Information Algebras and Their Applications

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Abstract. In this lecture we will present joint work with Ryan Thorngren on thermodynamic semirings and entropy operads, with Nicolas Tedeschi on Birkhoff factorization in thermodynamic semirings, ongoing work with Marcus Bintz on tropicalization of Feynman graph hypersurfaces and Potts model hypersurfaces, and their thermodynamic deformations, and ongoing work by the author on applications of thermodynamic semirings to models of morphology and syntax in Computational Linguistics.

Lecture Outline

This is an abstract for an invited talk in the Session on *Information and Topology* of the 2nd conference on *Geometric Science of Information*. The talk is based on joint work with Ryan Thorngren [20] and Nicolas Tedeschi [19], ongoing work with Marcus Bintz [4], and other ongoing work [18].

Tropical Semiring

The min-plus (or tropical) semiring $\mathbb{T}$ is $\mathbb{T} = \mathbb{R} \cup \{\infty\}$, with the operations $\oplus$ and $\odot$ given by

$$x \oplus y = \min\{x, y\},$$

with ∞ the identity element for $\oplus$ and with

$$x \odot y = x + y,$$

with 0 the identity element for $\odot$. The operations $\oplus$ and $\odot$ satisfy associativity and commutativity and distributivity of the product $\odot$ over the sum $\oplus$.

Thermodynamic Semirings (Information Algebras)

A notion of *thermodynamic semiring* was introduced in [20] and further developed in [19]. Thermodynamic semirings (or Information Algebras) are deformations of the min-plus algebra, where the product $\odot$ is unchanged, but the sum $\oplus$ is deformed to a new operation $\oplus_{\beta,S}$,

$$x \oplus_{\beta,S} y = \min_p \{px + (1-p)y - \frac{1}{\beta} S(p)\}. \tag{1}$$

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according to a binary entropy functional S and a deformation parameter β, which we interpret thermodynamically as an inverse temperature $\beta = 1/T$ (up to the Boltzmann constant which we set equal to 1). At zero temperature (that is, $\beta \to \infty$) one recovers the unperturbed idempotent addition. The algebraic properties (commutativity, left and right identity, associativity) of this operation correspond to properties of the entropy functional (symmetry $S(p) = S(1-p)$, minima $S(0) = S(1) = 0$, and extensivity $S(pq) + (1-pq)S(p(1-q)/(1-pq)) = S(p) + pS(q)$), namely the Khinchin axioms of the Shannon entropy.

More generally, the entropy functional considered in the deformation need not be the Shannon entropy: thermodynamic semirings associated to Rényi and Tsallis entropies have different algebraic properties: the lack of commutativity and associativity is measured in a way that relates to the corresponding axiomatic properties of these more general entropy functionals. In particular, as shown in [20], the general thermodynamic semirings have a natural interpretation in terms of non-extensive thermodynamics, [1,10].

The case where the deformation is achieved by the Shannon entropy was considered in [5] in relation to absolute arithmetic and $\mathbb{F}_1$-geometry. Thermodynamic semirings are also closely related to Maslov dequantization, see [24], and to statistical mechanics [22]. Applications to multifractals are also described in [20].

Entropy Operad

The theory of thermodynamic semirings was also presented in [20] in terms of a general operadic formulation of entropy functionals. A collection $\mathcal{S} = \{S_n\}_{n\in\mathbb{N}}$ of n-ary entropy functionals S_n satisfies a coherence condition if

$$S_n(p_1, \ldots, p_n) = S_m(p_{i_1}, \ldots, p_{i_m}),$$

whenever, for some $m < n$, we have $p_j = 0$ for all $j \notin \{i_1, \ldots, i_m\}$. Shannon, Rényi, Tsallis entropies satisfy this condition.

A collection $\mathcal{S} = \{S_n\}_{n\in\mathbb{N}}$ of coherent entropy functionals determines n-ary operations $C_{n,\beta,\mathcal{S}}$ on $\mathbb{R} \cup \{\infty\}$,

$$C_{n,\beta,\mathcal{S}}(x_1, \ldots, x_n) = \min_p \{\sum_{i=1}^n p_i x_i - \frac{1}{\beta} S_n(p_1, \ldots, p_n)\}, \tag{2}$$

with the minimum taken over $p = (p_i)$, with $\sum_i p_i = 1$. More generally, one obtains n-ary operations $C_{n,\beta,\mathcal{S},\mathcal{T}}(x_1, \ldots, x_n)$ with $\mathcal{S}$ as above and $\mathcal{T}$ planar rooted trees with n leaves. As shown in [20], these operations can be written as

$$C_{n,\beta,\mathcal{S},\mathcal{T}}(x_1, \ldots, x_n) = \min_p \{\sum_{i=1}^n p_i x_i - \frac{1}{\beta} S_{\mathcal{T}}(p_1, \ldots, p_n)\}, \tag{3}$$

with the $S_{\mathcal{T}}(p_1, \ldots, p_n)$ obtained from the S_j, for $j = 2, \ldots, n$. One obtains in this way an algebra over the A_∞-operad of rooted trees.

Birkhoff Factorization in Thermodynamic Semirings

As part of the "renormalization and computation" program developed in [14–16], Manin asked in [14] for an extension of the algebraic renormalization method based on Rota–Baxter algebras ([6–9]) to tropical semirings.

This is achieved in [19], by introducing Rota–Baxter structures of weight λ on min-plus semirings and on their thermodynamic deformation. A Rota–Baxter operator of weight λ is defined as a $\oplus$-additive (monotone) map T satisfying

$$T(x) \odot T(y) = T(T(x) \odot y) \oplus T(x \odot T(y)) \oplus T(x \odot y) \odot \log \lambda$$

when $\lambda > 0$, while for $\lambda < 0$ one has the identity

$$T(x) \odot T(y) \oplus T(x \odot y) \odot \log(-\lambda) = T(T(x) \odot y) \oplus T(x \odot T(y)).$$

In the thermodynamic case, the notion of Rota–Baxter operator is the same, but with $\oplus$ replaced by the deformed $\oplus_{\beta,S}$.

Suppose given a map $\psi : \mathcal{H} \to \mathbb{T}_{\beta,S}$ from a commutative graded Hopf algebra $\mathcal{H}$ to a thermodynamic semiring $\mathbb{T}_{\beta,S}$ with a Rota–Baxter operator of weight $+1$, such that $\psi(xy) = \psi(x) \odot \psi(y)$. It is shown in [19] that ψ has a unique Birkhoff factorization $\psi_+ = \psi_- \star \psi$, with $\psi_- = T(\tilde{\psi})$, where $\tilde{\psi}$ is the Bogolyubov preparation of ψ,

$$\tilde{\psi}(X) = \psi(X) \oplus_{\beta,S} \bigoplus_{\beta,S} \psi_-(X') \odot \psi(X'')$$

where the Hopf algebra coproduct is $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$. The product $\star$ in the Birkhoff factorization is defined as

$$(\psi_1 \star \psi_2)(X) = \bigoplus_{\beta,S} (\psi_1(X^{(1)}) \odot \psi_2(X^{(2)})),$$

where $\oplus_{\beta,S}, \odot$ are the semiring operations and $\Delta(X) = \sum X^{(1)} \otimes X^{(2)} = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$ is the Hopf algebra coproduct.

The Hopf algebra can be taken to be, for instance, a Hopf algebra of Feynman graphs as in [6] or of flow charts as in [14]. Rota–Baxter operators can be constructed using running time or memory size, in the case of flow charts, or Markov random fields, or the order of polynomial countability of the graph hypersurface (or infinity when not polynomially countable) in the case of Feynman graphs, [19].

Tropical Hypersurfaces

Tropical geometry is a version of algebraic geometry over min-plus (or max-plus) semirings, see [11,12] for a general introduction. In recent years, tropical geometry has also been studied in relation to algebraic statistics, [21].

A tropical polynomial is a function $p : \mathbb{R}^n \to \mathbb{R}$ of the form

$$p(x_1, \ldots, x_n) = \oplus_{j=1}^m a_j \odot x_1^{k_{j1}} \odot \cdots \odot x_n^{k_{jn}} =$$

$$\min\{a_1 + k_{11}x_1 + \cdots + k_{1n}x_n, a_2 + k_{21}x_1 + \cdots + k_{2n}x_n, \cdots, a_m + k_{m1}x_1 + \cdots + k_{mn}x_n\}.$$

A tropical hypersurface is the set of points where the piecewise linear tropical polynomial is non-differentiable.

Thermodynamic Tropicalization

When deforming the tropical semiring to a thermodynamic semiring, one can similarly consider polynomials of the form

$$p_{\beta,\mathcal{S}}(x_1, \ldots, x_n) = \oplus_{\beta,S,j} a_j \odot x_1^{k_{j1}} \odot \cdots \odot x_n^{k_{jn}} =$$

$$\min_{p=(p_j)} \{\sum_j p_j(a_j + k_{j1}x_1 + \cdots + k_{jn}x_n) - \frac{1}{\beta} S_n(p_1, \ldots, p_n)\},$$

where $\mathcal{S} = \{S_n\}$ is a coherent family of entropy functionals, or more generally

$$p_{\beta,\mathcal{S},\mathcal{T}}(x_1, \ldots, x_n) = \min_p \{\sum_j p_j(a_j + k_{j1}x_1 + \cdots + k_{jn}x_n) - \frac{1}{\beta} S_{\mathcal{T}}(p_1, \ldots, p_n)\}, \tag{4}$$

for $\mathcal{T}$ a rooted tree with n leaves and $S_{\mathcal{T}}(p_1, \ldots, p_n)$ the corresponding entropy functional determined by the S_k in $\mathcal{S}$ with $2 \leq k \leq n$.

In the case where the entropy function is the Shannon entropy, this deformation of the tropical polynomial can be related to Maslov dequantization, see [24].

Applications to Feynman Graph and Potts Model Hypersurfaces

In perturbative quantum field theory, Feynman integrals can be written as period integrals (up to renormalization of divergences) on the complement of certain hypersurfaces, defined by the vanishing of the graph polynomial

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin E(T)} t_e,$$

with the sum over spanning trees of the graph and variables t_e assigned to the edges of the graph.

The algebro-geometric and motivic properties of these hypersurfaces have been widely studied in recent years, see [17] for an overview. These algebro-geometric methods were recently extended to other hypersurfaces that arise as zeros of partition functions of Potts models in [2].

We will discuss properties of the tropicalization of graph hypersurfaces and of Potts model hypersurfaces and their thermodynamic deformations, based on ongoing work [4].

Lexicographic Semirings and Geometric Models in Linguistics

Min-plus type semirings are widely used, in the form of "lexicographic semirings", in computational models of morphology and syntax in Linguistics, [13,23]. Another application of thermodynamics semirings that we will discuss is based on ongoing work [18], where entropy deformations of these linguistics models are considered, as a way of introducing an inverse temperature parameter β in deterministic finite-state representations of n-gram models based on the tropical semiring. These are a tropical geometry version of the Viterbi sequence algorithm, as described in [21]. Introducing the deformation parameter β plays a role, in these models, analogous to the thermodynamic formalism of [3]. We will discuss some consequences of this approach.

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