Introduction to Noncommutative Geometry Methods in Physics

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Focus of the class:

- *Geometrization of physics*: general relativity, gauge theories, extra dimensions ...

- Focus on: *Standard Model* of elementary particle physics, and its extensions (right handed neutrinos, supersymmetry,
  Pati-Salaam, ...)

- Also focus on *Matter coupled to gravity*: curved backgrounds

- *Gravity and cosmology*: modified gravity models, cosmic topology, multifractal structures in cosmology, gravitational instantons

- *Mathematical perspective*: spectral and noncommutative geometry

- Other physics applications: condensed matter, statistical physics, etc.

What is a good geometric model of physics?

- *Simplicity*: difficult computations follow from simple principles

- *Predictive power*: recovers known physical properties and provides new insight on physics, from which new testable calculations

- *Elegance*: “entia non sunt multiplicanda praeter necessitatem” (Ockham’s razor)

Two Standard Models:

- Standard Model of Elementary Particles

- Standard Cosmological Model

For both theories looking for possible extensions: for particles right-handed-neutrino sector, supersymmetry, dark matter, ...; for cosmology modified gravity, brane-cosmology, other dark matter and dark energy models, inflation scenarios, ...

Both theories depend on parameters that are not fixed by the theory

Particle physics and cosmology interact (early universe models)

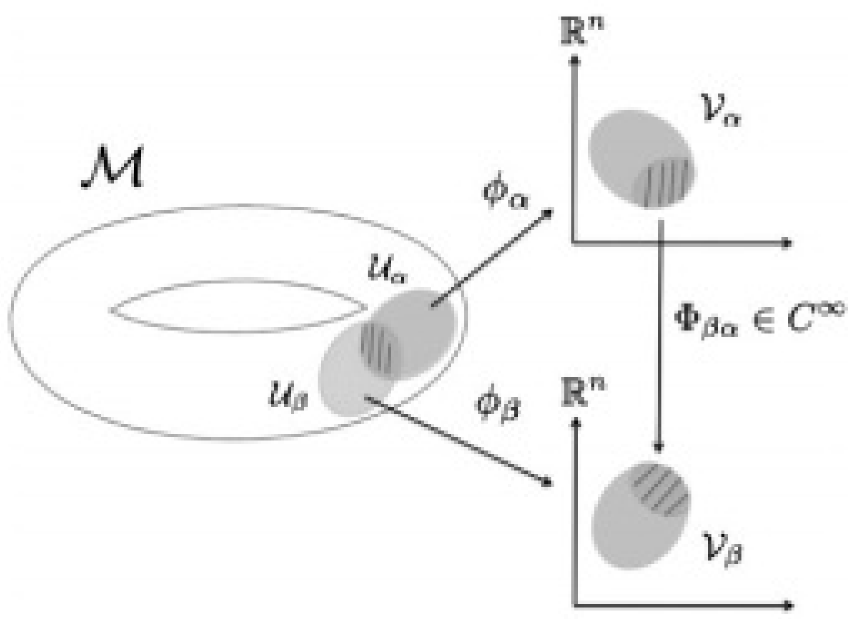
What input from geometry on these models?

Basic Mathematical Toolkit

Smooth Manifolds: M (with $\partial M = \emptyset$)
locally like $\mathbb{R}^n$: smooth atlas

$$M = \cup_i U_i, \quad \phi_i : U_i \stackrel{\simeq}{\to} \mathbb{R}^n$$

homeomorphisms, local coordinates $x_i = (x_i^\mu)$,
on $U_i \cap U_j$ change of coordinates $\phi_{ij} = \phi_j \circ \phi_i^{-1}$
$\mathcal{C}^\infty$-diffeomorphisms



Vector Bundles: $E \to M$ vector bundle rank N over manifold of dim n

- projection $\pi : E \to M$

- $M = \cup_i U_i$ with $\phi_i : U_i \times \mathbb{R}^N \stackrel{\simeq}{\to} \pi^{-1}(U_i)$ homeomorphisms with $\pi \circ \phi_i(x, v) = x$

- transition functions: $\phi_j^{-1} \circ \phi_i : (U_i \cap U_j) \times \mathbb{R}^N \to (U_i \cap U_j) \times \mathbb{R}^N$ with

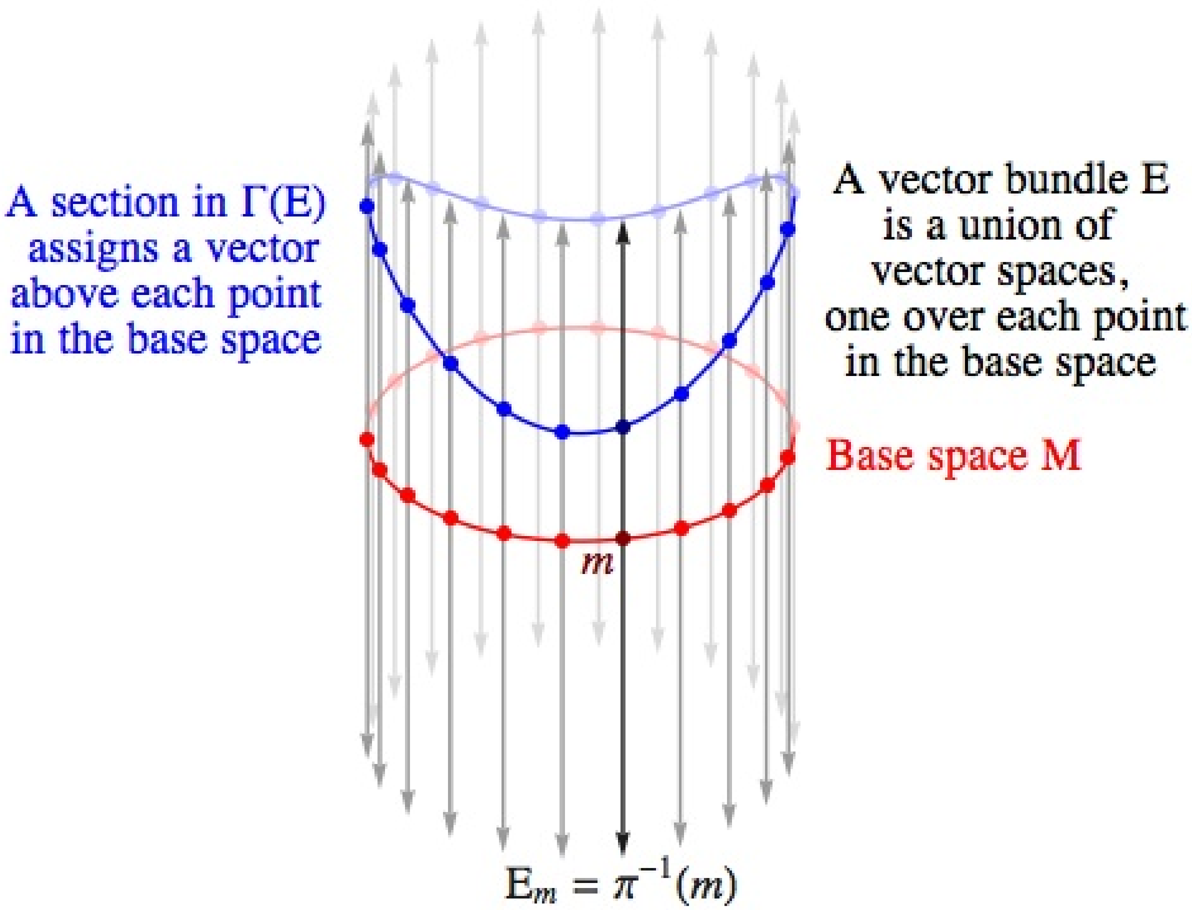
$$\phi_j^{-1} \circ \phi_i(x, v) = (x, \phi_{ij}(x)v)$$

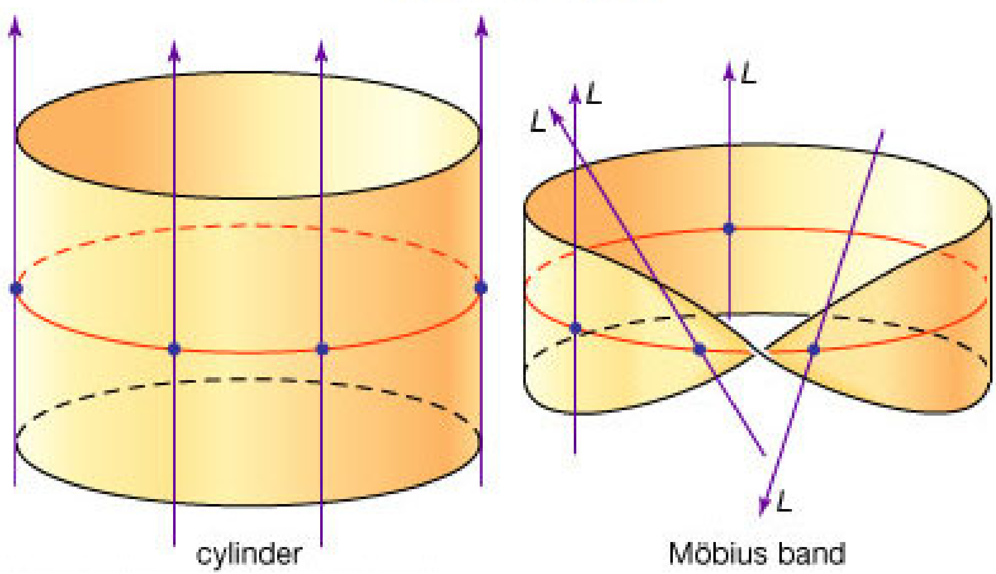
$$\phi_{ij} : U_i \cap U_j \to \mathrm{GL}_N(\mathbb{C})$$

  satisfy cocyle property: $\phi_{ii}(x) = id$ and $\phi_{ij}(x)\phi_{jk}(x)\phi_{ki}(x) = id$

- Sections: $s \in \Gamma(U, E)$ open $U \subseteq M$ maps $s : U \to E$ with $\pi \circ s(x) = x$

$$s(x) = \phi_i(x, s_i(x)) \quad \text{and } s_i(x) = \phi_{ij}(x)s_j(x)$$


A section in Γ(E)
assigns a vector
above each point
in the base space
A vector bundle E
is a union of
vector spaces,
one over each point
in the base space
Base space M
m
$E_m = \pi^{-1}(m)$


L
L
L
L
cylinder
Möbius band

Tensors and differential forms as sections of vector bundles

- Tangent bundle TM (tangent vectors), cotangent bundle T^*M (1-forms)

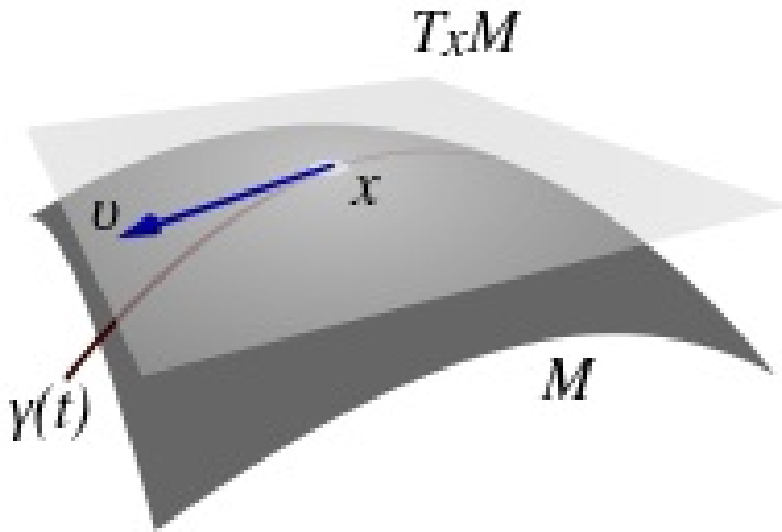
- Vector field: section $V = (v^\mu) \in \Gamma(M, TM)$

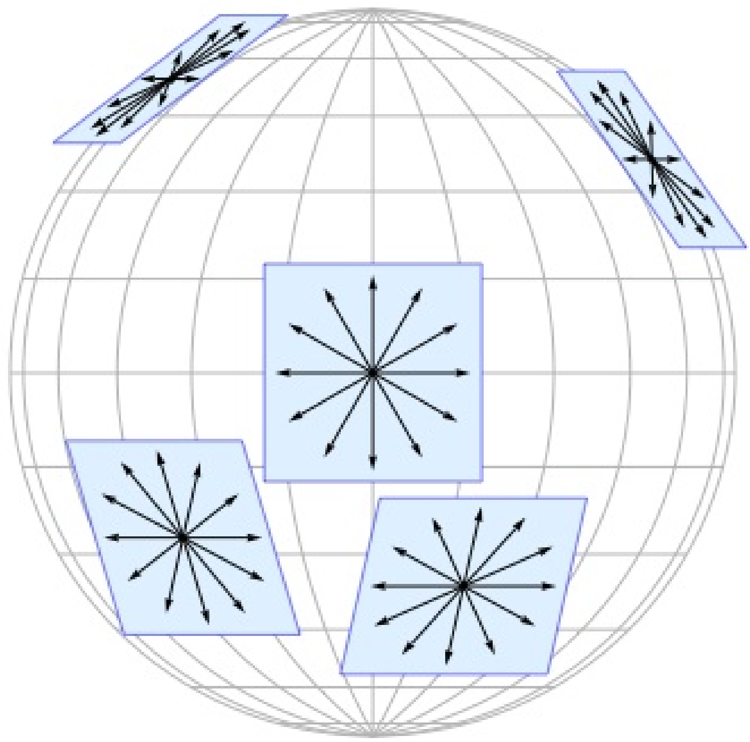
- metric tensor $g_{\mu\nu}$ symmetric tensor section of $T^*M \otimes T^*M$

- (p,q)-tensors: $T = (T^{i_1 \cdots i_p}_{j_1 \cdots j_q})$ section in $\Gamma(M, TM^{\otimes p} \otimes T * M^{\otimes q})$

- 1-form: section $\alpha = (\alpha_\mu) \in \Gamma(M, T^*M)$

- k-form $\omega \in \Gamma(M, \bigwedge^k (T^*M))$





Tangent bundle on a 2-sphere

Connections

- Linear map $\nabla : \Gamma(M, E) \to \Gamma(M, E \otimes T^*M)$ with *Leibniz rule*:

  $$\nabla(fs) = f\,\nabla(s) + s \otimes df$$

  for $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma(M, E)$

- local form: on $U_i \times \mathbb{R}^N$ section $s_i(x) = s^\alpha(x) e_\alpha(x)$ with $e_\alpha(x)$ local frame

  $$\nabla s_i = (ds_i^\alpha + \omega^\alpha{}_\beta s_i^\beta) e_\alpha$$

  with $\omega^\alpha{}_\beta e_\alpha = \nabla e_\alpha$

- $\omega = (\omega^\alpha{}_\beta)$ is an $N \times N$-matrix of 1-forms: 1-form with values in End(E)

- in local coordinates x^μ on U_i: $\omega^\alpha{}_\beta = \omega^\alpha{}_{\beta_\mu} dx^\mu$

- $\nabla_V s$ contraction of ∇s with vector field V

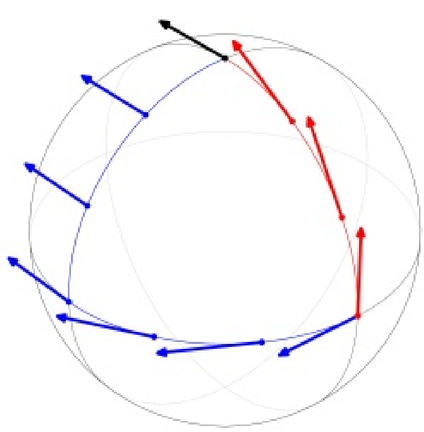
- *Curvature* of a connection ∇

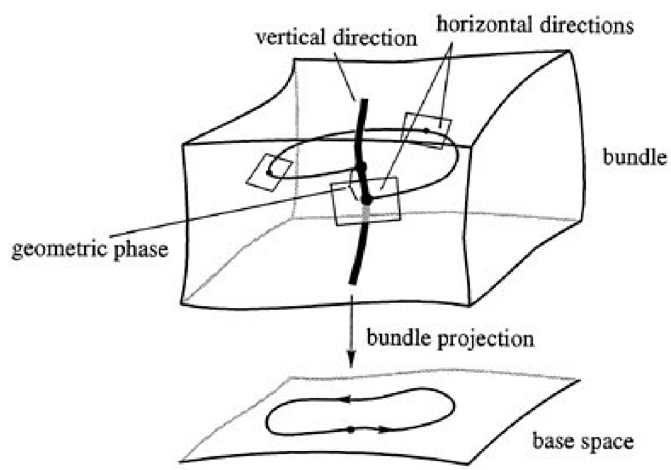
$$F_\nabla \in \Gamma(M, \mathrm{End}(E) \otimes \Lambda^2(T^*M))$$

$$F_\nabla(V,W)(s) = \nabla_V \nabla_W s - \nabla_W \nabla_V s - \nabla_{[V,W]} s$$

- ${F^\alpha}_{\beta_{\mu\nu}}$ curvature 2-form: $\Omega = d\omega + \omega \wedge \omega$

$${F^\alpha}_\beta = d{\omega^\alpha}_\beta + {\omega^\alpha}_\gamma \wedge {\omega^\gamma}_\beta$$





Action Functionals

Classical mechanics: equations of motion describe a path that is minimizing (or at least stationary) for the *action functional*... variational principle

$$S(q(t)) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t)\, dt$$

Lagrangian $L(q(t), \dot{q}(t), t)$

$$\delta S = \int_{t_0}^{t_1} \epsilon \frac{\partial L}{\partial q} + \dot{\epsilon} \frac{\partial L}{\partial \dot{q}} = \int_{t_0}^{t_1} \epsilon (\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}})$$

after integration by parts + boundary conditions $\epsilon(t_0) = \epsilon(t_1) = 0$

Euler–Lagrange equations: $\delta S = 0$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

equations of motion

Symmetries and conservation laws:
Noether's theorem

Example: Lagrangian invariant under translational symmetries in one direction q^k

$$\frac{\partial L}{\partial q^k} = 0 \;\Rightarrow\; \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} = 0$$

$p_k = \frac{\partial L}{\partial \dot{q}^k}$ momentum conservation

Important conceptual step in the "geometrization of physics" program: physical conserved quantity have geometric meaning (symmetries)

Action Functionals

General Relativity: Einstein field equations are variational equation $\delta S = 0$ for *Einstein–Hilbert action*

$$S(g_{\mu\nu}) = \int_M \frac{1}{2\kappa} R \sqrt{-g}\, d^4x$$

$M =$ 4-dimensional Lorentzian manifold $g_{\mu\nu} =$ metric tensor signature $(-,+,+,+)$

$$g = \det(g_{\mu\nu})$$

$$\kappa = 8\pi G c^{-4}$$

$G =$ gravitational constant, $c =$ speed of light in vacuum

$R =$ Ricci scalar

Ricci scalar R:

- Riemannian curvature $R^\rho{}_{\sigma\mu\nu}$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

- Levi-Civita connection (Christoffel symbols)

$$\Gamma^\rho_{\nu\sigma} = \frac{1}{2} g^{\rho\mu}(\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\nu\sigma})$$

  convention of summation over repeated indices for tensor calculus

- Ricci curvature tensor: contraction of Riemannian curvature

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$$

- Ricci scalar: further contraction (trace)

$$R = g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu$$

Variation

- Variation of Riemannian curvature

$$\begin{aligned} \delta R^\rho{}_{\sigma\mu\nu} = &\ \partial_\mu \delta\Gamma^\rho_{\nu\sigma} - \partial_\nu \delta\Gamma^\rho_{\mu\sigma} \\ + &\ \delta\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\sigma} \\ - &\ \delta\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\mu\sigma} \end{aligned}$$

  $\delta\Gamma^\rho_{\mu\lambda}$ is a tensor (difference of connections)

- Rewrite variation $\delta R^\rho{}_{\sigma\mu\nu}$ as

$$\delta R^\rho{}_{\sigma\mu\nu} = \nabla_\mu(\delta\Gamma^\rho_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^\rho_{\mu\sigma})$$

  with covariant derivative

$$\nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) = \partial_\lambda(\delta\Gamma^\rho_{\nu\mu}) + \Gamma^\rho_{\sigma\lambda}\delta\Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\nu\lambda}\delta\Gamma^\rho_{\sigma\mu}$$

- after contracting indices: Ricci tensor

$$\delta R_{\mu\nu} = \nabla_\rho(\delta\Gamma^\rho_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\rho_{\rho\mu})$$

- Ricci scalar variation $\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^\rho_{\rho\mu})$$

- so get variation

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}$$

- also have total derivative:

$$\sqrt{-g}\nabla_\mu A^\mu = \partial_\mu(\sqrt{-g}A^\mu)$$

so $\int_M \nabla_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^\rho_{\rho\mu})\sqrt{-g} = 0$

- Variation of determinant $g = \det(g_{\mu\nu})$: Jacobi formula

$$\frac{d}{dt}\log\det A(t) = \mathrm{Tr}(A^{-1}(t)\frac{d}{dt}A(t))$$

for invertible matrix A, so get

$$\delta g = \delta\det(g_{\mu\nu}) = g\, g^{\mu\nu}\,\delta g_{\mu\nu}$$

$$\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\,\delta g_{\mu\nu}$$

$\delta g^{\mu\nu} = -g^{\mu\sigma}(\delta g_{\sigma\lambda})g^{\lambda\nu}$ for inverse matrix

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \frac{-1}{2}g_{\mu\nu}\sqrt{-g}$$

Variation equation for Einstein–Hilbert action

$$\delta S = \int_M (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \, \delta g^{\mu\nu} \sqrt{-g} \, d^4x$$

stationary equation $\delta S = 0$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Einstein equation in vacuum

Other variants:

- Gravity coupled to matter $\mathcal{L}_M$ = matter Lagrangian

  $$S = \int_M (\frac{1}{2\kappa} R + \mathcal{L}_M) \sqrt{-g} \, d^4x$$

  energy–momentum tensor

  $$T_{\mu\nu} = -\frac{2\kappa}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}$$

  gives variational equations $\delta S = 0$:

  $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Action with cosmological constant:

$$S = \int_M (\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_M)\sqrt{-g}\, d^4x$$

gives variational equation $\delta S = 0$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

For more details see:
Sean Carroll, *Spacetime and Geometry*, Addison Wesley, 2004

Note: *Euclidean gravity* when $g_{\mu\nu}$ Riemannian: signature $(+,+,+,+)$ used in quantum gravity and quantum cosmology

Not all geometries admit Wick rotations between Riemannian and Loretzian signature (there are, for example, topological obstructions)

Action Functionals

Yang–Mills: gauge theories $SU(N)$

- Hermitian vector bundle E

- Lie algebra $su(N)$ generators $T^a = (T^{a\,\alpha}{}_\beta)$

$$\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = i f^{abc} T^c$$

  structure constants f^{abc}

- Connections: $\omega^\alpha{}_{\beta_\mu} = A^a_\mu \, T^{a\,\alpha}{}_\beta$
  gauge potentials A^a_μ

- Curvature:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu \wedge A^c_\nu$$

  covariant derivative $\nabla_\mu = \partial_\mu - igT^a A^a_\mu$

$$[\nabla_\mu, \nabla_\nu] = -igT^a F^a_{\mu\nu}$$

  (here g is a coupling constant)

- Yang–Mills Lagrangian:

$$\mathcal{L}(A) = -\frac{1}{2}\mathrm{Tr}(F^2) = -\frac{1}{4}F_a{}^{\mu\nu}F^a_{\mu\nu}$$

- Yang–Mills action:

$$S(A) = \int_M \mathcal{L}(A)\,\sqrt{-g}\,d^4x$$

- Equations of motion:

$$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{b\,\mu} F^c_{\mu\nu} = 0$$

  or equivalently $(\nabla^\mu F_{\mu\nu})^a = 0$ where $F_{\mu\nu} = T^a F^a_{\mu\nu}$

- case with $N = 1$: electromagnetism $U(1)$ equations give Maxwell's equations in vacuum

- unlike Maxwell in general Yang-Mills equations are non-linear

Lagrangian formalism in perturbative QFT

For simplicity consider example of scalar field theory in dim D

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi)$$

Lorentzian signature with $(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$; interaction part $\mathcal{L}_{int}(\phi)$ polynomial; action:

$$S(\phi) = \int_M \mathcal{L}(\phi)\sqrt{-g}d^D x$$

Classical solutions: stationary points of action; quantum case: sum over all confiurations weights by the action: oscillatory integral around classical

$$Z = \int e^{i\frac{S(\phi)}{\hbar}}\mathcal{D}[\phi]$$

Observables: $\mathcal{O}(\phi)$ function of the classical field; expectation value:

$$\langle\mathcal{O}\rangle = Z^{-1}\int \mathcal{O}(\phi)e^{i\frac{S(\phi)}{\hbar}}\mathcal{D}[\phi]$$

This ∞-dim integral *not* well defined: replace by a formal series expansion (perturbative QFT)

Euclidean QFT: metric $g_{\mu\nu}$ Riemannian

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{L}_{int}(\phi)$$

action $S(\phi) = \int_M \mathcal{L}(\phi)\sqrt{g}d^Dx$

$$Z = \int e^{-\frac{S(\phi)}{\hbar}} \mathcal{D}[\phi]$$

$$\langle \mathcal{O} \rangle = Z^{-1} \int \mathcal{O}(\phi) e^{-\frac{S(\phi)}{\hbar}} \mathcal{D}[\phi]$$

free-field part and interaction part

$$S(\phi) = S_0(\phi) + S_{int}(\phi)$$

interaction part as perturbation: free-field as Gaussian integral (quadratic form) $\Rightarrow$ integration of polynomials under Gaussians (repeated integration by parts): bookkeeping of terms, labelled by *graphs* (Feynman graphs)

Perturbative expansion:

- Feynman rules and Feynman diagrams

$$S_{eff}(\phi) = S_0(\phi) + \sum_\Gamma \frac{\Gamma(\phi)}{\#\text{Aut}(\Gamma)} \quad \text{(1PI graphs)}$$

$$\Gamma(\phi) = \frac{1}{N!} \int_{\sum_i p_i = 0} \widehat{\phi}(p_1) \cdots \widehat{\phi}(p_N) U(\Gamma(p_1, \dots, p_N)) dp_1 \cdots dp_N$$

$$U(\Gamma(p_1, \dots, p_N)) = \int I_\Gamma(k_1, \dots, k_\ell, p_1, \dots, p_N) d^D k_1 \cdots d^D k_\ell$$

$\ell = b_1(\Gamma)$ loops

- *Renormalization Problem*:
  Integrals $U(\Gamma(p_1, \dots, p_N))$ often divergent:

- *Renormalizable theory*: finitely many counterterms (expressed also as perturbative series) can be added to the Lagrangian to simultaneously remove all divergences from all Feynman integrals in the expansion

The problem with gravity:

- Gravity is not a renormalizable theory!

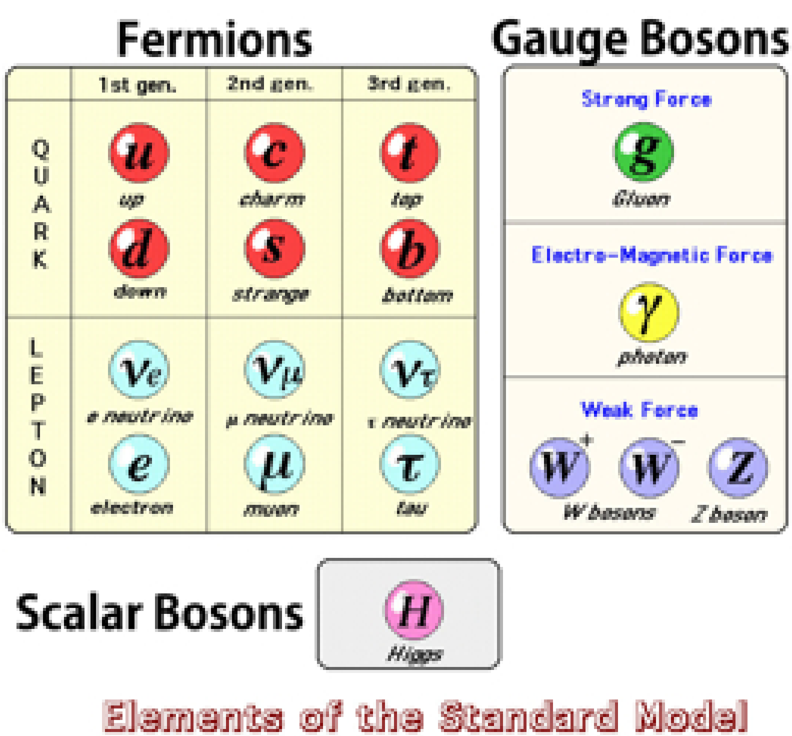
- Effective field theory, up to some energy scale below Planck scale (beyond, expect a different theory, maybe string theory maybe something else)

- Observation: some forms of "modified gravity" (higher derivatives) are renormalizable (but other desirable properties like unitarity can fail...)

- The models we will consider have some higher derivative terms in the gravity sector

Standard Model of Elementary Particles

- Forces: electromagnetic, weak, strong (gauge bosons: photon, $W^{\pm}$, Z, gluon)
- Fermions 3 generations: leptons (e,μ,τ and neutrinos) and quarks (up/down, charm/strange, top/bottom)
- Higgs boson



Parameters of the Standard Model

Minimal Standard Model: 19

- 3 coupling constants
- 6 quark masses, 3 mixing angles, 1 complex phase
- 3 charged lepton masses
- 1 QCD vacuum angle
- 1 Higgs vacuum expectation value; 1 Higgs mass

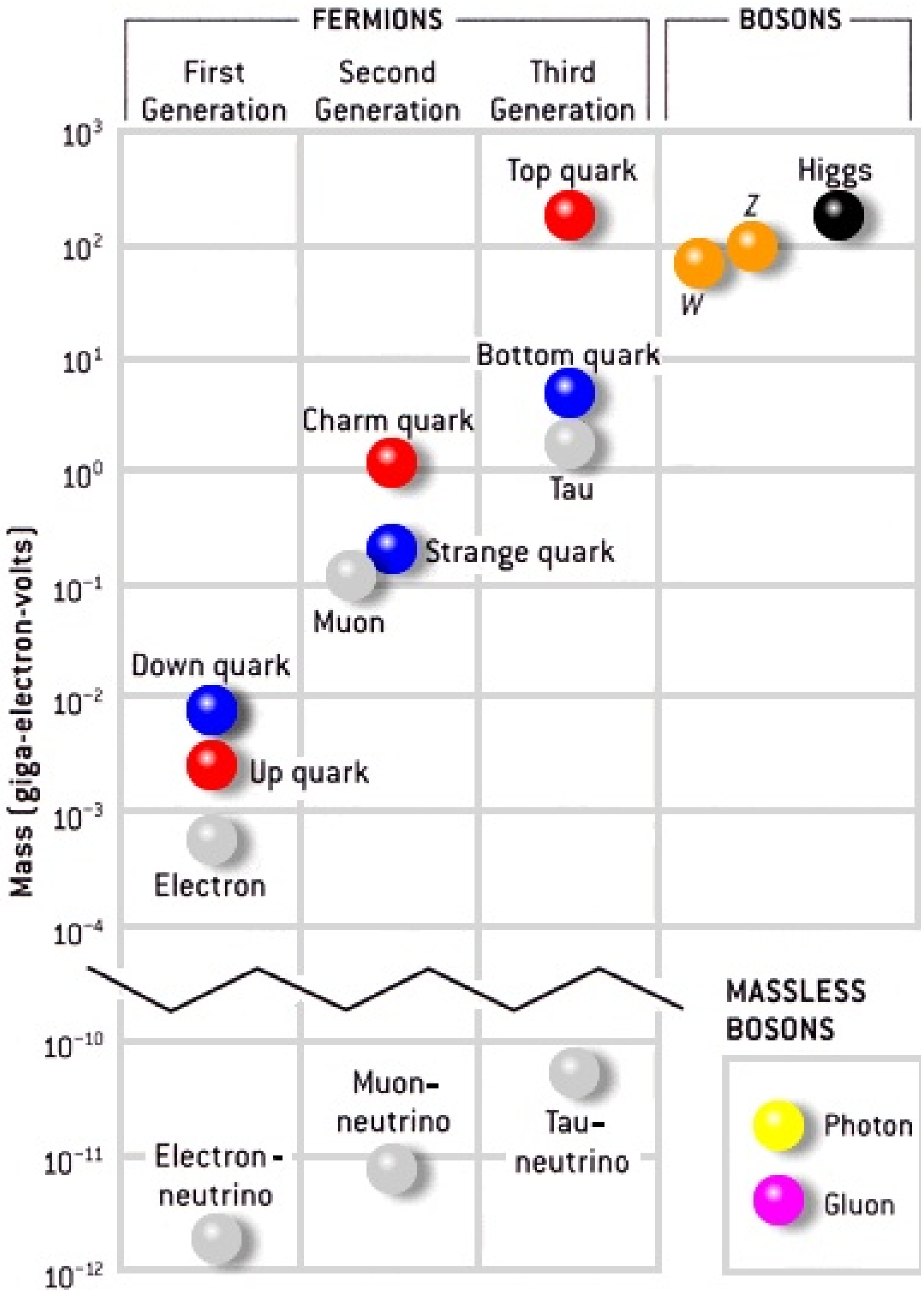
Extensions for neutrino mixing: 37

- 3 neutrino masses
- 3 lepton mixing angles, 1 complex phase
- 11 Majorana mass matrix parameters

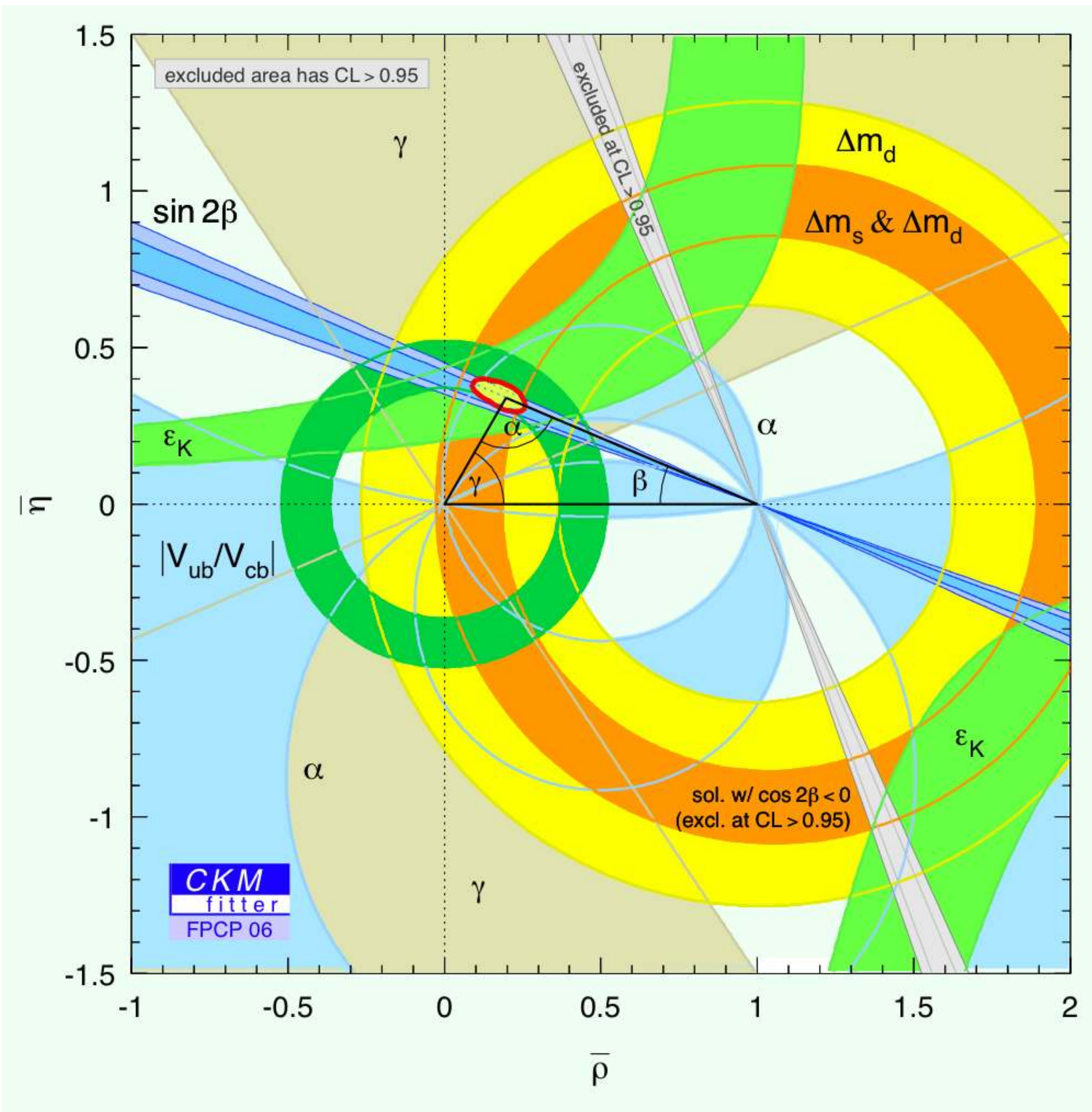
Constraints and relations? Values and constraints from experiments: but a priori theoretical reasons? Geometric space...

Note: parameters run with energy scale (renormalization group flow) so relations at certain scales versus relations at all scales

Experimental values of masses



Constraints on mixing angles (CKM matrix)



Standard Model Langrangian

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu -$$
$$\partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu -$$
$$igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) +$$
$$Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) -$$
$$A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) -$$
$$\frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu -$$
$$Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) +$$
$$g^2 s_w c_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H \partial_\mu H -$$
$$2M^2\alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 -$$
$$\beta_h\left(\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \frac{2M^4}{g^2}\alpha_h -$$
$$g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) -$$
$$\frac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\right) -$$
$$gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H -$$
$$\frac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) +$$
$$\frac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) +$$
$$\frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + M\,(\frac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+) -$$
$$ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) -$$
$$ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) -$$
$$\frac{1}{4}g^2 W^+_\mu W^-_\mu\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) -$$
$$\frac{1}{8}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-\right) - \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) - \frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) + \frac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- -$$
$$g^2 s_w^2 A_\mu A_\mu\phi^+\phi^- + \frac{1}{2}ig_s\,\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda(\gamma\partial +$$
$$m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j +$$
$$igs_w A_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right) + \frac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1 +$$

$$\gamma^5)\nu^\lambda)+(\bar{e}^\lambda\gamma^\mu(4s_w^2-1-\gamma^5)e^\lambda)+(\bar{d}_j^\lambda\gamma^\mu(\frac{4}{3}s_w^2-1-\gamma^5)d_j^\lambda)+(\bar{u}_j^\lambda\gamma^\mu(1-\frac{8}{3}s_w^2+\gamma^5)u_j^\lambda)\} + \frac{ig}{2\sqrt{2}}W_\mu^+\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5){U^{lep}}_{\lambda\kappa}e^\kappa) + (\bar{u}_j^\lambda\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d_j^\kappa)\right) + \frac{ig}{2\sqrt{2}}W_\mu^-\left((\bar{e}^\kappa {U^{lep}}^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}_j^\kappa C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u_j^\lambda)\right) + \frac{ig}{2M\sqrt{2}}\phi^+\left(-m_e^\kappa(\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m_\nu^\lambda(\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa}(1+\gamma^5)e^\kappa\right) + \frac{ig}{2M\sqrt{2}}\phi^-\left(m_e^\lambda(\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa}(1+\gamma^5)\nu^\kappa) - m_\nu^\kappa(\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa}(1-\gamma^5)\nu^\kappa\right) - \frac{g}{2}\frac{m_\nu^\lambda}{M}H(\bar{\nu}^\lambda\nu^\lambda) - \frac{g}{2}\frac{m_e^\lambda}{M}H(\bar{e}^\lambda e^\lambda) + \frac{ig}{2}\frac{m_\nu^\lambda}{M}\phi^0(\bar{\nu}^\lambda\gamma^5\nu^\lambda) - \frac{ig}{2}\frac{m_e^\lambda}{M}\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda) - \frac{1}{4}\,\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa - \frac{1}{4}\,\overline{\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa} + \frac{ig}{2M\sqrt{2}}\phi^+\left(-m_d^\kappa(\bar{u}_j^\lambda C_{\lambda\kappa}(1-\gamma^5)d_j^\kappa) + m_u^\lambda(\bar{u}_j^\lambda C_{\lambda\kappa}(1+\gamma^5)d_j^\kappa\right) + \frac{ig}{2M\sqrt{2}}\phi^-\left(m_d^\lambda(\bar{d}_j^\lambda C^\dagger_{\lambda\kappa}(1+\gamma^5)u_j^\kappa) - m_u^\kappa(\bar{d}_j^\lambda C^\dagger_{\lambda\kappa}(1-\gamma^5)u_j^\kappa\right) - \frac{g}{2}\frac{m_u^\lambda}{M}H(\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2}\frac{m_d^\lambda}{M}H(\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2}\frac{m_u^\lambda}{M}\phi^0(\bar{u}_j^\lambda\gamma^5 u_j^\lambda) - \frac{ig}{2}\frac{m_d^\lambda}{M}\phi^0(\bar{d}_j^\lambda\gamma^5 d_j^\lambda) + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu\bar{G}^a G^b g_\mu^c + \bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 - \frac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_w W_\mu^+(\partial_\mu\bar{X}^0 X^- - \partial_\mu\bar{X}^+ X^0) + igs_w W_\mu^+(\partial_\mu\bar{Y}X^- - \partial_\mu\bar{X}^+ Y) + igc_w W_\mu^-(\partial_\mu\bar{X}^- X^0 - \partial_\mu\bar{X}^0 X^+) + igs_w W_\mu^-(\partial_\mu\bar{X}^- Y - \partial_\mu\bar{Y}X^+) + igc_w Z_\mu^0(\partial_\mu\bar{X}^+ X^+ - \partial_\mu\bar{X}^- X^-) + igs_w A_\mu(\partial_\mu\bar{X}^+ X^+ - \partial_\mu\bar{X}^- X^-) - \frac{1}{2}gM\left(\bar{X}^+ X^+ H + \bar{X}^- X^- H + \frac{1}{c_w^2}\bar{X}^0 X^0 H\right) + \frac{1-2c_w^2}{2c_w}igM\left(\bar{X}^+ X^0\phi^+ - \bar{X}^- X^0\phi^-\right) + \frac{1}{2c_w}igM\left(\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-\right) + igMs_w\left(\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-\right) + \frac{1}{2}igM\left(\bar{X}^+ X^+\phi^0 - \bar{X}^- X^-\phi^0\right)\,.$$

Fundamental ideas of NCG models

- Derive the full Lagrangian from a *simple geometric input* by calculation

- Machine that inputs a (simple) geometry and produces a uniquely associated Lagrangian

- Very constrained: only certain theories can be obtained (only certain extensions of the minimal standard model)

- Simple action functional (spectral action) that reduces to SM + gravity in asymptotic expansion in energy scale

- Effective field theory: preferred energy scale (a unification energy)

The role of gravity:

- What if other forces were just gravity but seen from the perspective of a different geometry?

- This idea occurs in physics in different forms: holography AdS/CFT has field theory on boundary equivalent to gravity on bulk

- in NCG models action functional for gravity (spectral action) on an "almost-commutative geometry" gives gravity + SM on space-time manifold

... What is Noncommutative Geometry?