Spectral Action for Robertson–Walker metrics Part I: Brownian Bridge

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Reference

- Farzad Fathizadeh, Yeorgia Kafkoulis, Matilde Marcolli, *Bell polynomials and Brownian bridge in Spectral Gravity models on multifractal Robertson-Walker cosmologies*, arXiv:1811.02972

Other references

- A.H. Chamseddine, A. Connes, *Spectral action for Robertson-Walker metrics*, J. High Energy Phys. (2012) N.10, 101
- F. Fathizadeh, A. Ghorbanpour, M. Khalkhali, *Rationality of Spectral Action for Robertson-Walker Metrics*, JHEP (2014) N.12, 064

Spectral Action

$$\mathcal{S}_\Lambda = \mathrm{Tr}(f(\frac{D}{\Lambda})) = \sum_{\lambda \in \mathrm{Spec}(D)} f(\frac{\lambda}{\Lambda})$$

- D Dirac operator
- $\Lambda \in \mathbb{R}_+^*$ energy scale
- $f(x)$ test function (smooth approximation to cutoff function)
- Euclidean signature gravity

Asymptotic Expansion

- heat kernel expansion at $\tau \to 0^+$ for D^2 (Dirac Laplacian)

$$\mathrm{Tr}(e^{-\tau D^2}) \sim \sum_\alpha c_\alpha \tau^\alpha$$

- test function $f(x) = \int_0^\infty e^{-\tau x^2} d\mu(\tau)$ some measure μ normalized by $f(0) = \int_0^\infty d\mu(\tau)$
- asymptotic expansion of the spectral action (large Λ)

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\alpha<0} f_\alpha\, c_\alpha\, \Lambda^{-\alpha} + a_0 f(0) + \sum_{\alpha>0} f_\alpha\, c_\alpha\, \Lambda^{-\alpha}$$

- coefficients f_α given by

$$f_\alpha = \begin{cases} \int_0^\infty f(v) v^{-\alpha-1} dv & \alpha < 0 \\ (-1)^\alpha f^{(\alpha)}(0) & \alpha > 0,\ \alpha \in \mathbb{N} \end{cases}$$

Main Point: computing the expansion of the spectral action is the same problem as computing the coefficients of the heat kernel expansion of D^2

Robertson–Walker spacetime

- Topologically $S^3 \times \mathbb{R}$
- Metric (Euclidean)

$$ds^2 = dt^2 + a(t)^2 d\sigma^2$$

scaling factor $a(t)$, round metric $d\sigma^2$ on S^3

- Hopf coordinates on S^3

$$x = (t, \eta, \phi_1, \phi_2) \mapsto (t, \sin\eta\cos\phi_1, \sin\eta\sin\phi_2, \cos\eta\cos\phi_1, \cos\eta\sin\phi_2),$$

$$0 < \eta < \frac{\pi}{2}, \qquad 0 < \phi_1 < 2\pi, \qquad 0 < \phi_2 < 2\pi.$$

- Robertson-Walker metric in Hopf coordinates

$$ds^2 = dt^2 + a(t)^2 \left(d\eta^2 + \sin^2(\eta)\, d\phi_1^2 + \cos^2(\eta)\, d\phi_2^2 \right)$$

Dirac operator

• orthonormal coframe $\{\theta^a\}$

$$D = \sum_a \theta^a \nabla^S_{\theta_a}$$

• spin connection ∇^S with matrix of 1-forms $\omega = (\omega^a_b)$ with

$$\nabla \theta^a = \sum_b \omega^a_b \otimes \theta^b$$

• metric-compatibility and torsion-freeness (Levi–Civita connection)

$$\omega^a_b = -\omega^b_a, \qquad d\theta^a = \sum_b \omega^a_b \wedge \theta^b$$

• Dirac operator

$$D = \sum_{a,\mu} \gamma^a dx^\mu(\theta_a) \frac{\partial}{\partial x^\mu} + \frac{1}{4} \sum_{a,b,c} \gamma^c \omega^b_{ac} \gamma^a \gamma^b$$

with $\omega^b_a = \sum_c \omega^b_{ac} \theta^c$

• matrices γ^a Clifford action of θ^a on spin bundle:
$(\gamma^a)^2 = -I$
$\gamma^a\gamma^b + \gamma^b\gamma^a = 0$ for $a \neq b$

Dirac Laplacian on Robertson–Walker metrics

$$D^2 = -(\frac{\partial}{\partial t} + \frac{3a'(t)}{2a(t)})^2 + \frac{1}{a(t)^2}(\gamma^0 D_3)^2 - \frac{a'(t)}{a^2(t)}\gamma^0 D_3,$$

with

$$D_3 = \gamma^1(\frac{\partial}{\partial\chi} + \cot\chi) + \gamma^2\frac{1}{\sin\chi}(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta) + \gamma^3\frac{1}{\sin\chi\sin\theta}\frac{\partial}{\partial\phi}$$

Pseudo-differential Calculus: (manifold case)
to obtain *full* asymptotic expansion of the Spectral Action

- Dirac operator D and pseudodifferential symbol of D^2

$$\sigma(D^2)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$$

each p_k homogeneous of order k in ξ

- Cauchy integral formula

$$e^{-tD^2} = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} \, (D^2 - \lambda)^{-1} \, d\lambda$$

- Seeley de-Witt coefficients $(m = \dim M)$

$$\mathrm{Tr}(e^{-tD^2}) \sim_{t\to 0+} \quad t^{-m/2} \sum_{n=0}^{\infty} a_{2n}(D^2) t^n$$

Parametrix Method

• D^2 order 2 elliptic differential operator: exists a parametrix R_λ with

$$\sigma(R_\lambda) \sim \sum_{j=0}^{\infty} r_j(x,\xi,\lambda)$$

• $r_j(x,\xi,\lambda)$ pseudodifferential symbol order $-2-j$

$$r_j(x,t\xi,t^2\lambda) = t^{-2-j} r_j(x,\xi,\lambda)$$

• γ contour in $\mathbb{C}$ clockwise around $\mathbb{R}_-$

$$e^{-\tau D^2} = \frac{1}{2\pi i}\int_\gamma e^{-\tau\lambda}(D^2-\lambda)^{-1}\, d\lambda$$

• R_λ approximates $(D^2-\lambda)^{-1}$ with $\sigma((D^2-\lambda)R_\lambda) \sim 1$

- recursive equation:

$$\sigma((D^2-\lambda)R_\lambda) \sim ((p_2(x,\xi)-\lambda)+p_1(x,\xi)+p_0(x,\xi))\circ\left(\sum_{j=0}^{\infty} r_j(x,\xi,\lambda)\right) \sim 1$$

- recursively determine homog. pseudodifferential symbols r_j

$$r_n(x,\xi,\lambda) = -\sum \frac{1}{\alpha!}\partial_\xi^\alpha r_j(x,\xi,\lambda)\partial_x^\alpha p_k(x,\xi) r_0(x,\xi,\lambda)$$

with $r_0(x,\xi,\lambda) = (p_2(x,\xi)-\lambda)^{-1}$ and summation over $\alpha \in \mathbb{Z}_+^4$, $j \in 0,1,\dots,n-1$, $k \in \{0,1,2\}$ with $|\alpha|+j+2-k=n$, and p_k homogeneous components of symbol of D^2

• solution for R_λ constructed recursively:

$$r_0(x,\xi,\lambda) = (p_2(x,\xi) - \lambda)^{-1}$$

$$r_n(x,\xi,\lambda) = -\sum \frac{1}{\alpha!}\partial_\xi^\alpha r_j(x,\xi,\lambda)\, D_x^\alpha p_k(x,\xi)\, r_0(x,\xi,\lambda),$$

summation over all $\alpha \in \mathbb{Z}^4_{\geq 0}, j \in \{0,1,\ldots,n-1\}, k \in \{0,1,2\}$, with $|\alpha| + j + 2 - k = n$

Heat kernel coefficients from parametrix

$$\mathrm{Tr}(e^{-\tau D^2}) \sim_{\tau\to 0^+} \sum_{n=0}^{\infty} \frac{\tau^{(n-4)/2}}{16\pi^4} \int \mathrm{tr}(e_n(x))\, dvol_g$$

$$e_n(x)\cdot\sqrt{det(g)} = -\frac{1}{2\pi i}\int_\gamma e^{-\lambda} r_n(x,\xi,\lambda)\, d\lambda\, d\xi$$

Seeley-deWitt coefficients and Parametrix Method

$$a_{2n}(x, D^2) = \frac{(2\pi)^{-m}}{2\pi i} \int \int_\gamma e^{-\lambda} \operatorname{tr}\left(r_{2n}(x,\xi,\lambda)\right) \, d\lambda \, d^m\xi$$

• odd j coefficients vanish: $r_j(x,\xi,\lambda)$ odd function of ξ

coefficients prior to time-integration for Robertson–Walker

$$a_n(t) = \frac{1}{16\pi^4} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \operatorname{tr}(e_n) a^3(t) \sin(\eta) \cos(\eta) \, d\eta \, d\phi_1 \, d\phi_2$$

Pseudodifferential Symbol $\sigma_D(x,\xi)$ of Dirac operator D sum $q_1(x,\xi)+q_0(x,\xi)$ with $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)\in T^*_xM\simeq\mathbb{R}^4$ cotangent fiber at $x=(t,\eta,\phi_1,\phi_2)$

$$
\begin{aligned}
q_1(x,\xi) &= \begin{pmatrix}
0 & 0 & \frac{i\sec(\eta)\xi_4}{a(t)}-\xi_1 & \frac{i\xi_2}{a(t)}+\frac{\csc(\eta)\xi_3}{a(t)} \\
0 & 0 & \frac{i\xi_2}{a(t)}-\frac{\csc(\eta)\xi_3}{a(t)} & -\xi_1-\frac{i\sec(\eta)\xi_4}{a(t)} \\
-\xi_1-\frac{i\sec(\eta)\xi_4}{a(t)} & -\frac{i\xi_2}{a(t)}-\frac{\csc(\eta)\xi_3}{a(t)} & 0 & 0 \\
\frac{\csc(\eta)\xi_3}{a(t)}-\frac{i\xi_2}{a(t)} & \frac{i\sec(\eta)\xi_4}{a(t)}-\xi_1 & 0 & 0
\end{pmatrix}, \\
q_0(\xi) &= \begin{pmatrix}
0 & 0 & \frac{3ia'(t)}{2a(t)} & \frac{\cot(\eta)-\tan(\eta)}{2a(t)} \\
0 & 0 & \frac{\cot(\eta)-\tan(\eta)}{2a(t)} & \frac{3ia'(t)}{2a(t)} \\
\frac{3ia'(t)}{2a(t)} & \frac{\tan(\eta)-\cot(\eta)}{2a(t)} & 0 & 0 \\
\frac{\tan(\eta)-\cot(\eta)}{2a(t)} & \frac{3ia'(t)}{2a(t)} & 0 & 0
\end{pmatrix}.
\end{aligned}
\tag{1}
$$

Pseudodifferential symbol of square D^2 of Dirac operator:

$$\sigma_{D^2}(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi),$$

$$p_2(x,\xi) = q_1(x,\xi)\, q_1(x,\xi) = \left(\sum g^{\mu\nu}\xi_\mu\xi_\nu\right) I_{4\times 4}$$

$$= \left(\xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\csc^2(\eta)\xi_3^2}{a(t)^2} + \frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right) I_{4\times 4},$$

$$p_1(x,\xi) = q_0(x,\xi)\, q_1(x,\xi) + q_1(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i\frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_1}{\partial x_j}(x,\xi),$$

$$p_0(x,\xi) = q_0(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i\frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_0}{\partial x_j}(x,\xi).$$

Compute first terms for Robertson–Walker metric

$$a_0(t) = \frac{1}{2} a^3(t),$$

$$a_2(t) = \frac{a^3(t)}{4}\left(\frac{a''(t)}{a(t)} + \frac{(a'(t))^2 - 1}{a^2(t)}\right),$$

$$a_4(t) = \frac{1}{120}\left(3a^2(t)a^{(4)}(t) + 9a(t)a'(t)a^{(3)}(t) + 3a(t)(a'')^2(t) - 4(a')^2(t)a''(t) - 5a''(t)\right),$$

$$a_6(t) = -\frac{a'(t)^2 a''(t)}{240a^2(t)} - \frac{a'(t)^4 a''(t)}{84a^2(t)} + \frac{a''(t)^2}{120a(t)} + \frac{a'(t)^2 a''(t)^2}{21a(t)} - \frac{a''(t)^3}{90} + \frac{a'(t)a^{(3)}(t)}{240a(t)}$$
$$+ \frac{a'(t)a^{(3)}(t)}{84a(t)} - \frac{a'(t)a''(t)a^{(3)}(t)}{20} - \frac{a(t)a^{(3)}(t)^2}{1680} - \frac{a^{(4)}(t)}{240} - \frac{a'(t)^2 a^{(4)}(t)}{120}$$
$$+ \frac{a(t)a''(t)a^{(4)}(t)}{840} + \frac{a(t)a'(t)a^{(5)}(t)}{140} + \frac{a(t)^2 a^{(6)}(t)}{560}.$$

after the first few terms difficult to control the recursion: computationally hard

Examples from cosmological models

- inflation dominated universe $a(t) = e^{Ht}$

$$
\begin{aligned}
a_0(t) &= \frac{1}{2}e^{2Ht}, \\
a_2(t) &= \frac{2H^2e^{3Ht} - e^{Ht}}{4}, \\
a_4(t) &= 11H^4e^{3Ht} - 5H^2e^{Ht}, \\
a_6(t) &= \frac{-31}{2510}H^6e^{3Ht} + \frac{1}{240}H^4e^{Ht}.
\end{aligned}
$$

- radiation dominated universe $a(t) = (2Ht)^{1/2}$

$$
\begin{aligned}
a_0(t) &= \sqrt{2}(Ht)^{3/2}, \\
a_2(t) &= -\frac{\sqrt{2Ht}}{4}, \\
a_4(t) &= -\frac{\sqrt{2Ht}(-11H + 30t)}{1440t^2}, \\
a_6(t) &= \frac{-919 \cdot 2^{1/6}H^2t + 189 \cdot 2^{1/6}Ht^2 + 30 \cdot 6^{1/3}H(Ht)^{5/6}}{20160 \cdot 2^{2/3}t^5\sqrt{Ht}} \\
&+ \frac{21 \cdot 6^{1/3}t(Ht)^{5/6} + 126 \cdot 3^{2/3}H(Ht)^{7/6}}{20160 \cdot 2^{2/3}t^5\sqrt{Ht}}.
\end{aligned}
$$

- matter-dominated universe $a(t) = (\frac{3}{2}Ht)^{2/3}$

$$
\begin{aligned}
a_0(t) &= \frac{9}{8}H^2 t^2, \\
a_2(t) &= \frac{H^2}{8} - \frac{1}{4}\left(\frac{3}{2}\right)^{2/3}(Ht)^{2/3}, \\
a_4(t) &= \frac{1}{216}\frac{H^2}{t^2} + \frac{1}{72}\left(\frac{2}{3}\right)^{1/3}\frac{H^{2/3}}{t^{4/3}}, \\
a_6(t) &= \frac{5}{2916}\frac{H^2}{t^4} + \frac{11}{810 \cdot 2^{2/3} \cdot 3^{1/3}}\frac{H^{2/3}}{t^{10/3}}.
\end{aligned}
$$

- empty universe $a(t) = Ht$

$$
\begin{aligned}
a_0(t) &= \frac{1}{2}(Ht)^3, \\
a_2(t) &= \frac{H^3 t - Ht}{4},
\end{aligned}
$$

Rationality

property conjectured by Chamseddine–Connes, proved by Fathizadeh–Ghorbanpour–Khalkhali: coefficients are a rational function of $a(t)$ and derivatives with rational coefficients

$$a_{2m}(t) = \frac{Q_{2m}\left(a(t), a'(t), \ldots, a^{(2m)}(t)\right)}{a(t)^{2m-3}}$$

where $Q_{2m} \in \mathbb{Q}[x_0, x_1, \ldots, x_{2m}]$

$$Q_{2m}(x_0, x_1, \ldots, x_{2m}) = \sum_k c_{2m,k}\, x_0^{k_0} x_1^{k_1} \cdots x_{2m}^{k_{2m}}$$

with $c_{2m,k} \in \mathbb{Q}$ and for $k = (k_0, k_1, \ldots, k_{2m})$ in the summation

$$\text{either} \quad \sum_{j=0}^{2m} k_j = \sum_{j=0}^{2m} j k_j = 2m-2 \quad \text{or} \quad \sum_{j=0}^{2m} k_j = \sum_{j=0}^{2m} j k_j = 2m$$

Other computation method (Chamseddine–Connes)

- Dirac–Laplacian D^2 for Robertson–Walker metric

$$D^2 = -(\frac{\partial}{\partial t} + \frac{3a'(t)}{2a(t)})^2 + \frac{1}{a(t)^2}(\gamma^0 D_3)^2 - \frac{a'(t)}{a^2(t)}\gamma^0 D_3$$

$$D_3 = \gamma^1(\frac{\partial}{\partial \chi} + \cot\chi) + \gamma^2 \frac{1}{\sin\chi}(\frac{\partial}{\partial \theta} + \frac{1}{2}\cot\theta) + \gamma^3 \frac{1}{\sin\chi \sin\theta}\frac{\partial}{\partial \phi}$$

- $\gamma^0 D_3 = D_{S^3} \oplus -D_{S^3}$, Dirac operator on S^3
- Dirac spectrum on a round sphere S^d

$$\mathrm{Spec}(D_{S^d}) = \{\pm(k + d/2) \,:\, k \in \mathbb{Z}_+\}$$

multiplicities

$$\mu(\pm(k + d/2)) = 2^{[d/2]}\binom{k+d-1}{k}$$

- case of S^3

$$\mathrm{Spec}(D_{S^3}) = \{k + \frac{3}{2}\} \text{ multiplicities } \mu(k + \frac{3}{2}) = (k+1)(k+2)$$

• use basis of eigenfunctions of the Dirac operator on S^3 to decompose D^2 as direct sum of operators

$$H_n = -(\frac{d^2}{dt^2} - \frac{(n+\frac{3}{2})^2}{a^2} + \frac{(n+\frac{3}{2})a'}{a^2})$$

multiplicity $4(n+1)(n+2)$

• spectral action for test function $f(u) = e^{-su}$

$$\mathrm{Tr}(f(D^2)) \sim \sum_{n\geq 0} \mu(n)\,\mathrm{Tr}(f(H_n))$$

multiplicities $\mu(n) = 4(n+1)(n+2)$ and operator H_n

$$H_n = -\frac{d^2}{dt^2} + V_n(t),$$

$$V_n(t) = \frac{(n+\frac{3}{2})}{a(t)^2}((n+\frac{3}{2}) - a'(t))$$

Result of this approach

• to compute the spectral action for the Robertson–Walker metric need to evaluate the trace $\mathrm{Tr}(e^{-sH_n})$ which requires computing $e^{-sH_n}(t,t)$ (for coeffs prior to time integration)

Feynman–Kac formula

$$e^{-sH_n}(t,t) = \frac{1}{2\sqrt{\pi s}} \int \exp(-s \int_0^1 V_n(t+\sqrt{2s}\alpha(u))du)\, D[\alpha]$$

$D[\alpha]$ Brownian bridge integrals

Brownian bridge: Gaussian stochastic process characterized by the covariance

$$\mathbb{E}(\alpha(v_1)\alpha(v_2)) = v_1(1-v_2), \qquad 0 \le v_1 \le v_2 \le 1$$

Brownian motion and Brownian bridge

Reference for this background material:

- Barry Simon, *Functional Integration and Quantum Physics*, Academic Press, 1979

• Random walk: ν prob measure on $\{\pm 1\}$ with $\nu(-1) = \nu(1) = 1/2$ and $d\mu = \otimes_{n=1}^{\infty} d\nu_n$ on $\prod_{n=1}^{\infty}\{\pm 1\}$; take $X_n = \sum_{k=1}^{n} y_k$ with y_k the k-th coord; family of random variables $\{X_n\}_{n=1}^{\infty}$ random walk

• expectation $\mathbb{E}(y_n y_m) = \delta_{n,m}$ and $\mathbb{E}(X_n^2) = n$; rescaled $\tilde{X}_n = n^{-1/2} X_n$ approach a Gaussian random variable X_∞ (central limit theorem); for any continuous bounded function

$$\mathbb{E}(f(\tilde{X}_n)) \to (2\pi)^{-1/2} \int e^{-y^2/2} f(y)\, dy$$

• Brownian motion: $b(t) = \lim_{n\to\infty} n^{-1/2} X_{[nt]}$ integral part $[nt]$; Gaussian of variance t; since $X_n - X_m$ and X_m independent for $m < n$ get $b(t) - b(s)$ and $b(s)$ independent for $s < t$ so

$$\mathbb{E}(b(s)(b(t) - b(s))) = 0 \text{ hence } \mathbb{E}(b(s)b(t)) = s \; \forall s < t$$

• Wiener Process (Brownian motion) family $\{b(t)\}_{t\geq 0}$ of random variables with covariance $\mathbb{E}(b(s)b(t)) = \min\{s, t\}$

• characterization as unique invariant Gaussian Markov process up the changes of scale

Brownian motion and the Heat Kernel

• semigroup e^{-tH_0} with $H_0 = -\frac{1}{2}\frac{d^2}{dx^2}$ with integral kernel

$$P_t(x,y) = (2\pi)^{-1/2}\exp(-\frac{1}{2t}|x-y|^2)$$

• joint probability distribution of $b(s_1),\ldots,b(s_n)$ is

$$P_{t_1}(0,x_1)P_{t_2}(x_1,x_2)\cdots P_{t_n}(x_{n-1},x_n)$$

with $t_1 = s_1, t_2 = s_2 - s_1, \ldots, t_n = s_n - s_{n-1}$

• since $b(s_1), b(s_2) - b(s_1), \ldots, b(s_n) - b(s_{n-1})$ independent Gaussian random variables of variance t_k, joint distribution

$$P_{t_1}(0,y_1)P_{t_2}(0,y_2)\cdots P_{t_n}(0,y_n)$$

with $y_1 = x_1, y_2 = x_2 - x_1, \ldots, y_n = x_n - x_{n-1}$

Brownian bridge

• Gaussian process $\{\alpha(s)\}_{0\leq s\leq 1}$ with covariance

$$\mathbb{E}(\alpha(s)\alpha(t)) = s(1-t) \text{ for } 0 \leq s \leq t \leq 1$$

• relation to Brownian motion: $\alpha(s) = b(s) - sb(1)$

• setting for Feynman-Kac formula: operator $H = H_0 + V$ with potential and heat kernel e^{-tH}

• Trotter product formula:

$$\langle f, e^{-tH} g\rangle = \lim_{n\to\infty} \langle f, (e^{-tH_0/n} e^{-iV/n})^n g\rangle$$

• consequence of relation between Brownian motion and heat kernel of H_0:

$$\langle f_0, e^{-t_1 H_0} f_1 \cdots e^{-t_n H_0} f_n\rangle = \int f_0(\omega(s_0)) \cdots f_n(\omega(s_n))\, D[\omega]$$

with $D[\omega]$ Wiener measure; $t_k = s_k - s_{k-1}$ and $0 \leq s_0 < s_1 < \cdots < s_n$, with L^∞ functions, and $\omega(t) = x + b(t)$

- use previous two to write

$$\langle f, e^{-tH} g\rangle = \lim_{n\to\infty} \int \overline{f(\omega(0))} g(\omega(t)) \exp(-\frac{t}{n}\sum_{j=0}^{n-1} V(\omega(tj/n)))\, D[\omega]$$

$$\frac{t}{n}\sum_{j=0}^{n-1} V(\omega(tj/n)) \to \int_0^t V(\omega(s))\, ds$$

- resulting Feynman-Kac formula:

$$\langle f, e^{-tH} g\rangle = \int \overline{f(\omega(0))} g(\omega(t)) \exp(-\int_0^t V(\omega(s))\, ds)\, D[\omega]$$

- this gives $(e^{-tH} f)(0) = \int \exp(-\int_0^t V(b(s))\, ds)\, f(b(t))\, D[b]$
- Brownian bridge reformulation:

$$e^{-tH} = \frac{1}{2\sqrt{\pi t}} \int \exp(-t\int_0^1 V_n(t+\sqrt{2t}\alpha(u))du)\, D[\alpha]$$

Problem: technique used on Chamseddine–Connes for computing the Brownian bridge integrals becomes computationally intractable after the 10th or 12th term

New Method for computing the Brownian bridge integrals more efficiently and obtain the full expansion of the spectral action

notation $A(t) = 1/a(t)$ and $B(t) = A(t)^2$ so potential V_n

$$V_n(t) = x^2 A(t)^2 + xA'(t) = x^2 B(t) + xA'(t), \qquad \text{with } x = n + 3/2$$

Integral in Feynman–Kac formula becomes

$$-s \int_0^1 V_n(t + \sqrt{2s}\,\alpha(v))\, dv = -x^2 U - xV$$

where

$$U = s \int_0^1 A^2 \left(t + \sqrt{2s}\,\alpha(v)\right)\, dv = s \int_0^1 B \left(t + \sqrt{2s}\,\alpha(v)\right)\, dv$$

$$V = s \int_0^1 A' \left(t + \sqrt{2s}\,\alpha(v)\right)\, dv$$

Poisson Summation for summation on n index

$$f_s(x) := \left(x^2 - \frac{1}{4}\right) e^{-x^2 U - xV}$$

$$\int_{-\infty}^{\infty} f_s(x)\, dx = \frac{\sqrt{\pi}\, e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}}$$

Generating function for the full expansion of the spectral action

$$\frac{1}{\sqrt{\pi s}} \frac{\sqrt{\pi}\, e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}} = \frac{1}{\sqrt{s}} \frac{e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}}$$

then consider Laurent series expansion in the variable s

Laurent Series variable $\tau = s^{1/2}$

$$U = \tau^2 \sum_{n=0}^{\infty} \frac{u_n}{n!} \tau^n \quad \text{and} \quad V = \tau^2 \sum_{n=0}^{\infty} \frac{v_n}{n!} \tau^n$$

$$u_n = B^{(n)}(t)\, 2^{n/2} x_n(\alpha) = \left(\sum_{k=0}^{n} \binom{n}{k} A^{(k)}(t) A^{(n-k)}(t) \right) 2^{n/2}\, x_n(\alpha)$$

$$v_n = A^{(n+1)}(t) 2^{n/2}\, x_n(\alpha)$$

$$x_k(\alpha) = \int_0^1 \alpha(v)^k\, dv$$

Series Expansion: preliminaries

$$e^{\frac{V^2}{4U}}\, U^r\, V^m = \tau^{2(r+m)} \sum_{M=0}^{\infty} C_M^{(r,m)} \tau^M$$

where $C_M^{(r,m)}$ is given by the sum

$$\sum_{\substack{0\le k,p,N\le M\\ 0\le n\le M/2\\ N+2n=M\\ 1\le \ell_1,\dots,\ell_k,q_1,\dots,q_p\le N\\ \ell_1+\dots+\ell_k+q_1+\dots+q_p=N}} \frac{\binom{-n+r}{k}\binom{2n+m}{p}}{4^n n!}\, u_0^{-n+r-k}\, v_0^{2n+m-p}\, \frac{u_{\ell_1}\cdots u_{\ell_k} v_{q_1}\cdots v_{q_p}}{\ell_1!\cdots\ell_k!\, q_1!\cdots q_p!}$$

Heat Kernel Expansion $\tau = s^{1/2} \to 0^+$

$$\mathrm{Tr}(\exp(-\tau^2 D^2)) \sim \sum_{M=0}^{\infty} \tau^{2M-4} \int a_{2M}(t)\, dt,$$

$$a_{2M}(t) = \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} \left(C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)} \right) \right) D[\alpha]$$

for $M \in \mathbb{Z}_{\geq 0}$ (zero is neg index $2M-2$)

$$\frac{1}{\tau} \frac{e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}} =$$

$$\frac{1}{4} \sum_{M=0}^{\infty} \left(C_M^{(-5/2,2)} - C_M^{(-1/2,0)} \right) \tau^{M-2} + \frac{1}{2} \sum_{M=0}^{\infty} C_M^{(-3/2,0)} \tau^{M-4}.$$

Bell Polynomials

- Faà di Bruno formula for derivatives of composite functions

$$\frac{d^n}{dt^n} f(g(t)) = \sum_{m=1}^{n} f^{(m)}(g(t))\, B_{n,m}(g'(t), g''(t), \ldots, g^{(n-m+1)}(t))$$

- multivariable Bell polynomials $B_{\beta,k}(x_1, \ldots, x_{\beta-k+1})$

$$\sum_{\lambda} \frac{\beta!}{\lambda_1! \lambda_2! \cdots \lambda_{\beta-k+1}!} \left(\frac{x_1}{1!}\right)^{\lambda_1} \left(\frac{x_2}{2!}\right)^{\lambda_2} \cdots \left(\frac{x_{\beta-k+1}}{(\beta-k+1)!}\right)^{\lambda_{\beta-k+1}}$$

summation over sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers with

$$|\lambda|' := \sum_{i=1}^{\infty} i\, \lambda_i = \beta, \qquad |\lambda| := \sum_{i=1}^{\infty} \lambda_i = k.$$

Spectral Action and Bell Polynmials

• coefficients in the spectral action expansion expressible in terms of Bell polynomials

$$C_{2M}^{(r,m)} = \sum_{\substack{0\leq k,p\leq 2M \\ 0\leq n\leq M \\ 0\leq \beta\leq 2M-2n}} \left(\frac{\binom{-n+r}{k}\binom{2n+m}{p}\binom{2M-2n}{\beta}\, k!\, p!}{4^n\, n!\,(2M-2n)!} u_0^{-n+r-k} v_0^{2n+m-p} \times \right.$$

$$\left. B_{\beta,k}\left(u_1,\ldots,u_{\beta-k+1}\right) B_{2M-2n-\beta,\,p}\left(v_1,\ldots,v_{2M-2n-\beta-p+1}\right)\right).$$

Structure of Brownian Bridge Integrals

Step 1: integrals of monomials on the standard simplex

$$\Delta^n = \{(v_1, v_2, \dots, v_n) \in \mathbb{R}^n : 0 \le v_1 \le v_2 \le \cdots \le v_n \le 1\}.$$

monomial $v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$

$$\int_{\Delta^n} v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} \, dv_1 \, dv_2 \cdots dv_n =$$

$$\frac{1}{(k_1+1)(k_1+k_2+2)\cdots(k_1+k_2+\cdots+k_n+n)}$$

Similarly for $1 \le j_1 < j_2 < \cdots < j_k \le n$

$$\int_{\Delta^n} v_{j_1} v_{j_2} \cdots v_{j_k} \, dv_1 \, dv_2 \cdots dv_n = \frac{j_1(j_2+1)(j_3+2)\cdots(j_k+k-1)}{(n+k)!}$$

Step 2: Brownian Bridge and integration on the simplex

- Using variance property of Brownian Bridge: $(v_1, v_2, \ldots, v_{2n}) \in \Delta^{2n}$

$$\int \alpha(v_1)\alpha(v_2)\cdots\alpha(v_{2n})\, D[\alpha] = \sum v_{i_1}(1-v_{j_1})v_{i_2}(1-v_{j_2})\cdots v_{i_n}(1-v_{j_n})$$

summation over indices with $i_1 < j_1$, $i_2 < j_2$, $\ldots$, $i_n < j_n$, and $\{i_1, j_1, i_2, j_2, \ldots, i_n, j_n\} = \{1, 2, \ldots, 2n\}$

- equivalently for $(v_1, v_2, \ldots, v_{2n}) \in \Delta^{2n}$

$$\int \alpha(v_1)\alpha(v_2)\cdots\alpha(v_{2n})\, D[\alpha] =$$

$$\sum_{\sigma \in S^*_{2n}} v_{\sigma(1)}(1 - v_{\sigma(2)})v_{\sigma(3)}(1 - v_{\sigma(4)})\cdots v_{\sigma(2n-1)}(1 - v_{\sigma(2n)})$$

S^*_{2n} set of all permutations σ in symmetric group S_{2n} with $\sigma(1) < \sigma(2)$, $\sigma(3) < \sigma(4)$, $\ldots$, $\sigma(2n-1) < \sigma(2n)$

Brownian Bridge Integrals

- Notation: $\mathcal{J}_{k,n}$ = set of all k-tuples of integers $J = (j_1, j_2, \ldots, j_k)$ such that $1 \leq j_1 < j_2 < \cdots < j_k \leq n$; for $J \in \mathcal{J}_{k,n}$ and $\sigma \in S^*_{2n}$ define $\sigma_J(1)$, $\sigma_J(2)$, $\ldots$, $\sigma_J(n+k)$ by property that

$$\sigma_J(1) < \sigma_J(2) < \cdots < \sigma_J(n+k)$$

and that the set of such σ_J's is given by
$\{\sigma_J(1) < \sigma_J(2) < \cdots < \sigma_J(n+k)\}$

$$= \{\sigma(1), \sigma(3), \ldots, \sigma(2n-1), \sigma(2j_1), \ldots, \sigma(2j_k)\}$$

$$x_k(\alpha) = \int_0^1 \alpha(v)^k \, dv$$

- Brownian Bridge Integrals

$$\int x_1(\alpha)^{2n} \, D[\alpha] = \int \Big(\int_0^1 \alpha(v) \, dv \Big)^{2n} D[\alpha] =$$

$$(2n)! \sum_{\sigma \in S^*_{2n}} \sum_{k=0}^{n} \sum_{J \in \mathcal{J}_{k,n}} (-1)^k \frac{\sigma_J(1)\,(\sigma_J(2)+1)\cdots(\sigma_J(n+k)+n+k-1)}{(3n+k)!}$$

Monomial Brownian Bridge Integrals

- for $(v_1, v_2, \ldots, v_n) \in \Delta^n$ and for $i_1, i_2, \ldots, i_n \in \mathbb{Z}_{\geq 0}$ such that $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\geq 0}$

$$\int \alpha(v_1)^{i_1} \alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n} \, D[\alpha] =$$

$$\binom{|I|}{I}^{-1} \frac{|I|!}{(\sqrt{-1})^{|I|}} \frac{(-1/2)^{|I|/2}}{(|I|/2)!} \left(\sum \binom{|I|/2}{k_{m,j}} \sum_{r_1=0}^{K_1} \sum_{r_2=0}^{K_2} \cdots \sum_{r_n=0}^{K_n} \prod_{p=1}^{n} (-1)^{r_p} v_p^{i_p - r_p} \right),$$

with $I = (i_1, i_2, \ldots, i_n)$, first summation over non-negative integers $k_{j,m}$, $j, m = 1, 2, \ldots, n$ such that

$$\sum_{j,m=1}^{n} k_{j,m} = \frac{|I|}{2}, \qquad \sum_{m=1}^{n} (k_{j,m} + k_{m,j}) = i_j \text{ for all } j = 1, 2, \ldots, n$$

and for each $m = 1, 2, \ldots, n$,

$$K_m := k_{m,m} + \sum_{j=1}^{m-1} (k_{j,m} + k_{m,j})$$

Sketch of proof

$$\int \exp\left(\sqrt{-1}\sum_{j=1}^{n} u_j \alpha(v_j)\right) D[\alpha] = \exp\left(-\frac{1}{2}\sum_{j,m=1}^{n} c_{j,m} u_j u_m\right)$$

where the terms $c_{j,m}$ are given by

$$c_{j,m} = v_j(1-v_m) \quad \text{if} \quad j \leq m, \quad \text{and} \quad c_{j,m} = v_m(1-v_j) \quad \text{if} \quad m \leq j$$

Expanding gives

$$\frac{(\sqrt{-1})^{i_1+i_2+\cdots+i_n}}{(i_1+i_2+\cdots+i_n)!}\binom{i_1+i_2+\cdots+i_n}{i_1,i_2,\ldots,i_n}\int \alpha(v_1)^{i_1}\alpha(v_2)^{i_2}\cdots\alpha(v_n)^{i_n}\, D[\alpha] =$$

$$\frac{(-1/2)^{(i_1+i_2+\cdots+i_n)/2}}{((i_1+i_2+\cdots+i_n)/2)!}\left(\text{Coefficient of } u_1^{i_1}u_2^{i_2}\cdots u_n^{i_n} \text{ in } \Big(\sum_{j,m=1}^{n} c_{j,m}u_j u_m\Big)^{(i_1+i_2+\cdots+i_n)/2}\right)$$

$$= \frac{(-1/2)^{(i_1+i_2+\cdots+i_n)/2}}{((i_1+i_2+\cdots+i_n)/2)!}\sum \binom{(i_1+i_2+\cdots+i_n)/2}{k_{1,1},k_{1,2},\ldots,k_{1,n},k_{2,1},\ldots,k_{n,n}}\prod_{j,m=1}^{n} c_{j,m}^{k_{j,m}}$$

from which then can group terms as stated

Shuffle Product

- for $(v_1, v_2, \ldots, v_n) \in \Delta^n$ and $i_1, i_2, \ldots, i_n \in \mathbb{Z}_{\geq 0}$ with $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\geq 0}$

$$V^b(i_1, i_2, \ldots, i_n) := \int \alpha(v_1)^{i_1} \alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n} \, D[\alpha]$$

- extend V^b linearly to vector space generated by all words $(i_1, i_2, \ldots, i_n)$ in the letters $i_1, i_2, \ldots, i_n$
- Shuffle product $\alpha \sqcup\!\sqcup \beta$ of two words $\alpha = (i_1, i_2, \ldots, i_p)$ and $\beta = (j_1, j_2, \ldots, j_q)$ sum of $\binom{p+q}{p}$ words obtained by interlacing letters of these two words so that in each term the order of the letters of each word is preserved

• $2n = m_1 i_1 + m_2 i_2 + \cdots + m_r i_r$ even positive integer with $i_1, i_2, \ldots, i_r$ distinct positive integers and $m_1, m_2, \ldots, m_r$ positive integers

$$\int x_{i_1}(\alpha)^{m_1} x_{i_2}(\alpha)^{m_2} \cdots x_{i_r}(\alpha)^{m_r} \, D[\alpha] =$$

$$m! \int_{\Delta^{|m|}} V^b(\underbrace{(i_1, \ldots, i_1)}_{m_1} \sqcup \underbrace{(i_2, \ldots, i_2)}_{m_2} \sqcup \cdots \sqcup \underbrace{(i_r, \ldots, i_r)}_{m_r}) dv_1 dv_2 \cdots dv_{|m|}$$

$$m! = (m_1!)(m_2!) \cdots (m_r!), \qquad |m| = m_1 + m_2 + \cdots + m_r \, .$$

follows directly from writing

$$\int x_{i_1}(\alpha)^{m_1} x_{i_2}(\alpha)^{m_2} \cdots x_{i_r}(\alpha)^{m_r} \, D[\alpha]$$

$$= \int \Big(\int_0^1 \alpha(v_1)^{i_1} \, dv_1 \Big)^{m_1} \Big(\int_0^1 \alpha(v_2)^{i_2} \, dv_2 \Big)^{m_2} \cdots \Big(\int_0^1 \alpha(v_r)^{i_r} \, dv_r \Big)^{m_r} D[\alpha],$$

Brownian Bridge Integrals in the Coefficients of the Spectral Action

$$\int C_{2M}^{(r,m)}\, D[\alpha] =$$

$$\sum \left(\frac{\binom{-n+r}{k}\binom{2n+m}{p} k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b\big(\underbrace{(1,\dots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\dots,2)}_{\lambda_2+\mu_2} \sqcup \cdots \big)\, dv_1 \cdots dv_{k+p} \right.$$

$$\left. \times B(t)^{-n+r-k}\, (A'(t))^{2n+m-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!} \right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!} \right)^{\mu_i} \right)$$

summation is over integers $0 \le k, p \le 2M, 0 \le n \le M$, $0 \le \beta \le 2M - 2n$, and over sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ of non-negative integers for each choice of k, p, n, β, such that $|\lambda|' = \beta, |\lambda| = k$, $|\mu|' = 2M - 2n - \beta, |\mu| = p$

coefficients of the expansion of the spectral action of Robertson–Walker metric

$$a_{2M}(t) =$$

$$\frac{1}{2}\sum{}' \Bigg(\frac{\binom{-n-3/2}{k}\binom{2n}{p} k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b\big(\underbrace{(1,\dots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\dots,2)}_{\lambda_2+\mu_2} \sqcup \cdots\big)\, dv_1 \cdots dv_{k+p} \times$$

$$B(t)^{-n-(3/2)-k}\, \big(A'(t)\big)^{2n-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \Bigg)$$

$$+\frac{1}{4}\sum{}'' \Bigg(\Bigg(\binom{-n-5/2}{k}\binom{2n+2}{p} B(t)^{-5/2}\big(A'(t)\big)^2 - \binom{-n-1/2}{k}\binom{2n}{p} B(t)^{-1/2} \Bigg) \times$$

$$\frac{k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b\big(\underbrace{(1,\dots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\dots,2)}_{\lambda_2+\mu_2} \sqcup \cdots\big)\, dv_1 \cdots dv_{k+p} \times$$

$$B(t)^{-n-k}\, \big(A'(t)\big)^{2n-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \Bigg)$$

summation $\sum{}'$ is over all integers $0 \le k, p \le 2M, 0 \le n \le M,\ 0 \le \beta \le 2M - 2n$, and sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ of non-negative integers (for each choice of k, p, n, β) such that $|\lambda|' = \beta, |\lambda| = k,\ |\mu|' = 2M - 2n - \beta, |\mu| = p$; second summation $\sum{}''$ is over all integers $0 \le k, p \le 2M - 2, 0 \le n \le M - 1,\ 0 \le \beta \le 2M - 2 - 2n$, over all sequences $\lambda = (\lambda_1, \lambda_2, \dots),\ \mu = (\mu_1, \mu_2, \dots)$ of non-negative integers such that $|\lambda|' = \beta, |\lambda| = k,\ |\mu|' = 2M - 2 - 2n - \beta, |\mu| = p$

Faà di Bruno Hopf algebra

- structure of coefficients $a_{2M}(t)$ based on Bell Polynomials
- Bell polynomials and the Faà di Bruno formula have a Hopf algebra interpretation
- affine group scheme $G^{\rm diff}(A)$ formal diffeomorphisms tangent to the identity

$$f(t) = t + \sum_{n\geq 2} f_n\, t^n \in tA[[t]]$$

A a unital commutative algebra over a field $\mathbb{K}$, product given by composition

- Faà di Bruno Hopf algebra $G^{\rm diff}(A) = {\rm Hom}(\mathcal{H}_{\rm FdB}, A)$
- Connes–Kreimer Hopf algebra of renormalization in QFT

$$\mathcal{H}_{\rm FdB} \hookrightarrow \mathcal{H}_{CK} \quad \text{dually} \quad G_{CK} \twoheadrightarrow G^{\rm diff}$$

- description of spectral action in terms of Brownian bridge integrals suitable for treatment as a quantum theory
- expression in terms of Bell polynomials (with the Faà di Bruno Hopf algebra action) suggests Hopf-algebraic renormalization structure