Information Algebras and their Applications

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This lecture based on:

- M. Marcolli, R. Thorngren, *Thermodynamic semirings*, J. Noncommut. Geom. 8 (2014), no. 2, 337–392
- M. Marcolli, N. Tedeschi, *Entropy algebras and Birkhoff factorization*, J. Geom. Phys. 97 (2015) 243–265

Min-Plus Algebra (Tropical Semiring)

min-plus (or tropical) semiring $\mathbb{T} = \mathbb{R} \cup \{\infty\}$

- operations $\oplus$ and $\odot$

$$x \oplus y = \min\{x, y\} \quad \text{with identity } \infty$$

$$x \odot y = x + y \quad \text{with identity } 0$$

- operations $\oplus$ and $\odot$ satisfy:
  - associativity
  - commutativity
  - left/right identity
  - distributivity of product $\odot$ over sum $\oplus$

Thermodynamic semirings $\mathbb{T}_{\beta,S} = (\mathbb{R} \cup \{\infty\}, \oplus_{\beta,S}, \odot)$

• deformation of the tropical addition $\oplus_{\beta,S}$

$$x \oplus_{\beta,S} y = \min_p \{px + (1-p)y - \frac{1}{\beta} S(p)\}$$

β thermodynamic inverse temperature parameter
$S(p) = S(p, 1-p)$ binary information measure, $p \in [0,1]$

• for $\beta \to \infty$ (zero temperature) recovers unperturbed idempotent addition $\oplus$

• multiplication $\odot = +$ is undeformed

• for $S =$ Shannon entropy considered first in relation to $\mathbb{F}_1$-geometry in

- A. Connes, C. Consani, *From monoids to hyperstructures: in search of an absolute arithmetic*, arXiv:1006.4810

Khinchin axioms $\mathrm{Sh}(p) = -C(p \log p + (1-p) \log(1-p))$

- Axiomatic characterization of Shannon entropy $S(p) = \mathrm{Sh}(p)$
  1. symmetry $S(p) = S(1-p)$
  2. minima $S(0) = S(1) = 0$
  3. extensivity
     $S(pq) + (1-pq) S(p(1-q)/(1-pq)) = S(p) + pS(q)$
- correspond to algebraic properties of semiring $\mathbb{T}_{\beta,S}$
  1. commutativity of $\oplus_{\beta,S}$
  2. left and right identity for $\oplus_{\beta,S}$
  3. associativity of $\oplus_{\beta,S}$

$\Rightarrow$ $\mathbb{T}_{\beta,S}$ commutative, unital, associative iff $S(p) = \mathrm{Sh}(p)$

Khinchin axioms *n*-ary form

Given S as above, define $S_n : \Delta_{n-1} \to \mathbb{R}_{\geqslant 0}$ by

$$S_n(p_1,\dots,p_n) = \sum_{1\leqslant j\leqslant n-1} (1 - \sum_{1\leqslant i<j} p_i) S(\frac{p_j}{1-\sum_{1\leqslant i<j} p_i}).$$

Then Khinchin axioms:

1. (Continuity) $S(p_1,\dots,p_n)$ continuous in $(p_1,\dots,p_n) \in \Delta_n$ simplex
2. (Maximality) $S(p_1,\dots,p_n)$ maximum at the uniform $p_i = 1/n$
3. (Additivity/Extensivity) $p_i = \sum_{j=1}^{m_i} p_{ij}$ then

$$S(p_{11},\dots,p_{nm_n}) = S(p_1,\dots,p_n) + \sum_{i=1}^{n} p_i S(\frac{p_{i1}}{p_i},\dots,\frac{p_{im_i}}{p_i});$$

4. (Expandability) Δ_n face in Δ_{n+1}

$$S(p_1,\dots,p_n,0) = S(p_1,\dots,p_n)$$

Shannon entropy case:

$$x \oplus_{\beta,\mathrm{Sh}} y = \min_p \{px + (1-p)y - \frac{1}{\beta}\mathrm{Sh}(p)\}$$

equivalent form of $\oplus_{\beta,\mathrm{Sh}}$

$$x \oplus_{\beta,\mathrm{Sh}} y = -\beta^{-1} \log \left(e^{-\beta x} + e^{-\beta y} \right)$$

leads to relation with Maslov dequantization

Rényi entropy:

$$\mathrm{Ry}_\alpha(p_1,\ldots,p_n) := \frac{1}{1-\alpha}\log\left(\sum_i p_i^\alpha\right)$$

$$\lim_{\alpha\to 1}\mathrm{Ry}_\alpha(p_1,\ldots,p_n) = \mathrm{Sh}(p_1,\ldots,p_n)$$

- lack of associativity of $x \oplus_S y$, when $S = \mathrm{Ry}_\alpha$

$$\mathrm{Ry}_\alpha(p) = \frac{1}{1-\alpha}\log(p^\alpha + (1-p)^\alpha)$$

measured by the transformation $(p_1, p_2, p_3) \mapsto (p_3, p_2, p_1)$

Non-extensive thermodynamics:

• gas of particles with chemical potentials $\log x$ and $\log y$ and Hamiltonian (p mole fraction)

$$\mathcal{H} = p \log x + (1-p) \log y$$

• partition function $Z = e^{-F_{\rm eq}}$ with $F_{\rm eq}$ equilibrium value of free energy at temperature $T = 1/\beta$

$$x \oplus_{\beta,S} y = \max_p (e^{TS(p) + p \log x + (1-p) \log y})$$

partition sum of a two state system with energies x and y

• Extensive thermodynamics: independent subsystems A and B, combined system $A \star B$

$$S(A \star B) = S(A) + S(B)$$

• Non-extensive deformations (Tsallis)

$$S_q(A \star B) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B)$$

Tsallis entropy:

$$\mathrm{Ts}_\alpha(p) = \frac{1}{\alpha - 1}(1 - p^\alpha - (1-p)^\alpha)$$

reproduces Shannon entropy $\alpha \to 1$

• Tsallis entropy uniquely determined by symmetry $S(p) = S(1-p)$, minima $S(0) = S(1) = 0$, and α-deformed extensivity

$$S(p_1) + (1-p_1)^\alpha S(\frac{p_2}{1-p_1}) = S(p_1+p_2) + (p_1+p_2)^\alpha S(\frac{p_1}{p_1+p_2})$$

Tsallis thermodynamic semiring

- Tsallis thermodynamic semiring: commutativity, unitarity and associativity of α-deformed $\oplus_{\beta,S,\alpha}$

$$x \oplus_{\beta,S,\alpha} y = \min_p \{ p^\alpha x + (1-p)^\alpha y - \frac{1}{\beta} \mathrm{Ts}_\alpha(p) \}$$

- General idea: transform axiomatic characterizations of various entropy functionals into algebraic properties of corresponding thermodynamic deformations of min-plus algebras

Entropy Operads

• Operad: objects $\mathcal{C}(j)$ in a symmetric monoidal category: parameter space of j-ary operations with composition maps

$$\gamma : \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \to \mathcal{C}(j_1 + \cdots + j_k)$$

associative, unital, and equivariant under permutations

• $\mathcal{C}$-algebra A: an object with Sym_j-equivariant maps

$$\mathcal{C}(j) \otimes A^j \to A,$$

thought of as actions, associative and unital

• operad $\mathcal{P}$ of probabilities on finite sets $\mathcal{P}(j) = \Delta_j$ simplex

$$\gamma((p_i)_{i\in j} \otimes (q_{1l})_{l\in k_0} \otimes \cdots \otimes (q_{jl})_{l\in k_{j-1}}) = (p_i q_{il})_{l\in k_i, i\in j} \in \mathcal{C}(k_0 + \cdots + k_{j-1})$$

composite of subsystems

Information Algebras (over the entropy operad)

• object $\mathbb{R}_{\geqslant 0}$, morphisms $x \in \mathbb{R}_{\geq 0}$; action of operad $\mathcal{P}$: maps S from finite probabilities to non-negative real number with

1. For $p \in \mathcal{P}(n)$ and $q_i \in \mathcal{P}(m_i)$
$$S(p \circ (q_1, \ldots, q_n)) = S(p) + \sum_i p_i S(q_i);$$
2. $S((1)) = 0$;
3. for $p \in \mathcal{P}(n)$ and $\sigma \in \mathrm{Sym}_n$
$$S(\sigma p) = S(p)$$
4. $S : \mathcal{P}(n) \to \mathbb{R}_{\geqslant 0}$ continuous

Characterizes entropy functionals

• related work: J. Baez, T. Fritz, T. Leinster, *A characterization of entropy in terms of information loss*, Entropy 13 (2011) no. 11, 1945–1957.

von Neumann entropy and the tropical trace

- convex set of density matrices

$$\mathcal{M}^{(N)} = \{\rho \in M_{N\times N}(\mathbb{C}) \,|\, \rho^* = \rho,\ \rho \geq 0,\ \mathrm{Tr}(\rho) = 1\}$$

- von Neumann entropy

$$\mathcal{N}(\rho) = -\mathrm{Tr}(\rho \log \rho), \quad \text{for} \quad \rho \in \mathcal{M}^{(N)}$$

Shannon entropy in diagonal case

- matrices $M_{N\times N}(\mathbb{T})$ over $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$$(A \oplus B)_{ij} = \min\{A_{ij}, B_{ij}\} \quad \text{and} \quad (A \odot B)_{ij} = \oplus_k A_{ik} \odot B_{kj} = \min_k\{A_{ik} + B_{kj}\}$$

- tropical trace $\mathrm{Tr}^{\oplus}(A) = \min_i\{A_{ii}\}$
- also consider

$$\widetilde{\mathrm{Tr}}^{\oplus}(A) := \min_{U \in U(N)} \min_i\{(UAU^*)_{ii}\} \leq \mathrm{Tr}^{\oplus}(A)$$

Entropical trace: thermodynamic deformation of tropical trace

$$\mathrm{Tr}^{\oplus}_{\beta,S}(A) := \min_{\rho\in\mathcal{M}^{(N)}} \{\mathrm{Tr}(\rho A) - \beta^{-1} S(\rho)\}$$

Tr in the right-hand-side is the *ordinary* trace

• in particular $S(\rho) = \mathcal{N}(\rho)$ von Neumann entropy, but also other entropy functionals (e.g. quantum versions of Rényi and Tsallis)

• Note: $\mathrm{Tr}(\rho A) = \langle A \rangle$ expectation value of observable A

• zero temperature limit

$$\lim_{\beta\to\infty} \mathrm{Tr}^{\oplus}_{\beta,S}(A) = \widetilde{\mathrm{Tr}}^{\oplus}(A)$$

Kullback–Leibler divergence and von Neumann entropical trace

- relative entropy (Kullback–Leibler divergence)

$$S(\rho||\sigma) = \mathrm{Tr}(\rho(\log \rho - \log \sigma))$$

- von Neumann deformation and relative entropy

(for $A = A^*$, $A \geq 0$)

$$\mathrm{Tr}(\rho A) - \beta^{-1}\mathcal{N}(\rho) = \frac{1}{\beta} S(\rho||\sigma_{\beta,A}) - \frac{1}{\beta}\log Z_A(\beta)$$

$$\sigma_{\beta,A} = \frac{e^{-\beta A}}{Z_A(\beta)} \quad \text{with} \quad Z_A(\beta) = \mathrm{Tr}(e^{-\beta A})$$

- von Neumann entropical trace (for $A = A^*$, $A \geq 0$)

$$\mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(A) = -\frac{\log Z_A(\beta)}{\beta}$$

with $Z_A(\beta) = \mathrm{Tr}(e^{-\beta A})$: rhs above is Helmholtz free energy

• if for $A = A^*$, $A \geq 0$ is direct sum of two matrices A_1 and A_2

$$\mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(A) = \mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(A_1) \odot \mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(A_2)$$

$$= -\beta^{-1} \left(\log \mathrm{Tr}(e^{-\beta A_1}) + \log \mathrm{Tr}(e^{-\beta A_2}) \right)$$

deformation of states on C^*-algebras

• states $\mathcal{M} = \{\varphi : \mathcal{A} \to \mathbb{C} \text{ linear} \,|\, \varphi(1) = 1 \text{ and } \varphi(a^* a) \geq 0\}$

• relative entropy of states: in case of Gibbs states $\varphi(a) = \tau(a\xi)$, $\psi(a) = \tau(a\eta)$

$$S(\varphi||\psi) = \tau(\xi(\log \xi - \log \eta))$$

in general more complicated

• thermodynamic deformation of a state $\psi \in \mathcal{M}$

$$\psi_{\beta,S}(a) = \min_{\varphi \in \mathcal{M}} \{\varphi(a) + \beta^{-1} S(\varphi||\psi)\}$$

Example:

• noncommutative torus: C^*-algebra generated by two unitaries U, V with $VU = e^{2\pi i\theta} UV$

• canonical trace, $\tau(U^n V^m) = 0$ for $(n,m) \neq (0,0)$ and $\tau(1) = 1$

• Gibbs states $\varphi(a) = \tau(a\xi)$ positive elements $\xi \in \mathcal{A}_\theta$

• thermodynamic deformation of canonical trace

$$\tau_{\beta,S}(a) = \min_{\varphi \in \mathcal{M}_\tau} \{\varphi(a) + \beta^{-1} S(\varphi || \tau)\}$$

• KMS state $\varphi_{\beta,a}(b) = \frac{\tau(be^{-\beta a})}{\tau(e^{\beta a})}$ of time evolution $\sigma_t(b) = e^{ita} b e^{-ita}$

$$\tau_{\beta,S}(a) = \min_{\varphi \in \mathcal{M}_\tau} \{\beta^{-1} S(\varphi || \varphi_{\beta,a}) - \beta^{-1} \log \tau(e^{-\beta a})\} = -\beta^{-1} \log \tau(e^{-\beta a})$$

Helmholtz free energy

Algebraic Renormalization (Connes–Kreimer)

• Feynman graphs of a quantum field theory form a commutative Hopf algebra $\mathcal{H}$

• Feynman rules: morphism ϕ of *commutative algebras* from $\mathcal{H}$ to a target algebra $\mathcal{R}$ (e.g. Laurent series)

• subtraction of infinities: Birkhoff factorization procedure

• multiplicative decomposition of ϕ into divergences (counterterms) and renormalized valued, obtained using
- coproduct Δ of Hopf algebra $\mathcal{H}$
- Rota-Baxter operator (e.g. pole subraction) on $\mathcal{R}$

• in physics BPHZ renormalization

Connes–Kreimer Hopf algebra of Feynman graphs $\mathcal{H}$

- Free commutative algebra in generators Γ 1PI Feynman graphs
- Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

- Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

- Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Extended to gauge theories (van Suijlekom): Ward identities as Hopf ideals

Rota–Baxter algebras (Ebrahimi-Fard, Guo, Kreimer)

• Rota–Baxter algebra of weight $\lambda = -1$: $\mathcal{R}$ commutative unital algebra; $\mathcal{T} : \mathcal{R} \to \mathcal{R}$ linear operator with

$$\mathcal{T}(x)\mathcal{T}(y) = \mathcal{T}(x\mathcal{T}(y)) + \mathcal{T}(\mathcal{T}(x)y) + \lambda\mathcal{T}(xy)$$

• Example: $\mathcal{T}$ = projection onto polar part of Laurent series

• T determines splitting $\mathcal{R}_+ = (1 - \mathcal{T})\mathcal{R}$, $\mathcal{R}_-$ = unitization of $\mathcal{T}\mathcal{R}$; both $\mathcal{R}_\pm$ are algebras

• Feynman rule $\phi : \mathcal{H} \to \mathcal{R}$ commutative algebra homomorphism from CK Hopf algebra $\mathcal{H}$ to Rota–Baxter algebra $\mathcal{R}$ weight -1

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R})$$

• Note: ϕ does *not know* that $\mathcal{H}$ Hopf and $\mathcal{R}$ Rota-Baxter, only commutative algebras

- Birkhoff factorization $\exists \phi_\pm \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R}_\pm)$

$$\phi = (\phi_- \circ S) \star \phi_+$$

where $\phi_1 \star \phi_2(x) = \langle \phi_1 \otimes \phi_2, \Delta(x) \rangle$

- Connes-Kreimer inductive formula for Birkhoff factorization:

$$\phi_-(x) = -\mathcal{T}(\phi(x) + \sum \phi_-(x')\phi(x''))$$

$$\phi_+(x) = (1 - \mathcal{T})(\phi(x) + \sum \phi_-(x')\phi(x''))$$

where $\Delta(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$

- Bogolyubov-Parshchuk preparation

$$\tilde{\phi}(x) = \phi(x) + \sum \phi_-(x')\phi(x'')$$

Renormalization and Computation (Manin)
proposal for a "renormalization of the halting problem"

• Idea: treat noncomputable functions like infinities in QFT

• Renormalization = extraction of finite part from divergent Feynman integrals; extraction of "computable part" from noncomputables

• First step: build a Hopf algebra (flow charts, partial recursive functions) and a Feynman rule that detects the presence of noncomputability (infinities)

• Second step: BPHZ type subtraction procedure with values in a min-plus or max-plus algebra (computing time, memory size)

• Third step: meaning of the "renormalized part" and of the "divergences part" of the Birkhoff factorization in terms of theory of computation

Semirings of functions

• min-plus semirings $\mathbb{S} = C(X, \mathbb{T})$ with pointwise $\oplus$, $\odot$

• thermodynamic deformations $\mathbb{S}_{\beta,S} = C(X, \mathbb{T}_{\beta,S})$ with pointwise $\oplus_{\beta,S}$, $\odot$

Logarithmically related pairs $(\mathcal{R}, \mathbb{S})$

• $\mathcal{R}$ commutative ring (algebra); $\mathbb{S}$ min-plus semiring; with formal logarithm bijective map $\mathcal{L} : \mathrm{Dom}(\mathcal{L}) \subset \mathcal{R} \to \mathbb{S}$

$$\mathcal{L}(ab) = \mathcal{L}(a) + \mathcal{L}(b) = \mathcal{L}(a) \odot \mathcal{L}(b)$$

• thermodynamic deformation (Shannon entropy)

$$f_1 \oplus_{\beta,S} f_2 = -\beta^{-1} \log(E(-\beta f_1) + E(-\beta f_2))$$

with $E : \mathbb{S} \to \mathrm{Dom}(\mathcal{L}) \subset \mathcal{R}$ inverse of $\mathcal{L}$

Examples

• $\mathcal{R} = C(X, \mathbb{R})$ and $\mathrm{Dom}(\mathcal{L}) \subset \mathcal{R}$ given by $C(X, \mathbb{R}^*_+)$ with $\mathcal{L}(a) = -\beta^{-1} \log(a)$

$$f_1 \oplus_{\beta,S} f_2 = -\beta^{-1} \log(e^{-\beta f_1} + e^{-\beta f_2})$$

is $\mathbb{S}_{\beta,S} = C(X, \mathbb{T}_{\beta,S})$ with $S = \mathrm{Sh}$

• $\mathcal{R} = \mathbb{Q}[[t]]$ ring of formal power series, $\mathrm{Dom}(\mathcal{L}) \subset \mathcal{R}$ power series $\alpha(t) = \sum_{k \geq 0} a_k t^k$ with $a_0 = 1$, with formal log

$$\mathcal{L}(1 + \alpha) = \alpha - \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \alpha^k$$

$$\alpha_1 \oplus_{\beta,S} \alpha_2 = \beta^{-1} \mathcal{L}(E(-\beta\alpha_1) + E(-\beta\alpha_2))$$

formal exponential $E(\gamma) = \sum_{k \geq 0} \gamma^k / k!$

min-plus valued characters (algebraic Feynman rules)

• $\mathcal{H}$ commutative Hopf algebra; $\mathbb{S}$ be a min-plus semiring

• $\psi : \mathcal{H} \to \mathbb{S}$ satisfying $\psi(1) = 0$ and

$$\psi(xy) = \psi(x) + \psi(y), \quad \forall x, y \in \mathcal{H}$$

• main idea: "arithmetic of orders of magnitude" $\epsilon \to 0$
- leading term in $\epsilon^\alpha + \epsilon^\beta$ is $\epsilon^{\min\{\alpha,\beta\}}$
- leading term of $\epsilon^\alpha \epsilon^\beta$ is $\epsilon^{\alpha+\beta}$

• model characters and Birkhoff factorization on "order of magnitude" version of usual ones convolution of min-plus characters

$$(\psi_1 \star \psi_2)(x) = \min\{\psi_1(x^{(1)}) + \psi_2(x^{(2)})\} = \bigoplus (\psi_1(x^{(1)}) \odot \psi_2(x^{(2)}))$$

minimum over all pairs $(x^{(1)}, x^{(2)})$ in coproduct $\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$ in Hopf algebra $\mathcal{H}$

Birkhoff factorization of a min-plus character ψ

$$\psi_+ = \psi_- \star \psi$$

$\star$ convolution product, $\psi_\pm$ satisfying $\psi_\pm(xy) = \psi_\pm(x) + \psi_\pm(y)$

Note: does not require antipode, works also for $\mathcal{H}$ bialgebra

Rota–Baxter semirings

• $\mathbb{S}$ be a min-plus semiring, map $T : \mathbb{S} \to \mathbb{S}$ is $\oplus$-additive if monotone, $T(a) \leq T(b)$ for $a \leq b$ (pointwise)

• Rota–Baxter semiring $(\mathbb{S}, \oplus, \odot)$ weight $\lambda > 0$:
exists $\oplus$-additive map $T : \mathbb{S} \to \mathbb{S}$ with

$$T(f_1) \odot T(f_2) = T(T(f_1) \odot f_2) \oplus T(f_1 \odot T(f_2)) \oplus T(f_1 \odot f_2) \odot \log \lambda$$

• Rota–Baxter semiring $(\mathbb{S}, \oplus, \odot)$ weight $\lambda < 0$:
exists $\oplus$-additive map $T : \mathbb{S} \to \mathbb{S}$ with

$$T(f_1) \odot T(f_2) \oplus T(f_1 \odot f_2) \odot \log(-\lambda) = T(T(f_1) \odot f_2) \oplus T(f_1 \odot T(f_2))$$

Birkhoff factorization in min-plus semirings (weight $+1$)

- Bogolyubov-Parashchuk preparation

$$\tilde{\psi}(x) = \min\{\psi(x), \psi_-(x') + \psi(x'')\} = \psi(x) \oplus \bigoplus \psi_-(x') \odot \psi(x'')$$

(x', x'') ranges over non-primitive part of coproduct
$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$

- ψ_- defined inductively on lower degree x' in Hopf algebra

$$\psi_-(x) := T(\tilde{\psi}(x)) = T(\min\{\psi(x), \psi_-(x') + \psi(x'')\})$$

$$= T\left(\psi(x) \oplus \bigoplus \psi_-(x') \odot \psi(x'')\right)$$

- by $\oplus$-linearity of T same as

$$\psi_-(x) = \min\{T(\psi(x)), T(\psi_-(x') + \psi(x''))\}$$

$$= T(\psi(x)) \oplus \bigoplus T(\psi_-(x') \odot \psi(x''))$$

- then ψ_+ by convolution

$$\psi_+(x) := (\psi_- \star \psi)(x) = \min\{\psi_-(x), \psi(x), \psi_-(x') + \psi(x'')\}$$

$$= \min\{\psi_-(x), \tilde{\psi}(x)\} = \psi_-(x) \oplus \tilde{\psi}(x)$$

- key step: associativity and commutativity of $\oplus$ and $\oplus$-additivity of T, plus Rota-Baxter identity weight $+1$ gives

$$\psi_-(xy) = \psi_-(x) + \psi_-(y)$$

hence ψ_+ also as convolution

Birkhoff factorization in min-plus semirings (weight -1)

- $\psi : \mathcal{H} \to \mathbb{S}$ min-plus character, and $T : \mathbb{S} \to \mathbb{S}$ Rota-Baxter weight -1: there is a Birkhoff factorization $\psi_+ = \psi_- \star \psi$; if T satisfies $T(f_1 + f_2) \geq T(f_1) + T(f_2)$, then ψ_- and ψ_+ are also min-plus characters: $\psi_\pm(xy) = \psi_\pm(x) + \psi_\pm(y)$

- as before

$$\psi_-(x) := T(\tilde{\psi}(x)) \quad \text{and} \quad \psi_+(x) := (\psi_- \star \psi)(x) = \min\{\psi_-(x), \tilde{\psi}(x)\}$$

- Rota–Baxter identity of weight -1 gives

$$\psi_-(xy) = \min\{T(\tilde{\psi}(x) + \tilde{\psi}(y)), T(\tilde{\psi}(x)) + T(\tilde{\psi}(y)))\}$$

if $T(f_1 + f_2) \geq T(f_1) + T(f_2)$ then

$$\psi_-(xy) = T(\tilde{\psi}(x)) + T(\tilde{\psi}(y))) = \psi_-(x) + \psi_-(y)$$

Thermodynamic Rota–Baxter structures

• $\mathbb{S}_{\beta,S}$ thermodynamic Rota–Baxter semiring weight $\lambda > 0$: there is $\oplus_{\beta,S}$-additive map $T : \mathbb{S}_{\beta,S} \to \mathbb{S}_{\beta,S}$

$$T(f_1)\odot T(f_2) = T(T(f_1)\odot f_2)\oplus_{\beta,S} T(f_1\odot T(f_2))\oplus_{\beta,S} T(f_1\odot f_2)\odot \log\lambda$$

• $\mathbb{S}_{\beta,S}$ thermodynamic Rota–Baxter semiring weight $\lambda < 0$: there is $\oplus_{\beta,S}$-additive map $T : \mathbb{S}_{\beta,S} \to \mathbb{S}_{\beta,S}$

$$T(f_1)\odot T(f_2)\oplus_{\beta,S} T(f_1\odot f_2)\odot \log(-\lambda) = T(T(f_1)\odot f_2)\oplus_{\beta,S} T(f_1\odot T(f_2))$$

like previous case but with $\oplus$ replaced with deformed $\oplus_{\beta,S}$

• $(\mathcal{R},\mathbb{S})$ logarithmically related pair: $T:\mathbb{S}\to\mathbb{S}$ determines $\mathcal{T}:\mathcal{R}\to\mathcal{R}$ with $\mathcal{T}(e^{-\beta f}):=e^{-\beta T(f)}$, for $a=e^{-\beta f}$ in $\mathrm{Dom}(\log)\subset\mathcal{R}$

$\mathcal{T}$ Rota-Baxter weight λ_β on $\mathcal{R}\Leftrightarrow T$ Rota-Baxter weight λ on $\mathbb{S}_{\beta,S}$

with $S=\mathrm{Sh}$ and $\lambda_\beta=\lambda^{-\beta}$, for $\lambda>0$, or $\lambda_\beta=-|\lambda|^{-\beta}$ for $\lambda<0$

$$\mathcal{T}(e^{-\beta f_1})\mathcal{T}(e^{-\beta f_2})=\mathcal{T}(\mathcal{T}(e^{-\beta f_1})e^{-\beta f_2})+\mathcal{T}(e^{-\beta f_1}\mathcal{T}(e^{-\beta f_2}))$$
$$+\lambda_\beta\,\mathcal{T}(e^{-\beta f_1}e^{-\beta f_2})$$

• $\mathcal{T}$ is $\mathbb{R}$-linear iff T is $\oplus_{\beta,S}$-linear

Birkhoff factorization in thermodynamic Rota–Baxter semirings (weight +1)

- $T : \mathbb{S}_{\beta,S} \to \mathbb{S}_{\beta,S}$ Rota–Baxter of weight $\lambda = +1$
- Bogolyubov–Parashchuk preparation of $\psi : \mathcal{H} \to \mathbb{S}_{\beta,S}$

$$\tilde{\psi}_{\beta,S}(x) = \psi(x) \oplus_{\beta,S} \bigoplus_{\beta,S} \psi_-(x') + \psi(x'')$$

$$= -\beta^{-1} \log \left(e^{-\beta\psi(x)} + \sum e^{-\beta(\psi_-(x')+\psi(x''))} \right)$$

- $\phi_\beta(x) := e^{-\beta\psi(x)}$ in $\mathcal{R}$: Bogolyubov–Parashchuk preparation $\tilde{\phi}_\beta(x) = e^{-\beta\tilde{\psi}(x)}$

$$\tilde{\phi}_\beta(x) := \phi_\beta(x) + \sum \mathcal{T}(\tilde{\phi}_\beta(x'))\phi_\beta(x'')$$

with $\mathcal{T}(e^{-\beta f}) := e^{-\beta T(f)}$ and $\mathcal{T}(-e^{-\beta f}) := -\mathcal{T}(e^{-\beta f})$

- Birkhoff factorization $\psi_{\beta,+} = \psi_{\beta,-} \star_\beta \psi$

$$\psi_{\beta,-}(x) = T(\tilde{\psi}_\beta(x)) = -\beta^{-1} \log \left(e^{-\beta T(\psi(x))} + \sum e^{-\beta T(\psi_-(x') + \psi(x''))} \right)$$

$$\psi_{\beta,+}(x) = -\beta^{-1} \log \left(e^{-\beta \psi_{\beta,-}(x)} + e^{-\beta \tilde{\psi}_\beta(x)} \right)$$

satisfying $\psi_{\beta,\pm}(xy) = \psi_{\beta,\pm}(x) + \psi_{\beta,\pm}(y)$

- in limit $\beta \to \infty$ thermodynamic Birkhoff factorization converges to min-plus Birkhoff factorization

Entropical von Neumann trace and Rota–Baxter identity

- $(\mathcal{R}, \mathcal{T})$ ordinary Rota–Baxter algebra weight λ; same weight on matrices $M_n(\mathcal{R})$ by $\mathcal{T}(A) = (\mathcal{T}(a_{ij}))$, for $A = (a_{ij})$

- for $(M_n(\mathcal{R}), M_n(\mathbb{S}))$ logarithmically related, with $\mathcal{T}$ Rota–Baxter weight $+1$ on $\mathcal{R} \Rightarrow T : M_n(\mathbb{S}) \to M_n(\mathbb{S})$ with $\mathcal{T}(e^{-\beta A}) = e^{-\beta T(A)}$ satisfying

$$\begin{aligned} \mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(T(A) \boxplus T(B)) &= \mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(T(T(A) \boxplus B)) \\ &\oplus_{\beta,S} \mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(T(A \boxplus T(B))) \\ &\oplus_{\beta,S} \mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(T(A) \boxplus T(B)) \end{aligned}$$

where $\boxplus$ = direct sum of matrices, $\mathcal{N}$ = von Neumann entropy

Example: Witt rings

• commutative ring R, Witt ring $W(R) = 1 + tR[[t]]$: addition is product of formal power series, multiplication determined by

$$(1-at)^{-1} \star (1-bt)^{-1} = (1-abt)^{-1} \quad a,b \in R$$

• injective ring homomorphism $g : W(R) \to R^{\mathbb{N}}$, ghost coordinates coefficients of

$$t \frac{1}{\alpha} \frac{d\alpha}{dt} = \sum_{r \geq 1} \alpha_r t^r$$

for $\alpha = \exp(\sum_{r\geq 1} \alpha_r t^r / r)$

• component-wise addition and multiplication on $R^{\mathbb{N}}$

• linear operator $\mathcal{T} : R[[t]] \to R[[t]]$ is Rota–Baxter weight λ iff $\mathcal{T}_W : W(R) \to W(R)$ defined by $g(\mathcal{T}_W(\alpha)) = \mathcal{T}(g(\alpha))$ satisfies

$$\mathcal{T}_W(\alpha_1) \circledast \mathcal{T}_W(\alpha_2) = \mathcal{T}_W(\alpha_1 \circledast \mathcal{T}_W(\alpha_2)) +_W \mathcal{T}_W(\mathcal{T}_W(\alpha_1) \circledast \alpha_2)$$

$$+_W \lambda \, \mathcal{T}_W(\alpha_1 \circledast \alpha_2)$$

with $+_W$ addition of $W(R)$ and convolution product

$$\alpha \circledast \gamma := \exp\left(\sum_{n\geq 1} (\sum_{r+\ell=n} \alpha_r \gamma_\ell) \frac{t^n}{n} \right)$$

for $\alpha = \exp(\sum_{r\geq 1} \alpha_r t^r / r)$ and $\gamma = \exp(\sum_{r\geq 1} \gamma_r t^r / r)$

- Example: $\mathcal{R} = R^{\mathbb{N}}$ Rota–Baxter weight $+1$

$$\mathcal{T} : (a_1, a_2, \ldots, a_n, \ldots) \mapsto (0, a_1, a_1 + a_2, \ldots, \sum_{k=1}^{n-1} a_k, \ldots)$$

resulting Rota–Baxter $\mathcal{T}_W$ weight $+1$ on Witt ring $W(R)$

$$\mathcal{T}_W(\alpha) = \alpha \circledast \mathbb{I}$$

convolution product with multiplicative unit $\mathbb{I} = (1-t)^{-1}$

- Hasse–Weil zeta functions of varieties over $\mathbb{F}_q$

$$Z(X,t) = \exp\left(\sum_{r\geq 1} \#X(\mathbb{F}_{q^r}) \frac{t^r}{r} \right)$$

elements in Witt ring:

$$Z(X \sqcup Y, t) = Z(X,t) Z(Y,t) \quad \text{and} \quad Z(X \times Y, t) = Z(X,t) \star Z(Y,t)$$

Rota–Baxter operator weight $+1$

$$\mathcal{T}_W(Z(X,t)) = Z(X,t) \circledast Z(\mathrm{Spec}(\mathbb{F}_q), t)$$

Computation examples

• inclusion–exclusion "cost functions": $\Gamma = \Gamma_1 \cup \Gamma_2$, $\gamma = \Gamma_1 \cap \Gamma_2$

$$\psi(\Gamma) = \psi(\Gamma_1) + \psi(\Gamma_2) - \psi(\gamma)$$

determine $\psi : \mathcal{H} \to \mathbb{T}$ character $\psi(xy) = \psi(x) + \psi(y)$

• class of machines $\psi_n(\Gamma)$ step-counting function of n-th machine: when it outputs on computation Γ (Hopf algebra of flow charts)

• Rota–Baxter operator weight $+1$ of partial sums: Bogolyubov–Parashchuck preparation

$$\tilde{\psi}_n(\Gamma) = \min\{\psi_n(\Gamma), \psi_n(\Gamma/\gamma) + \sum_{k=1}^{n-1} \tilde{\psi}_k(\gamma)\}$$

• a graph Γ with $\psi_n(\Gamma) = \infty$ (n-th machine does not halt) can have $\tilde{\psi}_n(\Gamma) < \infty$ if both

- source of infinity was localized in $\gamma \smallsetminus \partial\gamma$, so $\psi_n(\Gamma/\gamma) < \infty$
- $\psi_k(\gamma) < \infty$ for all previous machines

"renormalization of computational infinities" in Manin's sense

Polynomial countability

• in perturbative quantum field theory: graph hypersurfaces

$$X_\Gamma = \{\Psi_\Gamma = 0\} \subset \mathbb{A}^{\#E_\Gamma}$$

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin E(T)} t_e$$

sum over spanning trees

• X variety over $\mathbb{Z}$, reductions X_p over $\mathbb{F}_p$

$$\text{counting function} \quad N(X,q) := \#X_p(\mathbb{F}_q)$$

Polynomially countable X if counting function polynomial $P_X(q)$

• Question: when are graph hypersurfaces X_Γ polynomially countable? or equivalently complements $Y_\Gamma = \mathbb{A}^{\# E_\Gamma} \setminus X_\Gamma$

• max-plus character $\psi : \mathcal{H} \to \mathbb{T}_{max}$ with $N(Y_\Gamma, q) \sim q^{\psi(\Gamma)}$ leading order if Y_Γ polynomially countable or $\psi(\Gamma) := -\infty$ if not

• when Y_Γ not polynomially countable

$$\tilde{\psi}(\Gamma) = \max\{\psi(\Gamma), \tilde{\psi}(\gamma) + \psi(\Gamma/\gamma)\}$$

$$= \max\{\psi(\Gamma), \sum_{j=1}^{N} \psi(\gamma_j) + \psi(\gamma_{j-1}/\gamma_j)\}$$

identifies chains of subgraphs and quotient graphs whose hypersurfaces are polynomially countable

Work in progress: tropical geometry

- tropical polynomial $p : \mathbb{R}^n \to \mathbb{R}$ piecewise linear

$$p(x_1, \ldots, x_n) = \oplus_{j=1}^m a_j \odot x_1^{k_{j1}} \odot \cdots \odot x_n^{k_{jn}} =$$

$$\min\{a_1 + k_{11}x_1 + \cdots + k_{1n}x_n, a_2 + k_{21}x_1 + \cdots + k_{2n}x_n, \cdots, a_m + k_{m1}x_1 + \cdots + k_{mn}x_n\}.$$

tropical hypersurface where tropical polynomial non-differentiable

- Entropical geometry: thermodynamic deformations of $\mathbb{T}$

$$p_{\beta,\mathcal{S}}(x_1, \ldots, x_n) = \oplus_{\beta,S,j} a_j \odot x_1^{k_{j1}} \odot \cdots \odot x_n^{k_{jn}} =$$

$$\min_{p=(p_j)} \{ \sum_j p_j (a_j + k_{j1}x_1 + \cdots + k_{jn}x_n) - \frac{1}{\beta} S_n(p_1, \ldots, p_n) \}$$

- Goal: entropical geometry of graph hypersurfaces of QFT