Arithmetic Structures in Spectral Models of Gravity

Matilde Marcolli

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References:

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• Wentao Fan, Farzad Fathizadeh, Matilde Marcolli, *Spectral Action for Bianchi type-IX cosmological models*, arXiv:1506.06779, J. High Energy Phys. (2015) 85, 28 pp.

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Spectral action models of gravity (modified gravity)

- Spectral triple: $(\mathcal{A}, \mathcal{H}, D)$

  1. unital associative algebra $\mathcal{A}$
  2. represented as bounded operators on a Hilbert space $\mathcal{H}$
  3. Dirac operator: self-adjoint $D^* = D$ with compact resolvent, with bounded commutators $[D, a]$

- prototype: $(C^\infty(M), L^2(M, S), \not{D}_M)$

- extends to non smooth objects (fractals) and noncommutative (NC tori, quantum groups, NC deformations, etc.)

- Spectral action (Chamseddine–Connes)

$$\mathcal{S}_{ST}(\Lambda) = \mathrm{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \mathrm{Spec}(D)} \mathrm{Mult}(\lambda) f(\lambda/\Lambda)$$

f = smooth approximation to cutoff

Robertson–Walker spacetime

- Topologically $S^3 \times \mathbb{R}$
- Metric (Euclidean)

$$ds^2 = dt^2 + a(t)^2 d\sigma^2$$

scaling factor $a(t)$, round metric $d\sigma^2$ on S^3

- Hopf coordinates on S^3

$$x = (t, \eta, \phi_1, \phi_2) \mapsto (t, \sin\eta\cos\phi_1, \sin\eta\sin\phi_2, \cos\eta\cos\phi_1, \cos\eta\sin\phi_2),$$

$$0 < \eta < \frac{\pi}{2}, \qquad 0 < \phi_1 < 2\pi, \qquad 0 < \phi_2 < 2\pi.$$

- Robertson-Walker metric in Hopf coordinates

$$ds^2 = dt^2 + a(t)^2 \left(d\eta^2 + \sin^2(\eta)\, d\phi_1^2 + \cos^2(\eta)\, d\phi_2^2 \right)$$

Dirac operator

$$D = \sum_{a,\mu} \gamma^a dx^\mu(\theta_a) \frac{\partial}{\partial x^\mu} + \frac{1}{4} \sum_{a,b,c} \gamma^c \omega^b_{ac} \gamma^a \gamma^b$$

with $\omega^b_a = \sum_c \omega^b_{ac} \theta^c$

• matrices γ^a Clifford action of θ^a on spin bundle:
$(\gamma^a)^2 = -I$
$\gamma^a\gamma^b + \gamma^b\gamma^a = 0$ for $a \neq b$

Pseudodifferential symbol of square D^2 of Dirac operator:

$$\sigma_{D^2}(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi),$$

$$p_2(x,\xi) = q_1(x,\xi)\, q_1(x,\xi) = \left(\sum g^{\mu\nu}\xi_\mu\xi_\nu\right) I_{4\times 4}$$

$$= \left(\xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\csc^2(\eta)\xi_3^2}{a(t)^2} + \frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right) I_{4\times 4},$$

$$p_1(x,\xi) = q_0(x,\xi)\, q_1(x,\xi) + q_1(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i \frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_1}{\partial x_j}(x,\xi),$$

$$p_0(x,\xi) = q_0(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i \frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_0}{\partial x_j}(x,\xi).$$

Parametrix Method and another method to compute coefficients

- D^2 order 2 elliptic differential operator: exists a parametrix R_λ with

$$\sigma(R_\lambda) \sim \sum_{j=0}^{\infty} r_j(x, \xi, \lambda)$$

- $r_j(x, \xi, \lambda)$ pseudodifferential symbol order $-2 - j$

$$r_j(x, t\xi, t^2\lambda) = t^{-2-j} r_j(x, \xi, \lambda)$$

- R_λ approximates $(D^2 - \lambda)^{-1}$ with $\sigma((D^2 - \lambda)R_\lambda) \sim 1$
- recursive equation:

$$\sigma((D^2 - \lambda)R_\lambda) \sim ((p_2(x, \xi) - \lambda) + p_1(x, \xi) + p_0(x, \xi)) \circ \left(\sum_{j=0}^{\infty} r_j(x, \xi, \lambda) \right) \sim 1$$

- solution for R_λ constructed recursively:

$$r_0(x,\xi,\lambda) = (p_2(x,\xi)-\lambda)^{-1}$$

$$r_n(x,\xi,\lambda) = -\sum \frac{1}{\alpha!}\partial_\xi^\alpha r_j(x,\xi,\lambda)\, D_x^\alpha p_k(x,\xi)\, r_0(x,\xi,\lambda),$$

summation over all $\alpha \in \mathbb{Z}^4_{\geq 0}, j \in \{0,1,\ldots,n-1\}, k \in \{0,1,2\}$, with $|\alpha| + j + 2 - k = n$

Seeley-deWitt coefficients and Parametrix Method

$$a_{2n}(x,D^2) = \frac{(2\pi)^{-m}}{2\pi i}\int\int_\gamma e^{-\lambda}\,\mathrm{tr}\,(r_{2n}(x,\xi,\lambda))\; d\lambda\, d^m\xi$$

- odd j coefficients vanish: $r_j(x,\xi,\lambda)$ odd function of ξ

A different method: Wodzicki residue

• Wodzicki residue: unique trace functional on algebra of pseudodifferential operators on smooth sections of vector bundle over smooth manifold

• classical pseudodifferential operator P_σ of order $d \in \mathbb{Z}$ local symbol

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{d-j}(x,\xi) \qquad (\xi \to \infty),$$

σ_{d-j} positively homogeneous order $d-j$ in ξ

• Residue:

$$\mathrm{Res}(P_\sigma) = \int_{S^*M} \mathrm{Tr}\left(\sigma_{-m}(x,\xi)\right)\, d^{m-1}\xi\, d^m x,$$

$S^*M = \{(x,\xi) \in T^*M; ||\xi||_g = 1\}$ cosphere bundle

• spectral formulation of residue: pseudodifferential operator P_σ, Laplacian Δ

$$P_\sigma \mapsto \mathrm{Res}_{s=0}\mathrm{Tr}(P_\sigma \Delta^{-s})$$

same up to a constant $c_m = 2^{m+1}\pi^m$

• Mellin transform (for simplicity $\mathrm{Ker}(\Delta)=0$):

$$\mathrm{Tr}(\Delta^{-s}) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}(e^{-t\Delta})\, t^s\, \frac{dt}{t}$$

• heat kernel expansion

$$\mathrm{Tr}\left(e^{-t\Delta}\right) = t^{-m/2}\sum_{n=0}^{N} a_{2n} t^n + O(t^{-m/2+N+1})$$

- find for any non-negative integer $n \leq m/2 - 1$:

$$\mathrm{Res}_{s=m/2-n}\mathrm{Tr}(\Delta^{-s}) = \frac{a_{2n}(\Delta)}{\Gamma(m/2-n)},$$

- in particular

$$\mathrm{Res}_{s=1}\mathrm{Tr}(\Delta^{-s}) = a_{m-2}(\Delta)$$

- in terms of Wodzicki residue:

$$a_{m-2}(\Delta) = \frac{1}{c_m}\mathrm{Res}(\Delta^{-1}) = \frac{1}{2^{m+1}\pi^m}\mathrm{Res}(\Delta^{-1})$$

applied to $\Delta = D^2$

• coefficient $a_2(D^2)$

$$a_2(D^2) = \frac{1}{c_4}\mathrm{Res}(D^{-2}) = \frac{1}{32\pi^4}\int_{S^*M} \mathrm{Tr}\left(\sigma_{-4}(D^{-2})\right)\, d^3\xi\, d^4x$$

• for other coefficients, introduce an auxiliary product space for correct counting of dimensions: use flat r-dimensional torus $\mathbb{T}^r = (\mathbb{R}/\mathbb{Z})^r$

$$\Delta = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r},$$

$\Delta_{\mathbb{T}^r}$ flat Laplacian on $\mathbb{T}^r$

$$a_{2+r}(D^2) = \frac{1}{2^5\, \pi^{4+r/2}}\mathrm{Res}(\Delta^{-1})$$

because Künneth formula gives

$$a_{2+r}((x,x'),\Delta) = a_{2+r}(x,D^2)a_0(x',\mathbb{T}^r) = 2^{-r}\pi^{-r/2}a_{2+r}(x,D^2)$$

with volume term only non-zero heat coefficient for flat metric

- obtain for higher order coefficients

$$a_{2+r}(D^2) = \frac{1}{2^5 \pi^{4+r/2}} \int \mathrm{Tr}\left(\sigma_{-4-r}(\Delta^{-1})\right)\, d^{3+r}\xi\, d^4x.$$

- writing $\sigma(\Delta^{-1}) \sim \sum_{j=-2}^{-\infty} \sigma_j(x,\xi)$ inductively

$$\begin{aligned} \sigma_{-2}(x,\xi) &= p_2'(x,\xi)^{-1}, \\ \sigma_{-2-n}(x,\xi) &= -\sum \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_j(x,\xi)\, D_x^\alpha p_k(x,\xi)\, \sigma_{-2}(x,\xi) \qquad (n>0), \end{aligned}$$

summation over all multi-indices non-negative integers α, $-2-n < j \leq -2, 0 \leq k \leq 2$, with $|\alpha| - j - k = n$

The a_2 term

- 1-density (unit cotangent sphere bundle integral)

$$\mathrm{wres}_x P_\sigma = \left(\int_{|\xi|=1} \mathrm{tr}\left(\sigma_{-m}(x,\xi)\right) |\sigma_{\xi,\, m-1}| \right) |dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{m-1}|$$

- Wodzicki residue of ΨDO P_σ

$$\mathrm{Res}\,(P_\sigma) = \int_M \mathrm{wres}_x P_\sigma$$

- $a_2(D^2)$ coefficient, with $(D^2)^{-1}$ parametrix

$$a_2 = \frac{1}{2^5\, \pi^4}\, \mathrm{Res}\left((D^2)^{-1}\right),$$

- dimension of manifold is 4: need term $\sigma_{-4}(x,\xi)$ homogeneous order -4 in expansion of symbol of $(D^2)^{-1}$

- computer calculation of $\text{tr}(\sigma_{-4}(x, \xi))$ takes a couple of pages to write out (sum of fractions involving trigonometric functions and powers of ξ_i, scaling factor $a(t)$ and derivative)
- important properties of resulting expression:
  - each term with an odd power of ξ_j in numerator will integrate to 0 in integration of 1-density
  - numerical coefficients of all terms in integrand are *rational numbers*
  - treat scaling factor $a(t)$ and derivative $a'(t)$, $a''(t)$ as affine variables $\alpha, \alpha_1, \alpha_2$ (integration without performing time integration)
  - there is a natural change of coordinates replacing trigonometric functions by polynomials: rational function

change of coordinates

$$u_0 = \sin^2(\eta), \qquad u_1 = \xi_1, \qquad u_2 = \xi_2,$$
$$u_3 = \csc(\eta)\,\xi_3, \qquad u_4 = \sec(\eta)\,\xi_4,$$

Then have

$$\xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\xi_3^2 \csc^2(\eta)}{a(t)^2} + \frac{\xi_4^2 \sec^2(\eta)}{a(t)^2} = u_1^2 + \frac{1}{a(t)^2}(u_2^2 + u_3^2 + u_4^2),$$
$$\cot^2(\eta) = \frac{1-u_0}{u_0},$$
$$\csc^2(\eta) = \frac{1}{u_0},$$
$$\sec^2(\eta) = \frac{1}{1-u_0},$$
$$\cot(\eta)\cot(2\eta) = \frac{\cot^2(\eta)}{2} - \frac{1}{2},$$
$$\csc^2(2\eta) = \frac{1}{4}\csc^2(\eta)\sec^2(\eta),$$
$$\tan^2(\eta) = \sec^2(\eta) - 1,$$
$$\tan(\eta)\cot(2\eta) = \frac{1}{2} - \frac{\tan^2(\eta)}{2},$$
$$\cot^2(2\eta) = \frac{\tan^2(\eta)}{8} + \frac{\cot^2(\eta)}{8} + \frac{1}{8}\csc^2(\eta)\sec^2(\eta) - \frac{3}{4}.$$

Also exponents of the variables ξ_j are even positive integers

a_2-term as a period integral $C \cdot \int_{A_4} \Omega^{\alpha}_{(\alpha_1,\alpha_2)}$ with $C \in \mathbb{Q}[(2\pi i)^{-1}]$

- Algebraic differential form

$$\Omega = f\,\widetilde{\sigma}_3,$$

in affine coordinates $(u_0, u_1, u_2, u_3, u_4) \in \mathbb{A}^5$, $\alpha \in \mathbb{G}_m$, and $(\alpha_1, \alpha_2) \in \mathbb{A}^2$

- functions $f(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2) = f_{(\alpha_1,\alpha_2)}(u_0, u_1, u_2, u_3, u_4, \alpha)$ $\mathbb{Q}$-linear combinations of rational functions

$$\frac{P(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2)}{\alpha^{2r} u_0^k (1-u_0)^m (u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2))^{\ell}}$$

where

$$P(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2) = P_{(\alpha_1,\alpha_2)}(u_0, u_1, u_2, u_3, u_4, \alpha)$$

polynomials in $\mathbb{Q}[u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2]$
with r, k, m and ℓ non-negative integers

• algebraic differential form $\widetilde{\sigma}_3 = \widetilde{\sigma}_3(u_0, u_1, u_2, u_3, u_4)$

$$\frac{1}{2}\Big(u_1\, du_0\, du_2\, du_3\, du_4 - u_2\, du_0\, du_1\, du_3\, du_4 + u_3\, du_0\, du_1\, du_2\, du_4 - u_4\, du_0\, du_1\, du_2\, du_3\Big)$$

• forms $\Omega^\alpha = \Omega^\alpha_{(\alpha_1,\alpha_2)}$ restricting to fixed value of $\alpha \in \mathbb{A}^1 \smallsetminus \{0\}$: two parameter family

• defined on the complement in $\mathbb{A}^5$ of union of two affine hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ and hypersurface $\widehat{CZ}_\alpha$ defined by vanishing of the quadratic form

$$Q_{\alpha,2} = u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2)$$

• $\mathbb{Q}$-semialgebraic set: subset S of some $\mathbb{R}^n$

$$S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, P(x_1, \dots, x_n) \geq 0\},$$

for some polynomial $P \in \mathbb{Q}[x_1, \dots, x_n]$, and complements, intersections, unions

• domain of integration $\mathbb{Q}$-semialgebraic set

$$A_4 = \left\{ (u_0, u_1, u_2, u_3, u_4) \in \mathbb{A}^5(\mathbb{R}) \,:\, \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1-u_0) u_4^2 = 1, \\ 0 < u_i < 1, \text{ for } i = 0, 1, 2 \end{array} \right\}$$

a_4-term and Wodzicki Residue

$$a_4 = \frac{1}{2^5\,\pi^5}\mathrm{Res}(\Delta_4^{-1})$$

need $\mathrm{tr}(\sigma_{-6}(\Delta_4^{-1}))$ of order -6 in expansion of symbol of Δ_4^{-1}

- general recursive procedure with auxiliary flat tori T^r

$$\Delta_{r+2} = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r}$$

$$\sigma_{-2}(\Delta_{r+2}^{-1}) = \left(p_2(x,\xi_1,\xi_2,\xi_3,\xi_4) + (\xi_5^2 + \cdots + \xi_{4+r}^2)I_{4\times 4}\right)^{-1}$$

then recursively $\sigma_{-2-n}(\Delta_{r+2}^{-1})$ given by

$$-\left(\sum_{\substack{0\leq j<n,\,0\leq k\leq 2\\ \alpha\in\mathbb{Z}^4_{\geq 0}\\ -2-j-|\alpha|+k=-n}} \frac{(-i)^{|\alpha|}}{\alpha!}\left(\partial_\xi^\alpha \sigma_{-2-j}(\Delta_{r+2}^{-1})\right)\left(\partial_x^\alpha p_k\right)\right)\sigma_{-2}(\Delta_{r+2}^{-1}).$$

a_4-term as a period integral $C \cdot \int_{A_6} \Omega^{\alpha}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$

- algebraic differential form

$$\Omega = f\, \widetilde{\sigma}_5,$$

in affine coordinates $(u_0, u_1, u_2, u_3, u_4, u_5, u_6) \in \mathbb{A}^7$, $\alpha \in \mathbb{G}_m$, and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{A}^4$

- functions $f_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha)$ $\mathbb{Q}$-linear combinations of rational functions

$$\frac{P(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4)}{\alpha^{2r} u_0^k (1-u_0)^m (u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2)^{\ell}}$$

where

$$P(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

polynomials in $\mathbb{Q}[u_0, u_1, u_2, u_3, u_4, u_5, u_6\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4]$ where r, k, m and ℓ non-negative integers

- algebraic form $\widetilde{\sigma}_5 = \widetilde{\sigma}_5(u_0, u_1, u_2, u_3, u_4, u_5, u_6)$

$$\begin{aligned} \widetilde{\sigma}_5 \;\; = \;\; & \frac{1}{2}\Big(u_1\, du_0\, du_2\, du_3\, du_4\, du_5\, du_6 - u_2\, du_0\, du_1\, du_3\, du_4\, du_5\, du_6 \\ & + u_3\, du_0\, du_1\, du_2\, du_4\, du_5\, du_6 - u_4\, du_0\, du_1\, du_2\, du_3\, du_5\, du_6 \\ & + u_5\, du_0\, du_1\, du_2\, du_3\, du_4\, du_6 - u_6\, du_0\, du_1\, du_2\, du_3\, du_4\, du_5 \Big). \end{aligned}$$

- forms $\Omega^\alpha = \Omega^\alpha_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ restricting to a fixed $\alpha \in \mathbb{A}^1 \smallsetminus \{0\}$: four-parameter family
- domain of definition complement in $\mathbb{A}^7$ of the union of the affine hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ and the hypersurface $\widehat{CZ}_\alpha$ defined by the vanishing of the quadratic form

$$Q_{\alpha,4} = u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2$$

- domain of integration $\mathbb{Q}$-semialgebraic set

$$A_6 = \left\{ (u_0, \ldots, u_6) \in \mathbb{A}^7(\mathbb{R}) \; : \; \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1 - u_0) u_4^2 + u_5^2 + u_6^2 = 1 \\ 0 < u_i < 1, \quad i = 0, 1, 2, 5, 6 \end{array} \right\}$$

- the change of variables used here

$$u_0 = \sin^2(\eta), \qquad u_1 = \xi_1, \qquad u_2 = \xi_2$$

$$u_3 = \csc(\eta)\,\xi_3, \qquad u_4 = \sec(\eta)\,\xi_4, \qquad u_5 = \xi_5, \qquad u_6 = \xi_6$$

higher order terms a_{2n}

$$a_{2n} = \frac{1}{2^5\,\pi^{3+n}} \mathrm{Res}(\Delta_{2n}^{-1})$$

using auxiliary flat tori T^{2n-2}

$$\Delta_{2n} = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^{2n-2}}$$

need term σ_{-2n-2} homogeneous of order $-2n-2$ in expansion of pseudodifferential symbol of parametrix Δ_{2n}^{-1}

- recursive argument for structure of term σ_{-2n-2}

- term $\mathrm{tr}(\sigma_{-2n-2})$ given by

$$\sum_{j=1}^{M_n} c_{j,2n}\, u_0^{\beta_{0,1,j}/2}\,(1-u_0)^{\beta_{0,2,j}/2}\,\frac{u_1^{\beta_{1,j}}\, u_2^{\beta_{2,j}}\,\cdots\, u_{2n+2}^{\beta_{2n+2,j}}}{Q_{\alpha,2n}^{\rho_{j,2n}}}\,\alpha^{k_{0,j}}\,\alpha_1^{k_{1,j}}\,\cdots\,\alpha_{2n}^{k_{2n,j}},$$

where

$$\alpha = a(t), \qquad \alpha_1 = a'(t), \qquad \alpha_2 = a''(t), \qquad \ldots \qquad \alpha_{2n} = a^{2n}(t),$$

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + \cdots + u_{2n+2}^2,$$

$$c_{j,2n} \in \mathbb{Q}, \quad \beta_{0,1,j}, \beta_{0,2,j}, k_{0,j} \in \mathbb{Z}, \quad \beta_{1,j}, \ldots, \beta_{2n+2,j}, \rho_{j,2n}, k_{1,j}, \ldots, k_{2n,j} \in \mathbb{Z}_{\geq 0}.$$

- using change of coordinates

$$u_0 = \sin^2(\eta), \qquad u_3 = \csc(\eta)\,\xi_3, \qquad u_4 = \sec(\eta)\,\xi_4$$

$$u_j = \xi_j, \qquad j = 1, 2, 5, 6, \ldots, 2n+2$$

a_{2n}-terms as period integrals $C \cdot \int_{A_{2n}} \Omega^{\alpha}_{\alpha_1,\ldots,\alpha_{2n}}$

- algebraic differential form

$$\Omega^{\alpha}_{\alpha_1,\ldots,\alpha_{2n}}(u_0, u_1, \ldots, u_{2n+2})$$

- domain of definition complement

$$\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1)$$

with hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ and $\widehat{CZ}_{\alpha,2n}$ the hypersurface defined by the vanishing of the quadric

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + \cdots + u_{2n+2}^2$$

- $\mathbb{Q}$-semialgebraic set A_{2n+2}

$$A_{2n+2} = \left\{ (u_0, \ldots, u_{2n+2}) \in \mathbb{A}^{2n+3}(\mathbb{R}) \; : \; \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1-u_0)u_4^2 + \sum_{i=5}^{2n+2} u_i^2 = 1 \\ 0 < u_i < 1, \quad i = 0, 1, 2, 5, 6, \ldots, 2n+2 \end{array} \right\}$$

Periods and Motives

• Main Idea: can constrain the type of numbers that can occur as *periods* $\int_\sigma \omega$ on a given algebraic variety X on the basis of information about the *motive* $\mathfrak{m}(X)$ of X

• Motives (Grothendieck) are a universal cohomology theory for algebraic varieties (morphisms: equivalence classes of algebraic cycles in the product)

- pure motives: smooth projective varieties
- mixed motives: more general varieties (quasi-projective, singular...)

in applications to physics one typically deals with *mixed motives*

Motives of algebraic varieties (Grothendieck) Universal cohomology theory for algebraic varieties (with realizations)

- *Pure motives*: smooth projective varieties with correspondences

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = q\mathrm{Corr}^{m-n}_{/\sim,\mathbb{Q}}(X,Y)\,p$$

Algebraic cycles mod equivalence (rational, homological, numerical), composition

$$\mathrm{Corr}(X,Y)\times\mathrm{Corr}(Y,Z)\to\mathrm{Corr}(X,Z)$$

$$(\pi_{X,Z})_*(\pi^*_{X,Y}(\alpha)\bullet\pi^*_{Y,Z}(\beta))$$

intersection product in $X\times Y\times Z$; with projectors $p^2=p$ and $q^2=q$ and Tate twists $\mathbb{Q}(m)$ with $\mathbb{Q}(1)=\mathbb{L}^{-1}$

Numerical pure motives: $\mathcal{M}_{num,\mathbb{Q}}(k)$ semi-simple abelian category (Jannsen)

• *Mixed motives*: varieties that are possibly singular or not projective (much more complicated theory!) Triangulated category $\mathcal{DM}$ (Voevodsky , Levine, Hanamura)

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1]$$

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2]$$

• *Mixed Tate motives* $\mathcal{DMT} \subset \mathcal{DM}$ generated by the $\mathbb{Q}(m)$

Over a number field: t-structure, abelian category of mixed Tate motives (vanishing result, M.Levine)

• Periods: $\int_\sigma \omega$ integrals of algebraic differential forms ω on a cycle σ defined by algebraic equations in an algebraic variety

Mixed Motives and Mixed Tate Motives

• there is a triangulated $\otimes$-category $\mathcal{DM}$ of mixed motives (Voevodsky , Levine, Hanamura)

$$\mathfrak{m}(U\cap V) \to \mathfrak{m}(U)\oplus\mathfrak{m}(V) \to \mathfrak{m}(X) \to \mathfrak{m}(U\cap V)[1] \quad \text{Mayer-Vietoris}$$

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1] \quad \text{Gysin}$$

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2] \quad \text{homotopy}$$

$$\mathfrak{m}(X)^\vee = \mathfrak{m}^c(X)(-d)[-2d] \quad \text{duality}$$

• Mixed Tate motives: triangulated $\otimes$-subcateory $\mathcal{DMT} \subset \mathcal{DM}$ generated by the Tate objects $\mathbb{Q}(m)$
$\mathbb{Q}(1)$ formal inverse of Lefschetz motive $\mathbb{L} = h^2(\mathbb{P}^1)$

• Method: to show $\mathfrak{m}(X)$ mixed Tate realize it in terms of distinguished triangles where two out of three terms are mixed Tate $\Rightarrow$ third one also is (or one is and one is not, then third also not)

• Over a number field: t-structure, abelian category of mixed Tate motives (vanishing result, M.Levine)

Motives and the Grothendieck ring of varieties

• Usually difficult to determine explicitly the motive of $\mathfrak{m}(X)$ in the triangulated category of *mixed motives*

• Simpler invariant (universal Euler characteristic for motives): class $[X_\Gamma]$ in the Grothendieck ring of varieties $K_0(\mathcal{V})$

- generators $[X]$ isomorphism classes
- $[X] = [X \setminus Y] + [Y]$ for $Y \subset X$ closed
- $[X] \cdot [Y] = [X \times Y]$

Tate motives: $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \subset K_0(\mathcal{M})$
(K_0 group of category of pure motives: virtual motives)

Universal Euler characteristics:

Any additive invariant of varieties: $\chi(X) = \chi(Y)$ if $X \cong Y$

$$\chi(X) = \chi(Y) + \chi(X \smallsetminus Y), \quad Y \subset X$$

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

values in a commutative ring $\mathcal{R}$ is same thing as a ring homomorphism

$$\chi : K_0(\mathcal{V}) \to \mathcal{R}$$

Examples:

- Topological Euler characteristic
- Couting points over finite fields
- Gillet–Soulé motivic $\chi_{mot}(X)$:

$$\chi_{mot} : K_0(\mathcal{V})[\mathbb{L}^{-1}] \to K_0(\mathcal{M}), \quad \chi_{mot}(X) = [(X, id, 0)]$$

for X smooth projective; complex $\chi_{mot}(X) = W^{\cdot}(X)$

Mixed Motives associated to spectral action coefficients as periods

$$\mathfrak{m}(\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1), \Sigma)$$

divisor Σ containing boundary of domain of integration A_{2n}

- motives of quadrics (Rost, Vishik)
  - hyperbolic form $\mathbb{H} := \langle 1, -1 \rangle$
  - $Q = d \cdot \mathbb{H}$ of dimension $2d$

$$\mathfrak{m}(Z_{d\mathbb{H}}) = \mathbb{Z}(d-1)[2d-2] \oplus \mathbb{Z}(d-1)[2d-2] \oplus \bigoplus_{i=0,\ldots,d-2,d,\ldots,2d-2} \mathbb{Z}(i)[2i]$$

  - $Q = d \cdot \mathbb{H} \perp \langle 1 \rangle$ in dimension $2d+1$

$$\mathfrak{m}(Z_{d\mathbb{H}\perp\langle 1\rangle}) = \bigoplus_{i=0,\ldots,2d-1} \mathbb{Z}(i)[2i]$$

  - if $\exists$ quadratic field extension $\mathbb{K}$ where Q hyperbolic

$$\mathfrak{m}(Z_Q) = \begin{cases} \mathfrak{m}_1 \oplus \mathfrak{m}_1(1)[2] & m = 2 \mod 4 \\ \mathfrak{m}_1 \oplus \mathcal{R}_{Q,\mathbb{K}} \oplus \mathfrak{m}_1(1)[2] & m = 0 \mod 4 \end{cases}$$

involving *forms of Tate motives*

- quadratic field extension $\mathbb{Q}(\sqrt{-1})$, assuming $\alpha \in \mathbb{Q}^*$

$$Q_{\alpha,2} = u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2)$$

change of variables

$$X = u_1 + \frac{i}{\alpha} u_2, \quad Y = u_1 - \frac{i}{\alpha} u_2, \quad Z = \frac{i}{\alpha}(u_3 + i u_4), \quad W = \frac{i}{\alpha}(u_3 - i u_4)$$

identification of Z_α with the Segre quadric

$$\{XY - ZW = 0\} \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

- similar for a_{2n}-term case

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2 + \cdots + u_{2n+1}^2 + u_{2n+2}^2$$

inductively: change of coordinates

$$X = u_{2n+1} + i u_{2n+2}, \qquad Y = u_{2n+1} - i u_{2n+2}$$

puts $Q_{\alpha,2n}$ in the form

$$Q_{\alpha,2n} = Q_{\alpha,2n-2}(u_1, \dots, u_{2n}) + XY.$$

classes in the Grothendieck ring

- $Z_{\alpha,2n}$ quadric in $\mathbb{P}^{2n+1}$ determined by $Q_{\alpha,2n}$

$$[\mathbb{P}^{2n+1} \smallsetminus Z_{\alpha,2n}] = \mathbb{L}^{2n+1} - \mathbb{L}^n$$

$$[\mathbb{A}^{2n+3} \smallsetminus \widehat{CZ}_{\alpha,2n}] = \mathbb{L}^{2n+3} - \mathbb{L}^{2n+2} - \mathbb{L}^{n+2} + \mathbb{L}^{n+1}$$

$$[\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1)] = \mathbb{L}^{2n+3} - 3\mathbb{L}^{2n+2} + 2\mathbb{L}^{2n+1} - \mathbb{L}^{n+2} + 3\mathbb{L}^{n+1} - 2\mathbb{L}^n$$

- based on an inductive argument using identities

1. $[\mathbb{A}^N \smallsetminus \hat{Z}] = (\mathbb{L} - 1)[\mathbb{P}^{N-1} \smallsetminus Z]$
2. $[\mathbb{A}^{N+1} \smallsetminus \widehat{CZ}] = (\mathbb{L} - 1)[\mathbb{P}^N \smallsetminus CZ]$
3. $[CZ] = \mathbb{L}\,[Z] + 1$
4. $[\mathbb{A}^{N+1} \smallsetminus \widehat{CZ}] = \mathbb{L}^{N+1} - \mathbb{L}(\mathbb{L} - 1)[Z] - \mathbb{L}$
5. $[\mathbb{A}^{N+1} \smallsetminus (\widehat{CZ} \cup H \cup H')] = \mathbb{L}^{N+1} - 2\mathbb{L}^N - (\mathbb{L} - 2)(\mathbb{L} - 1)[Z] - (\mathbb{L} - 2).$

with $Z \subset \mathbb{P}^{N-1}$, $\hat{Z} \subset \mathbb{A}^N$ affine cone, CZ projective cone in $\mathbb{P}^N$, H and H' affine hyperplanes with $H \cap H' = \emptyset$, intersections $\widehat{CZ} \cap H$ and $\widehat{CZ} \cap H'$ sections $\hat{Z}$ of cone

Mixed Tate

- mixed motive (over field $\mathbb{Q}(\sqrt{-1})$)

$$\mathfrak{m}(\mathbb{A}^{2n+3} \setminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1), \Sigma)$$

is mixed Tate

- over $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ quadratic form

$$Q_{\alpha,2n}|_{\mathbb{Q}(\sqrt{-1})} = (n+1) \cdot \mathbb{H},$$

so motive

$$\mathfrak{m}(Z_{\alpha,2n}|_{\mathbb{K}}) = \mathbb{Z}(n)[2n] \oplus \mathbb{Z}(n)[2n] \oplus \bigoplus_{i=0,\ldots,n-1,n+1,\ldots 2n} \mathbb{Z}(i)[2i]$$

- rest of the argument shown in example of a_2 for simplicity

- $\mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha)$ is mixed Tate

$$\mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha) \to \mathfrak{m}(\mathbb{P}^3) \to \mathfrak{m}(Z_\alpha)(1)[2] \to \mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha)[1]$$

Gysin distinguished triangle of the closed codim one embedding $Z_\alpha \hookrightarrow \mathbb{P}^3$

- projective cone CZ_α in $\mathbb{P}^4$: homotopy invariance for $\mathbb{A}^1$-fibration $\mathbb{P}^4 \smallsetminus CZ_\alpha \to \mathbb{P}^3 \smallsetminus Z_\alpha$

$$\mathfrak{m}_c^j(\mathbb{P}^4 \smallsetminus CZ_\alpha) = \mathfrak{m}_c^{j-2}(\mathbb{P}^3 \smallsetminus Z_\alpha)(-1)$$

motive $\mathfrak{m}(\mathbb{P}^4 \smallsetminus CZ_\alpha)$ also mixed Tate

- motive $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ mixed Tate: $\mathbb{P}^1$-bundle $\mathcal{P}$ compactification of $\mathbb{G}_m$-bundle

$$\mathcal{T} = \mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha \to X = \mathbb{P}^4 \smallsetminus CZ_\alpha$$

and Gysin distinguished triangle

$$\mathfrak{m}(\mathcal{T}) \to \mathfrak{m}(\mathcal{P}) \to \mathfrak{m}_c(\mathcal{P} \smallsetminus \mathcal{T})^*(1)[2] \to \mathfrak{m}(\mathcal{T})[1]$$

$\mathfrak{m}_c(\mathcal{P} \smallsetminus \mathcal{T})$ mixed Tate since $\mathcal{P} \smallsetminus \mathcal{T}$ two copies of X, so $\mathfrak{m}(\mathcal{T})$ mixed Tate

• union $\widehat{CZ}_\alpha \cup H_0 \cup H_1$ is mixed Tate: motives $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (H_0 \cup H_1))$ and $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ and motive of intersection $\mathfrak{m}(\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ are mixed Tate

$$\mathfrak{m}(U \cap V) \to \mathfrak{m}(U) \oplus \mathfrak{m}(V) \to \mathfrak{m}(U \cup V) \to \mathfrak{m}(U \cap V)[1]$$

Mayer-Vietoris distinguished triangle with $U = \mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha$ and $V = \mathbb{A}^5 \smallsetminus (H_0 \cup H_1)$

- $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ mixed Tate by previous
- $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (H_0 \cup H_1))$ also mixed Tate since $\mathfrak{m}(H_0 \cup H_1)$ is
- $\mathfrak{m}(\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ mixed Tate because intersection $\widehat{CZ}_\alpha \cap (H_0 \cup H_1)$ two sections of the cone and $\mathfrak{m}(\hat{Z}_\alpha)$ Tate
- then also $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ mixed Tate

• divisor Σ in $\mathbb{A}^5$ is a union of coordinate hyperplanes and their translates: mixed Tate

• $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1), \Sigma)$ also mixed Tate: distinguished triangle with $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ and $\mathfrak{m}(\Sigma)$

Conclusions:

- known since some time that in Quantum Field Theory the Feynman integrals in perturbative expansion are periods of motives
- expect algebro-geometric structures of this kind to occur elsewhere in physics
- spectral action coefficients for sufficiently *nice* (regular) spacetimes like Robertson–Walker or Bianchi IX find that indeed coefficients of the asymptotic expansion are also periods of motives
- in QFT only smaller Feynman diagrams (up to 8 loops for scalar field theory) give mixed Tate motives
- for spectral action of Robertson–Walker (or Bianchi IX) all the coefficients are mixed Tate periods