{w_3 w_1}{w_2} \sigma_2^2 + \frac{w_1 w_2}{w_3} \sigma_3^2$$

$w_i = w_i(\mu)$ cosmic time μ

- time-dependent conformal perturbation:

$$d\tilde{s}^2 = F ds^2 = F \left(w_1 w_2 w_3 \, d\mu^2 + \frac{w_2 w_3}{w_1} \sigma_1^2 + \frac{w_3 w_1}{w_2} \sigma_2^2 + \frac{w_1 w_2}{w_3} \sigma_3^2 \right)$$

with $F = F(\mu)$

- effect on Dirac operator:

$$\tilde{D} = \frac{1}{\sqrt{F}} D + \frac{3F'}{4F^{\frac{3}{2}} w_1 w_2 w_3} \gamma^0$$

D Dirac operator of unperturbed Bianchi IX

- spectral action expansion for $\tilde{D}$ from heat kernel

$$\mathrm{Tr}\left(\exp(-t\tilde{D}^2)\right) \sim t^{-2}\sum_{n=0}^{\infty}\tilde{a}_{2n}t^n, \qquad t \to 0^+$$

- rationality result for coefficients of the spectral action

$$\tilde{a}_{2n} = \frac{\tilde{Q}_{2n}\left(w_1, w_2, w_3, F, w_1', w_2', w_3', F', \ldots, w_1^{(2n)}, w_2^{(2n)}, w_3^{(2n)}, F^{(2n)}\right)}{F^{2n}(w_1 w_2 w_3)^{3n-1}}$$

$\tilde{Q}_{2n}$ polynomial with rational coefficients

- zeroth coefficient: volume form (cosmological term)

$$\tilde{a}_0 = 4F^2 w_1 w_2 w_3$$

- second coefficient $\tilde{a}_2$: Einstein-Hilbert action

$$-\frac{F}{3}\left(w_1^2+w_2^2+w_3^2\right)+\frac{F}{6}\left(\frac{w_1^2 w_2^2 - w_3'^2}{w_3^2}+\frac{w_1^2 w_3^2 - w_2'^2}{w_2^2}+\frac{w_2^2 w_3^2 - w_1'^2}{w_1^2}\right)$$

$$-\frac{F}{3}\left(\frac{w_1' w_2'}{w_1 w_2} + \frac{w_1' w_3'}{w_1 w_3} + \frac{w_2' w_3'}{w_2 w_3}\right) + \frac{F}{3}\left(\frac{w_1''}{w_1} + \frac{w_2''}{w_2} + \frac{w_3''}{w_3}\right) - \frac{F'^2}{2F} + F''$$

- much longer and more complicated explicit formula for $\tilde{a}_4$ (Weyl conformal gravity and Gauss-Bonnet gravity)

Gravitational Instantons

- now assuming conformally perturbed Bianchi IX is self-dual Einstin metric and use parameterization by theta functions
- two-parameter family with non-vanishing cosmological constant:

$$w_1[p,q](i\mu) = -\frac{i}{2}\vartheta_3(i\mu)\vartheta_4(i\mu)\frac{\partial_q \vartheta[p, q+\frac{1}{2}](i\mu)}{e^{\pi i p}\vartheta[p,q](i\mu)}$$

$$w_2[p,q](i\mu) = \frac{i}{2}\vartheta_2(i\mu)\vartheta_4(i\mu)\frac{\partial_q \vartheta[p+\frac{1}{2}, q+\frac{1}{2}](i\mu)}{e^{\pi i p}\vartheta[p,q](i\mu)}$$

$$w_3[p,q](i\mu) = -\frac{1}{2}\vartheta_2(i\mu)\vartheta_3(i\mu)\frac{\partial_q \vartheta[p+\frac{1}{2}, q](i\mu)}{\vartheta[p,q](i\mu)}$$

$$F[p,q](i\mu) = \frac{2}{\pi\Lambda}\frac{1}{(\partial_q \ln \vartheta[p,q](i\mu))^2} = \frac{2}{\pi\Lambda}\left(\frac{\vartheta[p,q](i\mu)}{\partial_q\vartheta[p,q](i\mu)}\right)^2$$

- one-parameter family with vanishing cosmological constant:

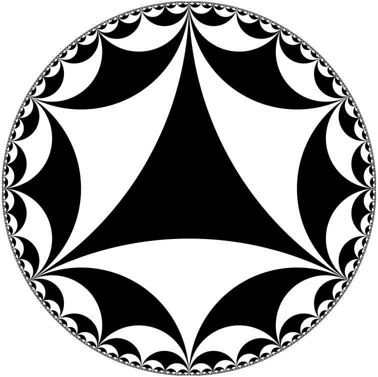
$$\begin{aligned} w_1[q_0](i\mu) &= \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_2(i\mu), \\ w_2[q_0](i\mu) &= \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_3(i\mu), \\ w_3[q_0](i\mu) &= \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_4(i\mu), \\ F[q_0](i\mu) &= C(\mu+q_0)^2, \end{aligned}$$

C arbitrary positive constant

What are modular forms?

• symmetries by a lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ in $\mathbb{C}$: elliptic curves, theta functions,...

• symmetry by a "hyperbolic lattice": fundamental domains of $\mathrm{SL}_2(\mathbb{Z})$ action on the hyperbolic plane: modular curve, modular forms

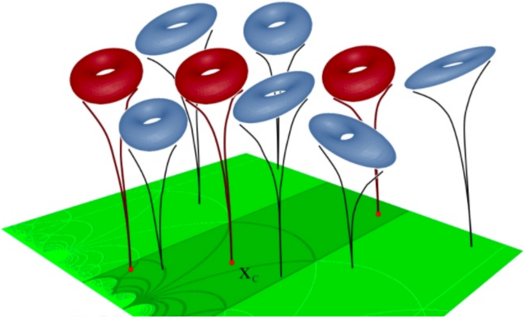


• the hyperbolic upper-half plane $\mathbb{H} = \{z \in \mathbb{C} : \Re(z) > 0\}$ with metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ (also model as Poincaré disk): isometries $\mathrm{PSL}_2(\mathbb{R})$ fractional linear transformations $g : z \mapsto \frac{az+b}{cz+d}$

• modular group: discrete subgroup $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ generators $S : z \mapsto -1/z$ and $T : z \mapsto z+1$

$$\Gamma = \langle S, T \,|\, S^2 = 1,\ (ST)^3 = 1 \rangle$$

• modular curve $X_\Gamma = \mathbb{H}/\Gamma$: moduli space of elliptic curves



Modular forms

• meromorphic functions on $\mathbb{H}$ satisfying

$$f(\frac{az+b}{cz+d}) = (cz+d)^{2k} f(z)$$

where $(cz+d)^{-2} = \frac{d(gz)}{dz}$ so modularity $f(gz)d(gz)^k = f(z)dz^k$

• on generators

$$f(z+1) = f(z), \quad f(-1/z) = z^{2k} f(z)$$

• holomorphic modular form (modular form): $f(z)$ holomorphic (including at infinity); cusp form: holomorphic and vanishing at infinity

Some significant examples

• Eisenstein series: sum over non-zero points of a lattice ($k > 1$)

$$G_k(z) = \sum_{(n,m)\in\mathbb{Z}^2\smallsetminus\{(0,0)\}} \frac{1}{(mz+n)^{2k}}$$

• modular discriminant: elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ as algebraic curve $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ with $g_2 = 60G_2$ and $g_3 = 140G_3$ when discriminant $\Delta \neq 0$

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau)$$

is a cusp modular form of weight 12

Modularity in the Spectral Action of Bianchi IX instantons

• generators of the modular group $\mathrm{PSL}_2(\mathbb{Z})$

$$T_1(\tau) = \tau + 1, \qquad S(\tau) = \frac{-1}{\tau}, \qquad \tau \in \mathbb{H}$$

• using behavior of theta functions and derivatives under modular transformations (two-parameter family):

$$\begin{aligned}
w_1[p,q](i\mu+1) &= w_1[p,q+p+\tfrac{1}{2}](i\mu), & w_1^{(n)}[p,q](i\mu+1) &= w_1^{(n)}[p,q+p+\tfrac{1}{2}](i\mu),\\
w_2[p,q](i\mu+1) &= w_3[p,q+p+\tfrac{1}{2}](i\mu), & w_2^{(n)}[p,q](i\mu+1) &= w_3^{(n)}[p,q+p+\tfrac{1}{2}](i\mu),\\
w_3[p,q](i\mu+1) &= w_2[p,q+p+\tfrac{1}{2}](i\mu), & w_3^{(n)}[p,q](i\mu+1) &= w_2^{(n)}[p,q+p+\tfrac{1}{2}](i\mu).
\end{aligned}$$

- for μ with $\Re(\mu) > 0$:

$$
\begin{aligned}
w_3[p,q](\frac{i}{\mu}) &= -\mu^2 w_1[-q,p](i\mu), \\
w_3'[p,q](\frac{i}{\mu}) &= \mu^4 w_1'[-q,p](i\mu) + 2\mu^3 w_1[-q,p](i\mu), \\
w_3''[p,q](\frac{i}{\mu}) &= -\mu^6 w_1''[-q,p](i\mu) - 6\mu^5 w_1'[-q,p](i\mu) - 6\mu^4 w_1[-q,p](i\mu), \\
w_3^{(3)}[p,q](\frac{i}{\mu}) &= \mu^8 w_1^{(3)}[-q,p](i\mu) + 12\mu^7 w_1''[-q,p](i\mu) + 36\mu^6 w_1'[-q,p](i\mu) \\
&\quad +24\mu^5 w_1[-q,p](i\mu), \\
w_3^{(4)}[p,q](\frac{i}{\mu}) &= -\mu^{10} w_1^{(4)}[-q,p](i\mu) - 20\mu^9 w_1^{(3)}[-q,p](i\mu) - 120\mu^8 w_1''[-q,p](i\mu) \\
&\quad -240\mu^7 w_1'[-q,p](i\mu) - 120\mu^6 w_1[-q,p](i\mu).
\end{aligned}
$$

- similar results for w_2 and w_3 under modular generator S

• conformal factor:

$$F[p,q](i\mu+1) = F[p,q+p+\frac{1}{2}](i\mu),$$
$$F^{(n)}[p,q](i\mu+1) = F^{(n)}[p,q+p+\frac{1}{2}](i\mu).$$

$$F[p,q](\frac{i}{\mu}) = -\mu^{-2}F[-q,p](i\mu),$$
$$F'[p,q](\frac{i}{\mu}) = F'[-q,p](i\mu) - 2\mu^{-1}F[-q,p](i\mu),$$
$$F''[p,q](\frac{i}{\mu}) = -\mu^{2}F''[-q,p](i\mu) + 2\mu F'[-q,p](i\mu) - 2F[-q,p](i\mu),$$
$$F^{(3)}[p,q](\frac{i}{\mu}) = \mu^{4}F^{(3)}[-q,p](i\mu),$$
$$F^{(4)}[p,q](\frac{i}{\mu}) = -\mu^{6}F^{(4)}[-q,p](i\mu) - 4\mu^{5}F^{(3)}[-q,p](i\mu).$$

• similar results for the case of the one-parameter family with vanishing cosmological constant

• modularity of spectral action coefficients:

$$\tilde{a}_0[p,q](i\mu+1) = \tilde{a}_0[p,q+p+\frac{1}{2}](i\mu)$$

$$\tilde{a}_2[p,q](i\mu+1) = a_2[p,q+p+\frac{1}{2}](i\mu)$$

$$\tilde{a}_4[p,q](i\mu+1) = \tilde{a}_4[p,q+p+\frac{1}{2}](i\mu)$$

$$\tilde{a}_0[p,q](\frac{i}{\mu}) = -\mu^2\tilde{a}_0[-q,p](i\mu)$$

$$\tilde{a}_2[p,q](\frac{i}{\mu}) = -\mu^2\tilde{a}_2[-q,p](i\mu)$$

$$\tilde{a}_4[p,q](\frac{i}{\mu}) = -\mu^2\tilde{a}_4[-q,p](i\mu)$$

Modularity of remaining coefficients $\tilde{a}_{2n}$

- Dirac operators $\tilde{D}^2[p,q]$, $\tilde{D}^2[p,q+p+\frac{1}{2}]$ and $\tilde{D}^2[-q,p]$ are isospectral
- heat kernel $K_t[p,q]$ of $\exp\left(-t\tilde{D}^2[p,q]\right)$ in terms of eigenvalues and eigenspinors $\Rightarrow$ modularity

$$
\begin{aligned}
K_t[p,q](i\mu_1+1, i\mu_2+1) &= K_t[p,q+p+\frac{1}{2}](i\mu_1, i\mu_2), \\
K_t[p,q](-\frac{1}{i\mu_1}, -\frac{1}{i\mu_2}) &= (i\mu_2)^2 K_t[-q,p](i\mu_1, i\mu_2).
\end{aligned}
$$

- then modularity of coefficients $\tilde{a}_{2n}$:

$$
\begin{aligned}
\tilde{a}_{2n}[p,q](i\mu+1) &= \tilde{a}_{2n}[p,q+p+\frac{1}{2}](i\mu), \\
\tilde{a}_{2n}[p,q](\frac{i}{\mu}) &= (i\mu)^2 \tilde{a}_{2n}[-q,p](i\mu).
\end{aligned}
$$

Vector valued modular forms

- coefficients satisfy:

$$\tilde{a}_{2n}[p+1,q] = \tilde{a}_{2n}[p,q+1] = \tilde{a}_{2n}[p,q],$$

- $\mathrm{PSL}_2(\mathbb{Z})$ action on $(p,q) \in \mathbb{R}/\mathbb{Z}^2$:

$$\begin{aligned} \tilde{S}(p,q) &= (-q,p) \\ \tilde{T}_1(p,q) &= (p, q+p+\frac{1}{2}) \end{aligned}$$

finite orbits $\mathcal{O}_{(p,q)}$ on rationals

- $\tilde{a}_{2n}[p',q'](i\mu)$, with $(p',q') \in \mathcal{O}_{(p,q)}$, vector-valued modular form of weight 2 for the modular group $\mathrm{PSL}_2(\mathbb{Z})$

- summing over orbits:

$$\tilde{a}_{2n}(i\mu; \mathcal{O}_{(p,q)}) = \sum_{(p',q')\in\mathcal{O}_{(p,q)}} \tilde{a}_{2n}[p',q'](i\mu)$$

is an ordinary modular form of weight 2 for $\mathrm{PSL}_2(\mathbb{Z})$

- Question: which modular form is it?
- analyze zeros and poles structure to find out
  - Example: for all n, modular form $\tilde{a}_{2n}(i\mu; \mathcal{O}_{(0,\frac{1}{3})})$ in one-dimensional space spanned by

$$\frac{G_{14}(i\mu)}{\Delta(i\mu)},$$

    with Δ modular discriminant (cusp form weight 12) and G_{14} is Eisenstein series weight 14
  - Example: for all n, modular form $\tilde{a}_{2n}(i\mu; \mathcal{O}_{(\frac{1}{6},\frac{5}{6})})$ in one-dimensional space spanned by

$$\frac{\Delta(i\mu)G_6(i\mu)}{G_4(i\mu)^4}$$