

**GEOMETRY OF INFORMATION:
CLASSICAL AND QUANTUM ASPECTS**

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ABSTRACT. In this article, we describe various aspects of categorification of the structures, appearing in information theory. These aspects include probabilistic models both of classical and quantum physics, emergence of F -manifolds, and motivic enrichments.

2010 Mathematics Subject Classification: 14G10; 18F30; 55R90

Comments: 44 pages

0. INTRODUCTION AND SUMMARY

This paper interconnects three earlier works by its (co)authors: [Ma99], [Mar19], and [CoMa20].

The monograph [Ma99] surveyed various versions and applications of the notion of varieties \mathcal{X} , whose tangent sheaf $\mathcal{T}_{\mathcal{X}}$ is endowed with a *commutative, associative and $\mathcal{O}_{\mathcal{X}}$ -bilinear multiplication*. Such \mathcal{X} got a generic name *F -manifolds* by that time. Attention of geometers and specialists in mathematical physics was drawn to them, in particular, because many deformation spaces of various origin are endowed with natural F -structures.

In [CoMa20], it was observed, that geometry of spaces of probability distributions on finite sets, “geometry of information”, that was developing independently for several decades, also led to some classes of F -manifolds.

The research in [Mar19] extended these constructions to the domain of *quantum probability distributions*, that we call “geometry of quantum information” here.

In this paper, we are studying categorical encodings of geometries of classical and quantum information, including its F -manifolds facets, appearing in Sec. 4, and further developed in Sec. 5. Sec. 1 is a brief survey of both geometries. In the categorical encoding we stress the aspects related to the *monoidal structures and dualities* of the relevant categories which we survey in Sec. 2 and 3. Finally, in Sec. 6 we introduce and study high level categorifications of information geometries lifting it to the level of motives.

1. CLASSICAL AND QUANTUM PROBABILITY DISTRIBUTIONS

1.1. Classical probability distributions on finite sets. ([CoMa20], 3.2.)
Let X be a finite set. Denote by R^X the \mathbf{R} -linear space of functions $(p_x) : X \rightarrow \mathbf{R}$.

By definition, a *classical probability distribution on X* is one point of the simplex Δ_X in R^X spanned by the end-points of basic coordinate vectors in R^X :

$$\Delta_X := \{(p_x) \in R^X \mid p_x \geq 0, \sum_x p_x = 1\}.$$

We denote by ${}^\circ\Delta_X$ its maximal open subset

$${}^\circ\Delta_X := \{(p_x) \in \Delta_X \mid \text{all } p_x > 0\}.$$

Sometimes it is useful to replace Δ_X by ${}^\circ\Delta_X$ in the definition above. Spaces of distributions become subspaces of *cones*. For a general discussion of the geometry of *cones*, see Sec. 3 below.

1.2. Quantum probability distributions on finite sets. The least restrictive environment, in which we can define quantum probability distributions on a finite set, according to Sec. 8 of [Mar19], involves an additional choice of finite dimensional Hilbert space \mathcal{V} . This means that \mathcal{V} is finite dimensional vector space over the field of complex numbers \mathbf{C} , endowed with a scalar product $\langle v, v' \rangle \in \mathbf{C}$ such that for any $v, v' \in \mathcal{V}$ and $a \in \mathbf{C}$ we have

$$\langle av, v' \rangle = a \langle v, v' \rangle, \quad \langle v, av' \rangle = \bar{a} \langle v, v' \rangle.$$

Here $a \mapsto \bar{a}$ means complex conjugation.

In particular, \langle, \rangle is \mathbf{R} -bilinear.

Whenever \mathcal{V} is chosen, we can define for any finite set X the finite dimensional Hilbert space $\mathcal{H}_X := \bigoplus_{x \in X} \mathcal{V}$, the direct sum of $\text{card } X$ copies of \mathcal{V} .

Finally, a *quantum probability distribution on X* is a linear operator $\rho_X : \mathcal{H}_X \rightarrow \mathcal{H}_X$ such that $\rho_X = \rho_X^*$ and $\rho_x \geq 0$. Here $*$ is the Hermitian conjugation. Such an operator is also called a *density matrix*.

Remarks. The space \mathcal{V} represents the quantum space of internal degrees of freedom of one point x . Its choice may be motivated by physical considerations,

if we model some physical systems. Mathematically, different choices of \mathcal{V} may be preferable when we pass to the study of categorifications: cf. below.

1.3. Categories of classical probability distributions ([Mar19], Sec. 2).

Let Y, X be two finite sets. Consider the real linear space $R^{Y,X}$ consisting of maps $Y \times X \rightarrow \mathbf{R}$: $(y, x) \mapsto S_{yx}$.

Such a map is called *a stochastic matrix*, if

- (i) $S_{yx} \geq 0$ for all (y, x) .
- (ii) $\sum_{y \in Y} S_{yx} = 1$ for all $x \in X$.

1.3.1. Proposition. (i) Consider pairs (X, P) , consisting of a finite set X and a probability distribution (p_x) on X (one point of the closed set of probability distributions, as above).

These pairs are objects of the category \mathcal{FP} , morphisms in which are stochastic matrices

$$\text{Hom}_{\mathcal{FP}}((X, P), (Y, Q)) := (S_{yx}),$$

They are related to the distributions P, Q by the formula $Q = SP$, i. e.

$$q_y = \sum_{x \in X} S_{yx} p_x,$$

where (q_y) is the classical probability distribution, assigning to $y \in Y$ the probability q_y .

Composition of morphisms is given by matrix multiplication.

Checking correctness of this definition is a rather straightforward task. In particular, for any $(p_x) \in P$ and any stochastic matrix (S_{yx}) , $q_y \geq 0$, and

$$\sum_{y \in Y} q_y = \sum_{x \in X} p_x \sum_{y \in Y} S_{yx} = \sum_{x \in X} p_x = 1,$$

so that SP is a probability distribution.

Associativity of composition of morphisms follows from associativity of matrix multiplication.

1.4. Categories of quantum probability distributions ([Mar19], Sec. 8).

We now pass to the quantum analogs of these notions.

Let again X be a finite set, now endowed as above, with a finite dimensional Hilbert space $\mathcal{H}_X = \oplus_{x \in X} \mathcal{V}$ and a density matrix $\rho_x : \mathcal{H}_X \rightarrow \mathcal{H}_X$.

Given a finite dimensional Hilbert space $\mathcal{V} \simeq \mathbf{C}^N$, consider an algebra $B(\mathcal{V})$ of linear operators on this space, containing the convex set $M_{\mathcal{V}}$ of density operators/matrices ρ as in Sec. 1.2 above, *satisfying the additional condition* $\text{Tr}(\rho) = 1$.

A linear map $\Phi : B(\mathcal{V}) \rightarrow B(\mathcal{V})$ is called *positive* if it maps positive elements $\rho \geq 0$ in $B(\mathcal{V})$ to positive elements and it is *completely positive* if for all $k \geq 0$ the operator $\Phi \otimes \text{Id}_k$ is positive on $B(\mathcal{V}) \otimes M_k(\mathbf{C})$. Completely positive maps form a cone $\mathcal{CP}_{\mathcal{V}}$: for all relevant information regarding cones, see Sec. 3 below.

A *quantum channel* is a trace preserving completely positive map $\Phi : M_{\mathcal{V}} \rightarrow M_{\mathcal{V}}$. Composition of quantum channels is clearly again a quantum channel. A quantum channel Φ can be represented by a matrix, the Choi matrix S_{Φ} which is obtained by writing $\rho' = \Phi(\rho)$ in the form

$$\rho'_{ij} = \sum_{ab} (S_{\Phi})_{ij,ab} \rho_{ab}.$$

The first pair of indices of $(S_{\Phi})_{ij,ab}$ defines the row of the matrix S_{Φ} and the second pair the column. Because quantum channels behave well under composition, they can be used to define morphisms of a category of finite quantum probabilities.

Quantum analog of the respective statistical matrix is the so called *stochastic Choi matrix* S_Q (see [Mar19], (8.1), where we replaced Marcolli's notation S_{Φ} by our S_Q , with Q for quantum).

These triples are objects of the category \mathcal{FQ} , morphisms in which can be represented by stochastic Choi matrices so that

$$(\rho_Y)_{ij} = \sum_{ab} (S_Q)_{ij,ab} (\rho_X)_{ab}.$$

We have omitted here a description of the encoding of stochastic Choi matrices involving the choices of bases in appropriate vector spaces, and checking the compatibility of bases changes with composition of morphisms.

As soon as one accepts this, the formal justification of this definition can be done in the same way as that of Proposition 1.3.1.

1.5. Monoidal categories. Speaking about monoidal categories, we adopt basic definitions, axiomatics, and first results about categories, sites, sheaves, and

their homological and homotopical properties developed in [KS06]. In particular, sets of objects and morphisms of a category always will be *small sets* ([KS06], p. 10).

Sometimes we have to slightly change terminology, starting with monoidal categories themselves. We will call a *monoidal category* here a family of data, called a *tensor category* during entire Chapter 4 of [KS06] and the rest of the book, with exception of two lines in remark 4.2.17, p. 102.

According to the Definition 4.2.1. of [KS06] (p. 96), a monoidal category is a triple $(\mathcal{H}, \otimes, a)$, where \mathcal{H} is a category, \otimes a bifunctor $\mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$, and a is an “associativity” isomorphism of triple functors, constructed from \otimes by two different bracketings.

The associativity isomorphism must fit into the commutative diagram (4.2.1) on p. 96 of [KS06].

According to the Def. 4.2.5, p. 98 of [KS06], a *unit object* $\mathbf{1}$ of a monoidal category \mathcal{H} is an object, endowed with an isomorphism $\rho : \mathbf{1} \otimes \mathbf{1} \rightarrow \mathbf{1}$ such that the functors $X \mapsto X \otimes \mathbf{1}$ and $X \mapsto \mathbf{1}$ are fully faithful. By default, our monoidal categories, or their appropriate versions, will be endowed by unit objects.

Lemma 4.2.6 of [KS06], pp. 98–100, collects all natural compatibility relations between the monoidal multiplication \otimes and the unit object $(\mathbf{1}, \rho)$, categorifying the standard properties of units in set–theoretical monoids.

1.6. Duality in monoidal environments. The remaining part of this Section contains a brief review, based upon [Ma18], of categorical aspects of monoidality related to dualities between monoidal categories with units.

They must be essential also for the understanding of quantum probability distributions, because generally the relevant constructions appeared during the study of various quantum models: see references in [Ma88], [Ma17], and a later development [MaVa20].

Let $(\mathcal{H}, \bullet, \mathbf{1})$ be a monoidal category, and let K be an object of \mathcal{H} . (Notice that here we changed the notation of the monoidal product, earlier \otimes , and replaced it by \bullet).

1.6.1. Definition. A functor $D_K : \mathcal{H} \rightarrow \mathcal{H}^{op}$ is a *duality functor*, if it is an *antiequivalence of categories*, such that for each object Y of \mathcal{H} the functor

$$X \mapsto \text{Hom}_{\mathcal{H}}(X \bullet Y, K)$$

is representable by $D_K(Y)$.

In this case K is called a dualizing object.

The data $(\mathcal{H}, \bullet, \mathbf{1}, K)$ are called a Grothendieck–Verdier (or GV–) category.

(ii) (\mathcal{H}, \circ, K) is a monoidal category with unit object K .

1.7. Example: quadratic algebras. Let k be a field. A *quadratic algebra* is defined as an associative graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$ generated by A_1 over $A_0 = k$, and such that the ideal of all relations between generators A_1 is generated by the subspace of quadratic relations $R(A) \subset A_1^{\otimes 2}$.

Quadratic algebras form objects of a category \mathcal{QA} , morphisms in which are homomorphisms of graded algebras identical on terms of degree 0. It follows that morphisms $f : A \rightarrow B$ are in canonical bijection between such linear maps $f_1 : A_1 \rightarrow B_1$ for which $(f_1 \otimes f_1)(R(A)) \subset R(B)$.

The main motivation for this definition was a discovery, that if in the study of a large class of quantum groups we replace (formal) deformations of the universal enveloping algebras of the relevant Lie algebras by (algebraic) deformations of the respective algebras of functions, then in many cases we land in the category \mathcal{QA} .

The monoidal product \bullet in \mathcal{QA} can be introduced directly as a lift of the tensor product of linear spaces of generators: $A \bullet B$ is generated by $A_1 \otimes B_1$, and its space of quadratic relations is $S_{23}(R(A) \otimes R(B))$, where the permutation map $S_{23} : A_1^{\otimes 2} \otimes B_1^{\otimes 2} \rightarrow (A_1 \otimes B_1)^{\otimes 2}$ sends $a_1 \otimes a_2 \otimes b_1 \otimes b_2$ to $a_1 \otimes b_1 \otimes a_2 \otimes b_2$.

Remark. Slightly generalizing these definition, we may assume that our algebras are $\mathbf{Z} \times \mathbf{Z}_2$ -graded, that is *supergraded*. This will lead to the appearance of additional signs \pm in various places. In particular, in the definition of S_{23} there will be sign $-$, if both a_2 and b_1 are odd.

This might become very essential in the study of quantum probability distributions where physical motivation comes from models of fermionic lattices.

We return now to our category \mathcal{QA} .

Define the dualization in \mathcal{QA} as a functor $A \mapsto A^!$ extending the linear dualization $A_1 \mapsto A_1^* := \text{Hom}(A_1, k)$ on the spaces of generators. The respective subspace of quadratic relations will be $R(A)^{\perp} \subset (A_1^*)^{\otimes 2}$, the orthogonal complement to $R(A)$.

1.7.1. Proposition. Consider $\mathbf{1} := k[\tau]/(\tau^2)$ as the object of \mathcal{QA} , and put $K := k[t]$.

Then $(\mathcal{QA}, \bullet, \mathbf{1}, K)$ is a *GV*-category.

More precisely, in the respective dual category $(\mathcal{QA}^{op}, \circ, K, \mathbf{1} = K^!)$ the “white product” \circ is another lift of the tensor product of linear generators $A_1 \otimes B_1$, with quadratic relations $S_{23}(R(A) \otimes B_1^{\otimes 2} + A_1^{\otimes 2} \otimes R(B))$.

For a proof, see [Ma88], pp. 19–28.

2. MONOIDAL DUALITY IN CATEGORIES OF CLASSICAL PROBABILITY DISTRIBUTIONS

2.1. Generalities. According to [KS06], (Examples 4.2.2 (vi) and (v), p.96), if a category \mathcal{C} admits finite products \times , resp. finite coproducts \sqcup , then both of them define monoidal structures on this category.

Both products and coproducts are defined by their universal properties in the Def. 2.2.1, p. 43, of [KS06].

If \mathcal{C} admits finite inductive limits and finite projective limits, then it has an initial object $\emptyset_{\mathcal{C}}$, which is the unit object for the monoidal product \sqcup . Any morphism $X \rightarrow \emptyset_{\mathcal{C}}$ is an isomorphism, and $\emptyset_{\mathcal{C}} \times X \simeq \emptyset_{\mathcal{C}}$ ([KS06], Exercise 2.26, p. 69).

When studying monoidal dualities in various categories of structured sets, it is useful to keep in mind the following archetypal example. For any set U , denote by $\mathcal{P}(U)$ the set of all non-empty subsets of U . Then, for any two sets without common elements X, Y , there exists a natural bijection

$$\mathcal{P}(X \cup Y) \rightarrow (\mathcal{P}(X) \times \mathcal{P}(Y)) \cup \mathcal{P}(X) \cup \mathcal{P}(Y). \quad (2.1)$$

Namely, if a non-empty subset $Z \subset X \cup Y$ has empty intersection with Y , resp. X , it produces the last two terms in the r.h.s. of (2.1). Otherwise, it produces a pair of non-empty subsets $Z \cap X \in \mathcal{P}(X)$ and $Z \cap Y \in \mathcal{P}(Y)$.

A more convenient version of (2.1) can be obtained, if one works in a category \mathcal{S} of pointed sets, and defines the set of subsets $\mathcal{P}(X)$ as previously, but adding to it the empty subset as the marked point. Then the union \cup in (2.1) is replaced by the smash product \vee , and the map (2.1) extended to a bifunctor of X, Y , becomes a “categorification” of the formula for an exponential map $e^{x+y} = e^x \cdot e^y$ underlying transitions between classical models in physics and quantum ones. In particular, (2.1) connects the unit element for direct product with the unit/zero element for the coproduct/smash product.

We will now describe a version of these constructions applied to sets of probability distributions.

2.1.1. A warning. If we construct a functor $\mathcal{C} \rightarrow \mathcal{D}$ or $\mathcal{C} \rightarrow \mathcal{D}^{op}$, sending \times to \sqcup and exchanging their units $\emptyset_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$, it *cannot be completed to a duality functor* in the sense of Def. 1.6.1 above: a simple count of cardinalities shows it.

However we will still consider such functors as weaker versions of monoidal dualities, and will not warn a reader about it anymore.

2.2. Category of classical probability distributions. If the cardinality of a finite set X is one, there is only one classical probability distribution on $X = \{x\}$, namely $p_x = 1$. In [Mar19], Sec. 2, such objects are called singletons.

Singletons are *zero objects* in \mathcal{FP} ([Mar19], Lemma 2.5), that is, they have the same categorical properties as the objects $\emptyset_{\mathcal{C}}$, described in 2.1.

Morphisms that factor through zero objects are generally called zero morphisms. In \mathcal{FP} , they are explicitly described as “target morphisms” by [Mar19], 2.1.2: they are such morphisms $\hat{Q} : (X, P) \rightarrow (Y, Q)$, for which $\hat{Q}_{ba} = Q_b$.

Category \mathcal{FP} is not large enough for us to be able to use essential constructions and results from [KS06], related to multiplicative/additive transitions sketched in Sec. 2.1 above.

M. Marcolli somewhat enlarged it by replacing in the definition of objects of \mathcal{FP} finite sets by pointed finite sets. In [Mar19], the resulting category is denoted \mathcal{PS}_* , so that \mathcal{FP} is embedded in it. Objects, morphisms sets in \mathcal{PS}_* , their compositions etc. are explicitly described in Def. 2.8, 2.9, 2.10 of [Mar19].

Objects of \mathcal{PS}_* are called *probabilistic pointed sets* in Def. 2.8.

Besides embedding, there is also a *forgetful functor* $\mathcal{PS}_* \rightarrow \mathcal{PF}$ ([Mar19], Remark 2.11).

This becomes a particular case of constructions in the categories of small sets in [Ks06], Ch. 1.

2.2.1. Coproducts of probabilistic pointed sets and classical probability distributions. Start with the usual smash product of pointed sets

$$(X, x) \vee (Y, y) := ((X \sqcup Y)/(X \times y \cup x \times Y), *),$$

where $*$ is the “smash” of the union of two coordinate axes $X \times y \cup x \times Y$. It induces on probabilistic pointed sets obtained from finite probability distributions

the product of statistically independent probabilities.

$$(X, P) \sqcup (Y, Q) = (X \times Y, p_{(x,y)} = p_x q_y).$$

([Mar19], Lemma 2.14) .

We will denote by $\emptyset_{\mathcal{F}}$ any object of \mathcal{PS}_* , consisting of a finite set and a point in it with prescribed probability 1. All these objects are isomorphic.

2.3. Theorem. (i) *The triple $(\mathcal{PS}_*, \sqcup, \emptyset_{\mathcal{F}})$ is a monoidal category with unit.*

(ii) *The triple $(\mathcal{FP}, \times, \{pt\})$ is a monoidal category with unit.*

For detailed proofs, see [Mar19], Sec. 2.

2.4. A generalization. Let $(\mathcal{C}, \sqcup, 0_{\mathcal{C}})$ be a monoidal category with unit/zero object.

Generalizing the passage from \mathcal{S}_* to \mathcal{PS}_* , M. Marcolli defines the category \mathcal{PC} , a probabilistic version of \mathcal{C} ([Mar19], Def. 2.18).

One object of \mathcal{PC} is a formal finite linear combination $\Lambda C := \sum_i \lambda_i C_i$, where C_i are objects of \mathcal{C} .

One morphism $\Phi : \Lambda C \rightarrow \Lambda' C'$ is a pair (S, F) , where S is a stochastic matrix with $SA = \Lambda'$, and $F_{ab,r} : C_b \rightarrow C'_a$ are real numbers in $[0, 1]$ such that $\sum_r \mu_r^{ab} = S_{ab}$.

As before, one can explicitly define a categorical coproduct \sqcup in \mathcal{PC} , so that it becomes a monoidal category with zero object.

3. MONOIDAL DUALITY IN CATEGORIES OF QUANTUM PROBABILITY DISTRIBUTIONS

3.1. Category of quantum probability distributions. We now return to the category \mathcal{FQ} of quantum probability distributions.

We will be discussing the relevant versions of monoidal dualities for enrichments of \mathcal{FQ} , based upon variable categories \mathcal{C} .

3.1.1. Definition. ([Mar19], Def. 8.2). *Given a category \mathcal{C} , its quantum probabilistic version \mathcal{QC} is defined as follows.*

One object of \mathcal{QC} is a finite family $((C_a, C_b), \rho := (\rho_{ab}))$, where

$$(C_a, C_b) \in \text{Ob } \mathcal{C} \times \text{Ob } \mathcal{C}, \quad a, b = 1, \dots, N \geq 1,$$

and $\rho = (\rho_{ab})$ is the Choi matrix of a quantum channel as in Sec. 1.4 above.

One morphism between two such objects, $((C_a, C_b), \rho)$ (source) and $((C'_a, C'_b), \rho')$ (target) is given by a Choi matrix, as in Sec. 1.4. above, entries of which now are morphisms in \mathcal{C} .

Composition of two morphisms is defined similarly to the classical case, so that the usual associativity diagrams lift to \mathcal{QC} .

3.2. Monoidal structures. Assume now that \mathcal{C} is endowed with a monoidal structure with unit/zero object. Then it can be lifted to \mathcal{QC} in the same way as in classical case: see [Mar19], Proposition 8.5.

We can therefore extend the (weak) duality formalism of Sec. 2 to the case of quantum probability distributions, keeping in mind the warning stated in 2.1.1.

4. CONVEX CONES AND F -MANIFOLDS

4.1. Convex cones. Let R be a finite-dimensional real linear space. A non-empty subset $V \subset R$ is called a *cone*, if

- (i) V is closed with respect to addition and multiplication by positive reals;
- (ii) The topological closure of V does not contain a real affine subspace R of positive dimension.

Basic example. Let $C \subset R$ be a convex open subset of R whose closure does not contain 0. Then the union of all half-lines in R , connecting 0 with a point of C , is an open cone in R .

Here convexity of C means that any segment of real line connecting two different points of C , is contained in C .

4.2. Convex cones of probability distributions on finite sets. ([CoMa20], 3.2.) Let X be a finite set. As in 1.1 above, we start with the real linear space R^X and denote by V the union of all oriented real half-lines from zero to one of the points of Δ_X , or else of ${}^\circ\Delta_X$.

Such cones V are called *open*, resp. *closed*, cones of classical probability distributions on X .

4.3. Characteristic functions of convex cones. Given a convex cone V in finite-dimensional real linear space R , construct its *characteristic function* $\varphi_V : V \rightarrow \mathbf{R}$ in the following way.

Let R' be the dual linear space, and $\langle x, x' \rangle$ the canonical scalar product between $x \in R$ and $x' \in R'$. Choose also a volume form vol' on R' invariant wrt translations in R' . Then put

$$\varphi_V(x) := \int e^{-\langle x, x' \rangle} vol'.$$

4.3.1. Claim. *Such a volume form and a characteristic function are defined up to constant positive factor.*

This follows almost directly from the definition.

Now we focus on the differential geometry of convexity.

A cone V is a smooth manifold, its tangent bundle \mathcal{T}_V can be canonically trivialised, $\mathcal{T}_V = V \times R$, in the following way: $\mathcal{T}_{V,x}$ is identified with R by the parallel transport sending $x \in V$ to $0 \in R$. Choose an affine coordinate system (x^i) in R and put

$$g_{ij} := \partial^2 \ln \varphi_V / \partial_{x^i} \partial_{x^j}.$$

4.4. Theorem ([Vi63]). $\sum_{i,j} g_{ij} dx^i dx^j$ is a Riemannian metric on V . The associated torsionless canonical connection on \mathcal{T}_V has components

$$\Gamma_{jk}^i = \frac{1}{2} \sum_l g^{il} \frac{\partial^3 \ln \varphi}{\partial_{x^j} \partial_{x^k} \partial_{x^l}}$$

where (g^{ij}) are defined by $\sum_j g^{ij} g_{jk} = \delta_k^i$.

Therefore, putting

$$\sum_j a^j \partial_{x^j} \circ \sum_k b^k \partial_{x^k} := \sum_{ijk} \Gamma_{jk}^i a^j b^k \partial_{x^i}$$

we define on \mathcal{T}_V a commutative \mathbf{R} -bilinear composition.

4.5. Convex cones of stochastic matrices and categories of classical probability distributions ([Mar19], Sec. 2). Our first main example are cones of stochastic matrices from 1.3 above.

Let Y, X be two finite sets. Consider the real linear space $R^{Y,X}$ consisting of maps $Y \times X \rightarrow \mathbf{R}$: $(y, x) \mapsto S_{yx}$, where S is a stochastic matrix. As was explained

above, such stochastic matrices can be considered as morphisms in the category \mathcal{FP} .

4.5.1. Proposition. *The sets of morphisms $\text{Hom}_{\mathcal{FP}}((X, P), (Y, Q))$ are convex sets.*

4.6. Convex cones of quantum probability distributions on finite sets. ([Mar19], Sec. 8 and others.) Using now stochastic Choi matrices, encoding morphisms between objects of the category \mathcal{FQ} , as was explained in 1.4. above., we will prove the following quantum analog of the Proposition 4.5.1.

4.6.1. Proposition. *The sets of morphisms in \mathcal{FQ} are convex sets.*

We leave both proofs as exercises for the reader.

4.6.2. Remarks: comparison between classical and quantum probability distributions. (i) The \mathbf{R} -linear spaces R or R^X from above correspond to \mathbf{C} -linear spaces \mathcal{H} or \mathcal{H}_X from [Mar19], Def. 8.1.

(ii) The \mathbf{R} -dual spaces R' correspond to the \mathbf{C} -antidual spaces \mathcal{H}^* from [Mar19].

The real duality pairing $\langle x, x' \rangle$ is replaced by the complex antiduality pairing: this means that, for $h \in \mathcal{H}, h^* \in \mathcal{H}$ and $a \in \mathbf{C}$, we have

$$\langle ax, x' \rangle = a \langle x, x' \rangle, \quad \langle x, ax' \rangle = \bar{a} \langle x, x' \rangle,$$

where $a \mapsto \bar{a}$ is the complex conjugation map.

(iii) This implies that the direct sum of real spaces $R \oplus R'$ on the classical side which is replaced by $\mathcal{H} \oplus \mathcal{H}^*$ on the quantum side, can be compared with the direct sum of also *real* subspaces of $\mathcal{H} \oplus \mathcal{H}^*$ corresponding to the *eigenvalues* $\pm i$ of the operator combining $h \mapsto h^*$.

A parametric variation of this structure should lead to *paracomplex geometry*, which entered the framework of geometry of quantum information in Sec. 4 of [CoMa20].

5. CLIFFORD ALGEBRAS AND FROBENIUS MANIFOLDS

5.1. Hilbert spaces over Frobenius algebras. Let \mathcal{A} be a commutative algebra over \mathbf{R} of finite dimension n , generated by n linearly independent elements B_1, \dots, B_q satisfying relations $B_i \cdot B_j = \gamma_{ij}^k B_k$.

Assume moreover, that \mathcal{A} is endowed with a nondegenerate bilinear form σ (Frobenius form), satisfying the associativity property $\sigma(ab, c) = \sigma(a, bc)$, and a homomorphism $\eta : \mathcal{A} \rightarrow \mathbf{R}$, whose kernel contains no non-zero left ideal of \mathcal{A} .

One can see that then $\sigma_{ij} := \sigma(B_i B_j) = \sum_s \gamma_{ij}^s \eta_s$, where $\eta_s \in \mathbf{R}$.

As usual, in such cases we will omit summation over repeating indices and write the r.h.s. simply as $\gamma_{ij}^s \eta_s$.

Denote by (B^b) the dual basis to (B_a) with respect to σ .

Now consider a right free \mathcal{A} -module $\mathcal{M}(\mathcal{A})$ of rank r . It has a natural structure of real rn -dimensional linear space. It can be also represented as the space of matrices over \mathcal{A} .

Generally, below we will be considering Hilbert spaces \mathcal{M} , endowed with a compatible action of \mathcal{A} .

5.1.1. Example. Consider a particular case $q = 2$. One can see that there exist three different algebras (up to isomorphism): *complex numbers*, *dual numbers*, and *paracomplex numbers*.

The respective bases, denoted (B_a) in Sec. 5.1 above, have traditional notations: $(1, i)$, $i^2 = -1$; $(1, \varepsilon)$, $\varepsilon^2 = 0$; and $(1, \epsilon)$, $\epsilon^2 = 1$.

5.2. General case: Clifford algebras. Let k be a field of characteristic $\neq 2$; V a finite dimensional linear space over k ; $Q : S^2(V) \rightarrow k$ a non-degenerate quadratic form on V . It defines the symmetric scalar product on V : $\langle u, v \rangle := \frac{1}{2}[Q(u+v) - Q(u) - Q(v)]$.

5.2.1. Definition. A Clifford algebra Cl over k is an associative unital k -algebra of finite dimension q , endowed with generators B_1, \dots, B_q satisfying relations

$$B_i B_j + B_j B_i = 2\langle B_i, B_j \rangle.$$

Notice, that Q is an implicit part of the structure in this definition.

If $Q = 0$, then Cl is the exterior algebra of V over k , hence linear dimension of Cl is 2^n , where $n = \dim V$ over k . This formula holds also for Q of arbitrary rank.

Finally, if $k = \mathbf{R}$ and Q is non-degenerate, it has a signature (p, q) , that is, Q in an appropriate basis has the standard form $v_1^2 + \dots + v_p^2 - v_{p+1}^2 - \dots - v_{p+q}^2$, hence $n = p + q$. We will denote the respective Clifford algebras by $Cl_{p,q}$.

This gives a complete list of (isomorphism classes of) Frobenius \mathbf{R} -algebras of finite dimension.

Clifford algebras have properties implying the existence of a symmetric scalar product on the vector space V . More precisely, $\langle u, v \rangle := \frac{1}{2}[Q(u + v) - Q(u) - Q(v)]$ where Q is the quadratic form associated to the Clifford algebra. Using this definition, we can obtain the characteristic function, which is defined in section 4.3. Recall that the characteristic function is explicitly given by

$$\varphi(x) := \int e^{-\langle x, x' \rangle} \text{vol}',$$

where R' is the dual linear space, and $\langle x, x' \rangle$ is the canonical scalar product between $x \in R$ and $x' \in R'$, and vol' is a volume form on R' invariant wrt translations in R' . Therefore, one establishes a direct relation between those Clifford algebras and the characteristic functions defined in section 4.3, and hence a relation to the F -manifolds.

5.3. The splitting theorem. Consider now a stochastic matrix (see subsections 1.3 and 1.4 above) acting upon a finite-dimensional Hilbert space \mathcal{H} .

5.3.1. Theorem. \mathcal{H} has a canonical splitting into into subsectors that are irreducible modules over respective Clifford algebras.

Proof. Let M be a manifold, endowed with an affine flat structure, a compatible metric g , and an even symmetric rank 3 tensor A . Define a multiplication operation \circ on the tangent sheaf by $\circ : T_M \times T_M \rightarrow T_M$. The manifold M is Frobenius if it satisfies local potentiality condition for A , i.e. locally everywhere there exists a potential function φ such that $A(X, Y, Z) = \partial_{X, Y, Z} \varphi$, where X, Y, Z are flat tangent fields and an associativity condition: $A(X, Y, Z) = g(X \circ Y, Z) = g(X, Y \circ Z)$ (see [Ma99]).

Clifford algebras can be considered under the angle of matrix algebras as Frobenius algebras. They are equipped with a symmetric bilinear form σ , such that $\sigma(a \cdot b, c) = \sigma(a, b \cdot c)$. We consider a module over this Frobenius algebra \mathcal{A} and construct the real linear space to which it is identified, denoted $E^{r,q}$.

Let us first discuss the rank 3 tensor A . We construct it on $E^{r,q}$, using the a (p, q) -tensor formula

$$T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} B_{a_1} \dots B_{a_p} B^{\beta_1} \dots B^{\beta_q}$$

in the adapted basis.

There exists a compatible metric, inherited from the non-degenerate, symmetric bilinear form defined on the algebra \mathcal{A} , given by $\sigma : \mathcal{A} \times \mathcal{A} \rightarrow k$, where $\sigma(B_i, B_j) = \gamma_{ij}^k B_k$. Call this metric g .

Now that the rank 3 tensor A and the metric g have been introduced, we discuss the multiplication operation ' \circ '. This multiplication operation is inherited from the multiplication on the algebra and given by $B_i \cdot B_j = \gamma_{ij}^k B_k$. It can be written explicitly by introducing a bilinear symmetric map $\bar{A} : E^{rq} \times E^{rq} \rightarrow E^{rq}$, which in local coordinates is

$$\bar{A} = A_{ab}^c = \sum_e A_{abe} g^{ec}, \quad g^{ab} = (g_{ab})^{-1}.$$

Here $A_{abc} := g_{cm} A_{ab}^m$. The multiplication is thus defined by:

$$Ag^{-1} : E^{rq} \times E^{rq} \rightarrow E^{rq}$$

with $X \circ Y = \bar{A}(X, Y)$ and X, Y are local flat tangent fields.

Since we have defined the multiplication operation, we can verify the associativity property. Indeed, recall that the metric is inherited from the Frobenius form σ , which satisfies $\sigma(a \cdot b, c) = \sigma(a, b \cdot c)$. Naturally, this property is inherited on E^{rq} , where this associativity relation is given by $g(X \circ Y, Z) = g(X, Y \circ Z)$ for X, Y, Z flat tangent fields.

Finally, by using the relation between \bar{A} , A and g , we can see that the potentiality property is satisfied.

See also Sec. 6.8 – 6.9 below.

6. MOTIVIC INFORMATION GEOMETRY

This last section is dedicated to the construction of the highest (so far) floor of the Babel Tower of categorifications of probabilities.

We investigate possible extensions of some aspects of the formalism of information geometry to a motivic setting, represented by various types of Grothendieck rings.

A notion of motivic random variables was developed in [Howe19], [Howe20], based on relative Grothendieck rings of varieties. In the setting of motivic Poisson summation and motivic height zeta functions, as in [Bilu18], [ChamLoe15],

[CluLoe10], [HruKaz09], one also considers other versions of the Grothendieck ring of varieties, in particular the Grothendieck ring of varieties with exponentials. A notion of information measures for Grothendieck rings of varieties was introduced in [Mar19b], where an analog of the Shannon entropy, based on zeta functions, is shown to satisfy a suitable version of the Khinchin axioms of information theory. We elaborate here some of these ideas with the goal of investigating motivic analogs of the Kullback–Leibler divergence and the Fisher–Rao information metric used in the context of information geometry (see [AmNag07]).

6.1. Grothendieck ring with exponentials and relative entropy. We show here that, in the motivic setting, it is possible to implement a version of Kullback–Leibler divergence based on zeta functions, using the Grothendieck ring of varieties with exponentials, defined in [ChamLoe15].

6.1.1. Definition. *The Grothendieck ring with exponentials $KExp(\mathcal{V}_K)$, over a field K , is generated by isomorphism classes of pairs (X, f) , where X is a K -variety, and f a morphism $f : X \rightarrow \mathbf{A}^1$. Two such pairs (X_1, f_1) and (X_2, f_2) are isomorphic, if there is an isomorphism $u : X_1 \rightarrow X_2$ of K -varieties such that $f_1 = f_2 \circ u$.*

The relations in $KExp(\mathcal{V}_K)$ are given by

$$[X, f] = [Y, f|_Y] + [U, f|_U],$$

for a closed subvariety $Y \hookrightarrow X$ and its open complement $U = X \setminus Y$, and the additional relation

$$[X \times \mathbf{A}^1, \pi_{\mathbf{A}^1}] = 0$$

where $\pi_{\mathbf{A}^1} : X \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$ is the projection on the second factor.

The ring structure is given by the product

$$[X_1, f_1] \cdot [X_2, f_2] = [X_1 \times X_2, f_1 \circ \pi_{X_1} + f_2 \circ \pi_{X_2}]$$

where $f_1 \circ \pi_{X_1} + f_2 \circ \pi_{X_2} : (x_1, x_2) \mapsto f_1(x_1) + f_2(x_2)$.

The original motivation for introducing the Grothendieck ring with exponentials was to provide a motivic version of exponential sums. Indeed, for a variety X over a finite field \mathbf{F}_q , with a morphism $f : X \rightarrow \mathbf{A}^1$, a choice of character $\chi : \mathbf{F}_q \rightarrow \mathbf{C}^*$ determines an exponential sum

$$\sum_{x \in X(\mathbf{F}_q)} \chi(f(x))$$

of which the class $[X, f] \in KExp(\mathcal{V})_{\mathbf{F}_q}$ is the motivic counterpart. The relation $[X \times \mathbf{A}^1, \pi_{\mathbf{A}^1}] = [X, 0] \cdot [\mathbf{A}^1, id] = 0$ corresponds to the property that, for any given character $\chi : \mathbf{F}_q \rightarrow \mathbf{C}^*$, one has $\sum_{a \in \mathbf{F}_q} \chi(a) = 0$.

Here we interpret the classes $[X, f]$ with $f : X \rightarrow \mathbf{A}^1$ as pairs of a variety and a potential (or Hamiltonian) $f : x \mapsto H_x = f(x)$. A family of commuting Hamiltonians is represented in this setting by a class $[X \times \mathbf{A}^1, F]$ with $F : X \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$, where for $\epsilon \in \mathbf{A}^1 \setminus \{0\}$ the function $f_\epsilon : X \rightarrow \mathbf{A}^1$ given by $f_\epsilon(x) = F(x, \epsilon)$ is our Hamiltonian $f_\epsilon : x \mapsto H_x(\epsilon) = f_\epsilon(x)$.

Note that, for this interpretation of classes $[X, f]$ as varieties with a potential (Hamiltonian) we do not need to necessarily impose the relation $[X \times \mathbf{A}^1, \pi_{\mathbf{A}^1}] = 0$. Thus, we can consider the following variant of the Grothendieck ring with exponentials.

6.1.2. Definition. *The coarse Grothendieck ring with exponentials $KExp^c(\mathcal{V}_K)$ is generated by isomorphism classes of pairs (X, f) of a K -variety X and a morphism $f : X \rightarrow \mathbf{A}^1$ as above.*

The relations etween them are generated by $[X, f] = [Y, f|_Y] + [U, f|_U]$, for a closed subvariety $Y \hookrightarrow X$ and its open complement $U = X \setminus Y$.

The product is $[X_1, f_1] \cdot [X_2, f_2] = [X_1 \times X_2, f_1 \circ \pi_{X_1} + f_2 \circ \pi_{X_2}]$.

The Grothendieck ring with exponentials $KExp(\mathcal{V}_K)$ is the quotient of this coarse version $KExp^c(\mathcal{V}_K)$ by the ideal, generated by $[\mathbf{A}^1, id]$.

We will be using motivic measures and zeta functions that come from a choice of character χ as above, for which the elements $[X \times \mathbf{A}^1, \pi_{\mathbf{A}^1}]$ will be in the kernel. So all the motivic measures we will be considering on $KExp^c(\mathcal{V}_K)$ will factor through the Grothendieck ring $KExp(\mathcal{V}_K)$ of Definition 6.1.1.

For a class $[X, f] \in KExp^c(\mathcal{V}_K)$, the symmetric products are defined as

$$[S^n(X, f)] := [S^n(X), f^{(n)}],$$

with $S^n(X)$ the symmetric product and $f^{(n)} : S^n(X) \rightarrow \mathbf{A}^1$ given by

$$f^{(n)}[x_1, \dots, x_n] = f(x_1) + \dots + f(x_n).$$

The analog of the Kapranov motivic zeta function in $KExp^c(\mathcal{V}_K)$ is given by

$$Z_{(X, f)}(t) = \sum_{n \geq 0} [S^n(X, f)] t^n.$$

Given a motivic measure $\mu : KExp^c(\mathcal{V}_K) \rightarrow R$, for some commutative ring R , one can consider the corresponding zeta function

$$\zeta_\mu((X, f), t) := \sum_{n \geq 0} \mu(S^n X, f^{(n)}) t^n.$$

As our basic example, consider the finite field case $K = \mathbf{F}_q$ and the motivic measures and zeta functions discussed in Section 7.8 of [ManMar21], where the motivic measure $\mu_\chi : KExp^c(\mathcal{V}_K) \rightarrow \mathbf{C}$ is determined by a choice of character $\chi : \mathbf{F}_q \rightarrow \mathbf{C}^*$,

$$\mu_\chi(X, f) = \sum_{x \in X(\mathbf{F}_q)} \chi(f(x)),$$

and the associated zeta function is given by (see Proposition 7.8.1 of [ManMar21])

$$\begin{aligned} \zeta_\chi((X, f), t) &= \sum_n \sum_{\underline{x} \in S^n(X)(\mathbf{F}_q)} \chi(f^{(n)}(\underline{x})) t^n = \\ &= \exp \left(\sum_{m \geq 1} N_{\chi, m}(X, f) \frac{t^m}{m} \right), \end{aligned}$$

with

$$N_{\chi, m}(X, f) = \sum_{\alpha} \sum_{r|m} r a_{\alpha, r} \alpha^{\frac{m}{r}},$$

where for given $\alpha \in \mathbf{C}$ and $r \in \mathbf{N}$, we have $a_{\alpha, r} = \text{card } X_{\alpha, r}$ for the level sets

$$X_{\alpha, r} := \{x \in X \mid [k(x) : \mathbf{F}_q] = r \text{ and } \chi(f(x)) = \alpha\}.$$

We can then consider two possible variations, with respect to which we want to compute a relative entropy through a Kullback–Leibler divergence: the variation of the Hamiltonian, obtained through a change in the function $f : X \rightarrow \mathbf{A}^1$, and the variation in the choice of the character χ in this motivic measure μ_χ . We'll show how to simultaneously account for both effects.

To mimic the thermodynamic setting described in the previous subsection, consider functions $F : X \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$ of the form $F(x, \epsilon) = f_\epsilon(x) = f(x) + \epsilon \cdot h(x)$ for

given morphisms $f, h : X \rightarrow \mathbf{A}^1$, with $\epsilon \in \mathbf{G}_m$ acting on $\mathbf{G}_a = \mathbf{A}^1$ by multiplication and $F(x, 0) = f(x)$.

Given a motivic measure $\mu_\chi : KExp^c(\mathcal{V})_{\mathbf{F}_q} \rightarrow \mathbf{C}$ associated to the choice of a character $\chi : \mathbf{F}_q \rightarrow \mathbf{C}^*$, and a class $[X, f]$ in $KExp^c(\mathcal{V})_{\mathbf{F}_q}$, we consider the “probability distribution”

$$P_{n, \underline{x}} = \frac{\chi(f^{(n)}(\underline{x})) t^n}{\zeta_\chi((X, f), t)},$$

for $\underline{x} \in S^n(X)(\mathbf{F}_q)$. We write $P_{n, \underline{x}}^{(f, \chi)}$ when we need to emphasise the dependence on the morphism $f : X \rightarrow \mathbf{A}^1$ and the character χ . We leave the t -dependence implicit. Of course, this is not a probability distribution in the usual sense, since it takes complex rather than positive real values, though it still satisfies the normalisation condition. We still treat it formally like a probability so that we can consider an associated notion of Kullback–Leibler divergence $KL(P||Q)$.

Given a choice of a branch of the logarithm, to a character $\chi \in Hom(G, \mathbf{C}^*)$, with G a locally compact abelian group, we can associate a group homomorphism $\log \chi : G \rightarrow \mathbf{C}$.

6.1.3. Proposition. *Given $[X, f] \in KExp^c(\mathcal{V})_{\mathbf{F}_q}$, consider a class $[X \times \mathbf{A}^1, F]$ with $F(x, 0) = f$ and $F(x, \epsilon) =: f_\epsilon(x)$, and characters $\chi, \chi' : \mathbf{F}_q \rightarrow \mathbf{C}^*$. The Kullback–Leibler divergence then is*

$$KL(\zeta_\chi((X, f), t) || \zeta_\chi((X, f_\epsilon), t)) := KL(P^{(f, \chi)} || P^{(f_\epsilon, \chi)}) = \sum_{n, \underline{x}} P_{n, \underline{x}}^{(f, \chi)} \log \frac{P_{n, \underline{x}}^{(f, \chi)}}{P_{n, \underline{x}}^{(f_\epsilon, \chi)}} = \log \langle \chi_\epsilon(h) \rangle - \langle \log \chi_\epsilon(h) \rangle,$$

where $\langle \cdot \rangle$ is the expectation value with respect to $P^{(f, \chi)}$.

Similarly, with $\psi = \chi^{-1} \cdot \chi'$, we have

$$KL(\zeta_\chi((X, f), t) || \zeta_{\chi'}((X, f), t)) := KL(P^{(f, \chi)} || P^{(f, \chi')}) = \sum_{n, \underline{x}, \chi} P_{n, \underline{x}, \chi} \log \frac{P_{n, \underline{x}, \chi}}{P_{n, \underline{x}, \chi'}} = \log \langle \psi(f) \rangle - \langle \log \psi(f) \rangle.$$

Proof. We write the zeta function as

$$\zeta_{\chi}((X, f_{\epsilon}), t) = \sum_n \sum_{\underline{x} \in S^n(X)(\mathbf{F}_q)} \chi(f^{(n)}(\underline{x})) \chi(\epsilon h^{(n)}(\underline{x})) t^n,$$

so that, if we formally regard as above

$$P_{n,\underline{x}} = \frac{\chi(f^{(n)}(\underline{x})) t^n}{\zeta_{\chi}((X, f), t)}$$

as our “probability distribution”, we have

$$\frac{\zeta_{\chi}((X, f_{\epsilon}), t)}{\zeta_{\chi}((X, f), t)} = \sum_n \sum_{\underline{x} \in S^n(X)(\mathbf{F}_q)} P_{n,\underline{x}} \chi(\epsilon h^{(n)}(\underline{x})) = \langle \chi(\epsilon h) \rangle,$$

computing the expectation value $\langle \chi(\epsilon h) \rangle$ with respect to the distribution $P_{n,\underline{x}}$.

Then, setting as above

$$P_{n,\underline{x},\epsilon} = P_{n,\underline{x}}^{(f_{\epsilon}, \chi)} = \frac{\chi(f^{(n)}(\underline{x})) \chi(\epsilon h^{(n)}(\underline{x})) t^n}{\zeta_{\chi}((X, f_{\epsilon}), t)}$$

we obtain

$$\log P_{n,\underline{x},\epsilon} = \log P_{n,\underline{x}} - \log \frac{\zeta_{\chi}((X, f_{\epsilon}), t)}{\zeta_{\chi}((X, f), t)} + \log \chi(\epsilon h^{(n)}(\underline{x})).$$

Thus, the Kullback–Leibler divergence gives

$$KL(\zeta_{\chi}((X, f), t) || \zeta_{\chi}((X, f_{\epsilon}), t)) := \sum_{n,\underline{x}} P_{n,\underline{x}} \log \frac{P_{n,\underline{x}}}{P_{n,\underline{x},\epsilon}} =$$

$$\log \frac{\zeta_{\chi}((X, f_{\epsilon}), t)}{\zeta_{\chi}((X, f), t)} - \sum_{n,\underline{x}} P_{n,\underline{x}} \log \chi_{\epsilon}(h^{(n)}(\underline{x})),$$

where χ_{ϵ} is the character $\chi_{\epsilon} = \chi \circ \epsilon$. We can write this equivalently as

$$KL(\zeta_{\chi}((X, f), t) || \zeta_{\chi}((X, f_{\epsilon}), t)) = \log \langle \chi_{\epsilon}(h) \rangle - \langle \log \chi_{\epsilon}(h) \rangle.$$

Similarly, given two characters $\chi, \chi' : \mathbf{F}_q \rightarrow \mathbf{C}^*$, let $\psi = \chi^{-1} \cdot \chi'$, the Kullback–Leibler divergence is given by

$$KL(\zeta_\chi((X, f), t) || \zeta_{\chi'}((X, f), t)) = \sum_{n, \underline{x}, \chi} P_{n, \underline{x}, \chi} \log \frac{P_{n, \underline{x}, \chi}}{P_{n, \underline{x}, \chi'}},$$

where $\log P_{n, \underline{x}, \chi'} = \log P_{n, \underline{x}, \chi} + \log \psi(f^{(n)}(\underline{x}))$, so that

$$\begin{aligned} KL(\zeta_\chi((X, f), t) || \zeta_{\chi'}((X, f), t)) &= \log \frac{\zeta_{\chi'}((X, f), t)}{\zeta_\chi((X, f), t)} - \sum_{n, \underline{x}} P_{n, \underline{x}} \log \psi(f^{(n)}(\underline{x})) = \\ &= \log \langle \psi(f) \rangle - \langle \log \psi(f) \rangle \end{aligned}$$

as stated. ■

The notion of Kullback–Leibler divergence for zeta functions considered above requires that the comparison is made at the same class $[X, f]$ in $KExp^c(\mathcal{V}_K)$. If one wants to introduce the possibility of comparing the zeta functions at two different classes $[X_1, f_1]$ and $[X_2, f_2]$ via a Kullback–Leibler divergence, it makes sense to compare them over the fibered product, namely the natural space where the morphisms f_1, f_2 agree. Namely we define $KL(\zeta_\chi((X_1, f_1), t) || \zeta_{\chi'}((X_2, f_2, \epsilon), t))$ to be given by

$$KL(\zeta_\chi(X_1 \times_{f_1, f_2} X_2, f), t) || \zeta_{\chi'}(X_1 \times_{f_1, f_2} X_2, f_\epsilon), t),$$

using the pullback $X_1 \times_{f_1, f_2} X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}$ with $f = f_1 \circ \pi_1 = f_2 \circ \pi_2 : X_1 \times_{f_1, f_2} X_2 \rightarrow \mathbf{A}^1$ and $f_\epsilon = F(\cdot, \epsilon)$ for some $F : X_1 \times_{f_1, f_2} X_2 \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$ with $F(\cdot, 0) = f$.

6.2. Shannon entropy and Hasse–Weil zeta function. A notion of Shannon information in the context of Grothendieck rings of varieties was proposed in [Mar19b], based on regarding zeta functions as physical partition functions. We show here that this can be regarded as a special case of the construction described above.

First observe that the usual Grothendieck ring of K -varieties $K_0(\mathcal{V}_K)$ embeds in the coarse Grothendieck ring with exponentials $KExp^c(\mathcal{V}_K)$ by mapping $[X]$ to $[X, 0]$. For varieties over a finite field $K = \mathbf{F}_q$, the measure μ_χ associated to the trivial character $\chi = 1$ is just the counting measure $\mu_1(X, f) = \text{card } X(\mathbf{F}_q)$, hence

the zeta function ζ_{μ_1} restricted to $K_0(\mathcal{V}_K) \subset KExp^c(\mathcal{V}_K)$ is the Hasse-Weil zeta function of X ,

$$Z^{HW}(X, t) = \exp \left(\sum_m \text{card } X(\mathbf{F}_{q^m}) \frac{t^m}{m} \right).$$

This can be seen by writing $Z^{HW}(X, t)$ in terms of effective zero-cycles as

$$Z^{HW}(X, t) = \sum_{\alpha} t^{\deg \alpha},$$

and further writing the latter in the form

$$Z^{HW}(X, t) = \sum_{n \geq 0} \text{card } S^n(X)(\mathbf{F}_q) t^n.$$

In the above expression, one can regard the quantity

$$P(\alpha) := \frac{t^{\deg \alpha}}{Z^{HW}(X, t)}$$

as a probability measure assigned to the zero-cycle α , hence one can consider the Shannon information of this distribution, which is given by

$$S(X, t) := - \sum_{\alpha} P(\alpha) \log P(\alpha) = \log Z^{HW}(X, t) + Z^{HW}(X, t)^{-1} H(X, t),$$

$$H(X, t) := - \sum_{\alpha} t^{\deg \alpha} \log(t^{\deg \alpha}).$$

To compare this to the Kullback-Leibler divergence introduced in the previous section, one can equivalently regard the above expression as the Shannon entropy of the distribution $P_{n, \underline{x}} = P_{n, \underline{x}}^{(f=0, \chi=1)}$,

$$P_{n, \underline{x}} = \frac{t^n}{Z^{HW}(X, t)},$$

for all $n \geq 0$ and all $\underline{x} \in S^n(X)(\mathbf{F}_q)$,

$$S(X, t) = - \sum_{n, \underline{x}} P_{n, \underline{x}} \log P_{n, \underline{x}}.$$

It is customary to make a change of variables $t = q^{-s}$ and write the Hasse–Weil zeta function as $Z^{HW}(X, q^{-s})$. We correspondingly write $S(X, s)$ for the Shannon entropy defines as above, after setting $t = q^{-s}$.

6.2.1. Lemma. *For a variety X over \mathbf{F}_q the Shannon entropy associated to the Hasse–Weil zeta function is given by*

$$S(X, s) = \left(1 - s \frac{d}{ds}\right) Z^{HW}(X, q^{-s}).$$

Proof. With the associated probability distribution $P_{n,x} = \frac{q^{-sn}}{Z^{HW}(X, q^{-s})}$ we have

$$S(X, s) = - \sum_{n,x} \frac{q^{-sn}}{Z^{HW}(X, q^{-s})} (\log q^{-sn} - \log Z^{HW}(X, q^{-s})) =$$

$$\log Z^{HW}(X, q^{-s}) + Z^{HW}(X, q^{-s})^{-1} \sum_n \text{card } S^n(X)(\mathbf{F}_q) q^{-sn} sn \log q.$$

■

6.2.2. Thermodynamical interpretation of the Shannon entropy. Lemma 6.2.1 shows that the Shannon entropy $S(X, s)$ agrees with the usual thermodynamical entropy

$$S = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \log Z(\beta)$$

of a physical system with partition function $Z(\beta)$ at inverse temperature $\beta > 0$, and free energy $F = -\log Z(\beta)$. We identify here the Hasse–Weil zeta function $Z^{HW}(X, q^{-s})$ with the partition function of a physical system with Hamiltonian H with energy levels $\text{Spec}(H) = \{n \log q\}_{n \geq 0}$ with degeneracies $\text{card } S^n(X)(\mathbf{F}_q)$, so that

$$\text{Tr}(e^{-\beta H}) = Z^{HW}(X, q^{-\beta}).$$

The expression given above for the thermodynamical entropy is the same as the Shannon entropy of the probability distribution $P_n = \frac{e^{-\beta \lambda_n}}{Z(\beta)}$, for $Z(\beta) = \text{Tr}(e^{-\beta H})$ with $\text{Spec}(H) = \{\lambda_n\}$, since we have

$$S = - \sum_n P_n \log P_n = \sum_n P_n \log Z(\beta) + \beta \sum_n P_n \lambda_n$$

where $\sum_n P_n \lambda_n = \frac{\partial}{\partial \beta} \log Z(\beta)$.

6.2.3. Lemma. *The entropy of $\text{Spec}(\mathbf{F}_q)$,*

$$S(\text{Spec}(\mathbf{F}_q), s) = - \left(1 - s \frac{d}{ds} \right) \log(1 - q^{-s}),$$

is the thermodynamical entropy of a physical system with non-degenerate energy levels $\{k \log q\}_{k \geq 0}$. The entropy of an affine space \mathbf{A}^n over \mathbf{F}_q ,

$$S(\mathbf{A}_{\mathbf{F}_q}^n, s) = - \left(1 - s \frac{d}{ds} \right) \log(1 - q^{-s+n}),$$

is the thermodynamical entropy of a physical system with energy levels $\{k \log q\}_{k \geq 0}$ with degeneracies q^{kn} . The entropy of a projective space \mathbf{P}^n over \mathbf{F}_q ,

$$S(\mathbf{P}_{\mathbf{F}_q}^n, s) = - \left(1 - s \frac{d}{ds} \right) \sum_{k=0}^n \log(1 - q^{-s+k}),$$

is the thermodynamical entropy of a composite system consisting of $n+1$ independent subsystems, all of them with energy levels $\{k \log q\}_{k \geq 0}$, where the j -th system has energy levels with degeneracies q^{kj} .

Proof. This follows directly from Lemma 6.2.1, since the Hasse-Weil zeta functions are respectively given by

$$\begin{aligned} Z^{HW}(\text{Spec}(\mathbf{F}_q), q^{-s}) &= \exp\left(\sum_m \frac{q^{-sm}}{m}\right) = \exp(-\log(1 - q^{-s})) \\ &= \frac{1}{1 - q^{-s}} = \sum_{k \geq 0} q^{-sk}. \end{aligned}$$

$$\begin{aligned} Z^{HW}(\mathbf{A}^n, q^{-s}) &= \exp\left(\sum_m \frac{q^{mn} q^{-sm}}{m}\right) = \exp(-\log(1 - q^n q^{-s})) \\ &= \frac{1}{1 - q^{-s+n}} = \sum_{k \geq 0} q^{kn} q^{-sk}. \end{aligned}$$

$$Z^{HW}(\mathbf{P}^n, q^{-s}) = Z^{HW}(\mathbf{A}^0 \cup \mathbf{A}^1 \cup \cdots \cup \mathbf{A}^n, q^{-s}) = \frac{1}{(1 - q^{-s}) \cdots (1 - q^{-s+n})}.$$

The case of \mathbf{P}^n reflects the usual property of additivity of the Shannon entropy over independent subsystems. ■

The following equivalent description of the Shannon entropy of \mathbf{P}^n over \mathbf{F}_p will become useful in the next subsection.

A matrix $M = (M_{ij}) \in M_{n \times n}(\mathbf{Z})$ is *reduced* if it is a lower triangular with $0 \leq M_{ij} \leq M_{jj}$ for $i \geq j$. Let Red_n denote the set of reduced matrices. For a given positive integer $m \in \mathbf{N}$ let $Red_n(m) = \{M \in Red_n \mid \det(M) = m\}$.

6.2.4. Lemma. *The entropy $S(\mathbf{P}_{\mathbf{F}_p}^{n-1}, s)$ can also be identified with the thermodynamical entropy of a physical system with energy levels $\{k \log p\}_{k \geq 0}$ and degeneracies $D_k = \text{card } Red_n(p^k)$.*

Proof. If a reduced matrix M has $\det(M) = p^k$ then the j -th diagonal entry is p^{k_j} with $\sum_j k_j = k$. For a given diagonal entry p^{k_j} there is a total of $p^{k_j(k-j)}$ possibilities in the j -th column satisfying $0 \leq M_{ij} \leq M_{jj}$ for $i \geq j$. Thus we can write the multiplicities as $D_k = \sum_{k_1 + \cdots + k_n = k} D_{k_1, \dots, k_n}$ with $D_{k_1, \dots, k_n} = p^{k_1(n-1)} p^{k_2(n-2)} \cdots p^{k_{n-1}}$. The partition function of such a system is given by

$$\begin{aligned} \text{Tr}(e^{-sH}) &= \sum_{k \geq 0} p^{-sk} \sum_{k_1 + \cdots + k_n = k} D_{k_1, \dots, k_n} \\ &= \sum_{k \geq 0} p^{-sk} \sum_{k_1 + \cdots + k_n = k} p^{k_1(n-1)} p^{k_2(n-2)} \cdots p^{k_{n-1}} \\ &= \sum_{k_1, \dots, k_n} p^{k_1(n-1-s)} p^{k_2(n-2-s)} \cdots p^{k_{n-1}(1-s)} p^{-k_n s} \\ &= \prod_{\ell=0}^{n-1} \frac{1}{1 - p^{-s+\ell}} = Z^{HW}(\mathbf{P}_{\mathbf{F}_p}^{n-1}, p^{-s}). \end{aligned}$$

This identifies the thermodynamical entropy of this system with the Shannon entropy $S(\mathbf{P}_{\mathbf{F}_p}^{n-1}, s)$. ■

6.3. The case of varieties over \mathbf{Z} . Consider a variety X over \mathbf{Z} and denote by X_p the reduction of X mod p . For simplicity, we will consider here only the

case where there are no primes of bad reduction. The (non-completed) L -function is then given by

$$L(X, s) = \prod_p Z^{WH}(X_p, p^{-s}),$$

while the completed L -function includes a contribution of the archimedean prime ([Se70]):

$$L^*(X, s) = L(X, s) \cdot L_\infty(X, s),$$

$$L_\infty(X, s) := \prod_{i=0}^{\dim X} L_\infty(H^i(X), s)^{(-1)^{i+1}},$$

$$L_\infty(H^i(X), s) := \prod_{p < q} \Gamma_{\mathbf{C}}(s - p)^{h^{p,q}} \prod_p \Gamma_{\mathbf{R}}(s - p)^{h^{p,+}} \Gamma_{\mathbf{R}}(s - p + 1)^{h^{p,-}},$$

where $h^{p,q}$ are the Hodge numbers of the complex variety $X_{\mathbf{C}}$, with $h^{p,\pm}$ the dimension of the $(-1)^p$ -eigenspace of the involution on $H^{p,p}$ induced by the real structure, and

$$\Gamma_{\mathbf{C}}(s) := (2\pi)^{-s} \Gamma(s), \quad \Gamma_{\mathbf{R}}(s) = 2^{-1/2} \pi^{-s/2} \Gamma(s/2).$$

In the more general cases of number fields with several archimedean places, corresponding to the embeddings of the number field in \mathbf{C} , the archimedean places given by real embeddings have an archimedean factor as above and the archimedean places given by complex embeddings have a similar one, that also depends on the Hodge structure, of the form

$$\prod_{p,q} \Gamma_{\mathbf{C}}(s - \min(p, q))^{h^{p,q}}.$$

As is shown in [Se70], the form of these archimedean local factors is dictated by the expected form of the conjectural functional equation for the completed L -function $L^*(X, s)$.

The non-completed L -function $L(X, s)$ can be understood as the partition function of a physical system consisting of a countable family of independent subsystems, one for each prime, with partition function $Z^{WH}(X_p, p^{-s})$, hence the additivity of

the Shannon entropy over independent subsystems prescribes that the associated entropy should be

$$S_{\mathbf{Z}}(X, s) := \sum_p S(X_p, s) = H_{\mathbf{Z}}(X, s) + \log L(X, s),$$

$$H_{\mathbf{Z}}(X, s) := \sum_p Z^{HW}(X_p, p^{-s})^{-1} H(X_p, p^{-s}).$$

This can be equivalently written as

$$S_{\mathbf{Z}}(X, s) = \left(1 - s \frac{d}{ds}\right) \log L(X, s).$$

The following statement is a direct consequence of Lemma 6.2.1, Lemma 6.2.3 and the above definition of the entropy $S_{\mathbf{Z}}(X, s)$.

6.3.1. Lemma. *The Shannon entropy of the non-completed L-function for $\text{Spec}(\mathbf{Z})$,*

$$S_{\mathbf{Z}}(\text{Spec}(\mathbf{Z}), s) = \left(1 - s \frac{d}{ds}\right) \log \zeta(s)$$

is the thermodynamical entropy of a physical system with non-degenerate energy levels $\{\log k\}_{k \geq 1}$.

Such physical systems are realised, for instance, by the Julia system of [Ju90] or by the Bost–Connes system of [BoCo95].

The Shannon entropy of the non-completed L-function for an affine space \mathbf{A}^n over \mathbf{Z} ,

$$S_{\mathbf{Z}}(\mathbf{A}^n, s) = \left(1 - s \frac{d}{ds}\right) \log \zeta(s - n),$$

is the thermodynamical entropy of a physical system with energy levels $\{\log k\}_{k \geq 1}$ with degeneracies k^n . The Shannon entropy of the non-completed L-function for a projective space \mathbf{P}^n over \mathbf{Z} ,

$$S_{\mathbf{Z}}(\mathbf{P}^n, s) = \left(1 - s \frac{d}{ds}\right) \sum_{m=0}^n \log \zeta(s - m),$$

is the thermodynamical entropy of a physical system with energy levels $\{\log k\}_{k \geq 1}$ and degeneracies $D_k = \text{card Red}_n(k)$.

Such physical systems are realised by the GL_n -versions of the Bost–Connes system considered in [Shen16].

When one considers also the archimedean places, one can regard the completed partition function $L^*(X, s)$ in a similar way as the partition function of a composite system consisting of independent subsystems for each non-archimedean and archimedean prime. The contributions of the non-archimedean primes are given, as above, by systems with partition function given by the Hasse–Weil zeta function $Z^{HW}(X_p, p^{-s})$, while the contribution of the archimedean places has partition function $L_\infty(X, s)$. The additivity of the Shannon entropy over independent subsystems again prescribes that we assign entropy

$$S_{\mathbf{Z}}^*(X, s) := \sum_p S(X_p, s) + S_\infty(X, s),$$

where $S_\infty(X, s)$ is the entropy of a system with partition function $L_\infty(X, s)$,

$$S_\infty(X, s) := \left(1 - s \frac{d}{ds}\right) \log L_\infty(X, s).$$

The difficulty here is in interpreting the expression

$$L_\infty(H^i(X), s) := \prod_{p < q} \Gamma_{\mathbf{C}}(s - p)^{h^{p,q}} \prod_p \Gamma_{\mathbf{R}}(s - p)^{h^{p,+}} \Gamma_{\mathbf{R}}(s - p + 1)^{h^{p,-}}$$

as a partition function (see also [Ju90]). This problem is closely related to the well known arithmetic problem of obtaining an interpretation of the archimedean factors that parallels the corresponding form of the non-archimedean ones, see [Den91], [Manin95]. Note that the logarithmic derivative of these non-archimedean local factors, which determines the associated entropy, has a Lefschetz trace formula interpretation as proved in Sec. 7 of [CCM07].

6.4. Shannon entropy for other motivic measures. Denote by $Z^{mot}(X, t)$ the Kapranov motivic zeta function (see [Kap00]):

$$Z^{mot}(X, t) = \sum_{n=0}^{\infty} [S^n X] t^n.$$

It is a formal power series in $K_0(\mathcal{V})[[t]]$.

Starting with a motivic measure, given by a ring homomorphism $\mu : K_0(\mathcal{V}) \rightarrow R$ with values in a commutative ring R , we can define the zeta function ζ_μ by applying the motivic measure μ to the coefficients of the Kapranov zeta function:

$$\zeta_\mu(X, t) = \sum_{n=0}^{\infty} \mu(S^n X) t^n.$$

In particular, a motivic measure $\mu : K_0(\mathcal{V}) \rightarrow R$ is called *exponentiable*, if the zeta function ζ_μ defines a *ring* homomorphism $\zeta_\mu : K_0(\mathcal{V}) \rightarrow W(R)$ to the Witt ring of R : see [Ram15], [RamTab15].

For any μ , the zeta function ζ_μ defines an additive map, which means that it satisfies the inclusion–exclusion property

$$\zeta_\mu(X \cup Y, t) = \frac{\zeta_\mu(X, t) \zeta_\mu(Y, t)}{\zeta_\mu(X \cap Y, t)},$$

where the product of power series is the *addition* in the Witt ring.

The exponentiability means that one also has

$$\zeta_\mu(X \times Y, t) = \zeta_\mu(X, t) \star \zeta_\mu(Y, t),$$

where \star is the *product* in the Witt ring, which is uniquely determined by

$$(1 - at)^{-1} \star (1 - bt)^{-1} = (1 - abt)^{-1}, \quad a, b \in R.$$

The motivic measure given by counting points over a finite field, with ζ_μ the Hasse–Weil zeta function, is exponentiable.

Given a motivic measure $\mu : K_0(\mathcal{V}) \rightarrow R$ with associated zeta function

$$\zeta_\mu(X, t) = \sum_{n=0}^{\infty} \mu(S^n X) t^n,$$

one can define the respective Shannon entropy as

$$S_\mu(X, t) := (1 - t \log t \frac{d}{dt}) \log \zeta_\mu(X, t),$$

where the $\log t$ factor occurs due to a change of variables $t = \lambda^{-s}$ with respect to the thermodynamic entropy with inverse temperature $\beta = \log \lambda$.

As discussed in [Mar19b], the entropy S_μ satisfies an analog of the Khinchin axioms for the usual Shannon entropy, where the extensivity property of the Shannon entropy on composite system is expressed as the inclusion–exclusion property

$$S_\mu(X \cup Y, t) = S_\mu(X, t) + S_\mu(Y, t) - S_\mu(X \cap Y, t).$$

As we discussed at the beginning of Section 6.2.2, this definition of entropy is justified by interpreting the zeta function $\zeta_\mu(X, t)$ as a partition function and its logarithm $-\log \zeta_\mu(X, t)$ as the associated free energy. In the following subsection we return to the relative entropy (Kullback–Leibler divergence) introduced in Section 6.1, and we discuss the corresponding thermodynamical interpretation.

6.5. Thermodynamics of Kullback–Leibler divergence. We recalled above the thermodynamical entropy

$$S = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \log Z(\beta)$$

of a physical system with partition function given by the zeta function $Z(\beta)$ at inverse temperature $\beta > 0$ with free energy $F = -\log Z(\beta)$.

The Kullback–Leibler divergence

$$KL_\phi(P||Q) = \sum_{x \in X} P_x \log \frac{P_x}{Q_x}$$

of two probabilities P, Q on the same set X can also be interpreted in terms of free energy and Gibbs free energy. If $Q_x = \frac{e^{-\beta H_x}}{Z(\beta)}$ with $Z(\beta) = \sum_x e^{-\beta H_x}$ the partition function, and P a given probability distribution on the configuration space, the Gibbs free energy is given by

$$G(P) = -\log Z(\beta) + \sum_x P_x \log \frac{P_x}{Q_x},$$

hence $KL(P||Q) = G(P) + \log Z(\beta)$. It follows that the usual free energy is minimization of the Gibbs energy over configuration probabilities: since $KL(P||Q) \geq 0$, we have

$$\min_P G(P) = -\log Z(\beta).$$

In the typical approach of mean field theory, when computation of the free energy of a system is not directly accessible, one considers a trial Hamiltonian \tilde{H} with probability distribution $P_x = \tilde{Z}(\beta)^{-1} e^{-\beta \tilde{H}_x}$, where $\tilde{Z}(\beta) := \sum_x e^{-\beta \tilde{H}_x}$. Then $\log P_x = -\log \tilde{Z}(\beta) - \beta \tilde{H}_x$, and the Helmholtz free energy

$$-\sum_x P_x \log P_x = \log \tilde{Z}(\beta) + \beta \langle \tilde{H} \rangle = (1 - \beta \frac{\partial}{\partial \beta}) \log \tilde{Z}(\beta)$$

satisfies

$$\sum_x P_x \log \frac{P_x}{Q_x} = \log \frac{Z(\beta)}{\tilde{Z}(\beta)} + \beta \langle H - \tilde{H} \rangle.$$

In the mean field theory setting, it is common to also assume that the trial Hamiltonian satisfies $\langle H \rangle = \langle \tilde{H} \rangle$ with the averages computed with respect to the probability distribution P_x , so that the identity above would reduce just to

$$\sum_x P_x \log \frac{P_x}{Q_x} = -\log \tilde{Z}(\beta) + \beta \langle \tilde{H} \rangle + \log Z(\beta) - \beta \langle H \rangle = \log \frac{Z(\beta)}{\tilde{Z}(\beta)}.$$

However, we will not make this assumption.

If we consider the case of a 1-parameter family of *commuting* Hamiltonians $H(\epsilon)$ that depends analytically on ϵ , such that

$$H(\epsilon) = \tilde{H} + \epsilon \frac{\partial \tilde{H}}{\partial \epsilon} \Big|_{\epsilon=0} + O(\epsilon^2),$$

we have

$$\sum_x P_x H_x(\epsilon) \sim \sum_x P_x \tilde{H}_x + \epsilon \sum_x P_x \frac{\partial H_x(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0}.$$

The generalized force corresponding to the variable ϵ is given by $L_x = -\frac{\partial H_x(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0}$. Then

$$\langle L \rangle = \sum_x P_x L_x = \frac{1}{\beta} \frac{\partial}{\partial \epsilon} \log Z_\epsilon(\beta) \Big|_{\epsilon=0},$$

where $Z_\epsilon(\beta) = \sum_x e^{-\beta H(\epsilon)}$. For $P_x(\epsilon) = Z_\epsilon(\beta)^{-1} e^{-\beta H(\epsilon)}$ We have

$$\log P_x(\epsilon) = -\log Z_\epsilon(\beta) - \beta(\tilde{H}_x + \epsilon L_x + O(\epsilon^2)).$$

Thus, we can write the Kullback–Leibler divergence in this case as

$$\begin{aligned} \sum_x P_x \log \frac{P_x}{P_x(\epsilon)} &= \sum_x P_x \log P_x + \log Z_\epsilon(\beta) + \beta \sum_x P_x \tilde{H}_x + \epsilon \beta \sum_x P_x L_x + O(\epsilon^2) = \\ &= -(1 - \beta \frac{\partial}{\partial \beta}) \log \tilde{Z}(\beta) + \log Z_\epsilon(\beta) + \beta \sum_x P_x \tilde{H}_x + \epsilon \frac{\partial}{\partial \epsilon} \log Z_\epsilon(\beta)|_{\epsilon=0} + O(\epsilon^2) = \\ &= \log \frac{Z_\epsilon(\beta)}{\tilde{Z}(\beta)} + \epsilon \frac{\partial}{\partial \epsilon} \log Z_\epsilon(\beta)|_{\epsilon=0} + O(\epsilon^2). \end{aligned}$$

This thermodynamical point of view on the Kullback–Leibler divergence has the advantage that one can express its leading term purely in terms of partition functions, suggesting how to develop a motivic analog defined in terms of zeta and L -functions. This motivates the definition of Kullback–Leibler divergence that we used in Proposition 6.1.3.

6.6. Fisher–Rao metric. Recall that, for a family of probability distributions $P(\gamma) = (P_x(\gamma))$ on a set X , depending differentiably on parameters $\gamma = (\gamma_1, \dots, \gamma_r)$, the Fisher–Rao information metric is defined as

$$g_{ij}(\gamma) := \sum_x P_x(\gamma) \frac{\partial \log P_x(\gamma)}{\partial \gamma_i} \frac{\partial \log P_x(\gamma)}{\partial \gamma_j}.$$

Assuming that the probability distributions $P_x(\gamma)$ are of the form

$$P_x(\gamma) = \frac{e^{-\beta H_x(\gamma)}}{Z_\gamma(\beta)}, \quad Z_\gamma(\beta) = \sum_x e^{-\beta H_x(\gamma)},$$

for a family of commuting Hamiltonians $H(\gamma)$, we can write as above

$$\log P_x(\gamma) = -\log Z_\gamma(\beta) - \beta H_x(\gamma).$$

If we consider, the generalized forces

$$L_{x,i} = -\frac{\partial H_x(\gamma)}{\partial \gamma_i},$$

we can write the Fisher-Rao metric as

$$g_{ij}(\gamma) = \frac{\partial \log Z_\gamma(\beta)}{\partial \gamma_i} \frac{\partial \log Z_\gamma(\beta)}{\partial \gamma_j} + \beta^2 \sum_x P_x(\gamma) L_{x,i} L_{x,j}.$$

Equivalently, the Fisher-Rao metric can be obtained as the Hessian matrix of the Kullback-Leibler divergence

$$g_{ij}(\gamma_0) = \frac{\partial^2}{\partial \gamma_i \partial \gamma_j} KL(P(\gamma) || P(\gamma_0))|_{\gamma=\gamma_0}.$$

Another way of characterizing the Fisher-Rao metric tensor in the case of classical information theory is designed for easier comparison with the case of quantum information, and will also be more directly useful in our setting.

One identifies classical probability distributions with diagonal density matrices. This is equivalent to considering pairs $\rho, \rho' \in M_N$ of commuting density matrices, $[\rho, \rho'] = 0$, so that

$$KL(\rho || \rho') = Tr(\rho (\log \rho - \log \rho')) = \sum_i P_i \log \frac{P_i}{Q_i},$$

where ρ and ρ' are simultaneously diagonalized to $\rho = \text{diag}(P_i)$ and $\rho' = \text{diag}(Q_i)$. Since we have $KL(\rho || \rho') = Tr(\rho (\log \rho - \log \rho')) = \infty$ whenever $0 \in \text{Spec}(\rho')$ we can restrict to the case where ρ' is invertible.

Let h be an (infinitesimal) increment, so that $\rho + h \in \mathcal{M}^{(N)}$. In particular, this implies $Tr(h) = 0$. We are interested in computing the relative entropy $KL(\rho + h || \rho)$, up to second order terms in h . Again, we assume ρ is invertible, to ensure $KL(\rho + h || \rho) < \infty$.

6.6.1. Lemma. *In the classical case, or equivalently whenever $[\rho, h] = 0$, we simply have*

$$KL(\rho + h || \rho) = \frac{1}{2} Tr(h \rho^{-1} h) + O(h^3),$$

where the right hand side is the classical Fisher metric.

Proof. We have

$$KL(\rho + h || \rho) = Tr((\rho + h) \log(\rho + h)) - Tr((\rho + h) \log \rho)$$

$$\begin{aligned}
&= \operatorname{Tr}(\rho \log(\rho(I + \rho^{-1}h))) + \operatorname{Tr}(h \log(\rho(I + \rho^{-1}h))) - \operatorname{Tr}(\rho \log \rho) - \operatorname{Tr}(h \log \rho) \\
&= \operatorname{Tr}(\rho \log(I + \rho^{-1}h)) + \operatorname{Tr}(h \log(I + \rho^{-1}h)).
\end{aligned}$$

Expanding up to second order $\log(I + \rho^{-1}h) = \rho^{-1}h - \frac{1}{2}\rho^{-1}h\rho^{-1}h + O(h^3)$, we obtain

$$KL(\rho+h||\rho) = \operatorname{Tr}(\rho\rho^{-1}h) - \frac{1}{2}\operatorname{Tr}(\rho\rho^{-1}h\rho^{-1}h) + \operatorname{Tr}(h\rho^{-1}h) + O(h^3) = \frac{1}{2}\operatorname{Tr}(h\rho^{-1}h),$$

where the condition $\operatorname{Tr}(h) = 0$ gives the vanishing of the first order term. ■

6.7. Fisher–Rao tensor and zeta functions. We now return to our setting in the coarse Grothendieck ring of varieties with exponentials $K\operatorname{Exp}^c(\mathcal{V})_K$ and we discuss how to obtain a Fisher–Rao tensor based on the notion of Kullback–Leibler divergence discuss above.

Given a variety X over a field K , we define as in [DenLoe01] the arc space $\mathcal{L}(X)$ of X as the projective limit of the truncated arc spaces $\mathcal{L}_m(X)$, where $\mathcal{L}_m(X)$ is the variety over K whose L -rational points, for a field L containing K , are the $L[[u]]/u^{m+1}$ -rational points of X . In particular, $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X)$ is the tangent bundle of X . Consider a morphism $f : X \rightarrow \mathbf{A}^1$ and the induced morphisms $f_m : \mathcal{L}_m(X) \rightarrow \mathcal{L}_m(\mathbf{A}^1)$. Points in $\mathcal{L}(\mathbf{A}^1)$ can be identified with power series $\alpha(u)$ in $L[[u]]$, for some field extension L , or respectively in $L[[u]]/u^{m+1}$ in the case of $\mathcal{L}_m(\mathbf{A}^1)$.

Given a pair (X, f) with $f : X \rightarrow \mathbf{A}^1$, we can consider the associated pairs $(\mathcal{L}_m(X), f_m)$ with the induced morphism $f_m : \mathcal{L}_m(X) \rightarrow \mathcal{L}_m(\mathbf{A}^1)$ and similarly for the arc space $\mathcal{L}(X)$. For $K = \mathbf{F}_q$, consider characters χ_m given by group homomorphisms

$$\chi_m : \mathbf{F}_q[[u]]/u^{m+1} \rightarrow \mathbf{C}^*,$$

so that we can compute

$$\mu_{\chi_m}(\mathcal{L}_m(X), f_m) = \sum_{\varphi \in \mathcal{L}_m(X)(\mathbf{F}_q)} \chi(f_m(\varphi)).$$

The respective zeta function will be of the form

$$\zeta_{\chi_m}((\mathcal{L}_m(X), f_m), t) = \sum_{n \geq 0} \mu_{\chi_m}(S^n((\mathcal{L}_m(X), f_m))) t^n$$

$$= \sum_n \sum_{\underline{\varphi} \in S^n(\mathcal{L}_m(X))(\mathbf{F}_q)} \chi_m(f_m^{(n)}(\underline{\varphi})) t^n,$$

where the symmetric products are given by $(S^n((\mathcal{L}_m(X), f_m))) = (\mathcal{L}_m(S^n(X)), f_m^{(n)})$ with the morphism $f_m^{(n)} : \mathcal{L}_m(S^n(X)) \rightarrow \mathcal{L}_m(\mathbf{A}^1)$ induced by $f^{(n)} : S^n(X) \rightarrow \mathbf{A}^1$.

For the purpose of constructing a Fisher-Rao tensor, it suffices to work with the tangent bundle, namely with the first truncated arc space $\mathcal{L}_1(X)$.

6.7.1. Lemma. *Let $K = \mathbf{F}_q$ be a finite field. Consider a class $[\mathcal{L}_1(X), f_1] \in KExp^c(\mathcal{V})_K$ and a character $\chi_1 : K[u]/u^2 \rightarrow \mathbf{C}^*$. Then, for all $n \in \mathbf{N}$, the summands $h_{n,\underline{\varphi}} := \chi_1(f_1^{(n)}(\underline{\varphi}))t^n$ satisfy $\sum_{\underline{\varphi}} h_{n,\underline{\varphi}} = 0$.*

Proof. Elements $\varphi \in \mathcal{L}_1(X)(K)$ can be interpreted as specifying a point $x \in X(K)$ and a tangent vector $v \in T_x(X)$, with $f(\varphi) \in \mathcal{L}_1(\mathbf{A}^1)$, for a given $f : X \rightarrow \mathbf{A}^1$, correspondingly specifying a point $f(x)$ and a tangent vector $df_x(v) \in T_{f(x)}\mathbf{A}^1$. We can identify group homomorphisms $\chi_1 : \mathbf{F}_q[u]/u^2 \rightarrow \mathbf{C}^*$ with pairs of characters $\chi, \chi' : \mathbf{F}_q \rightarrow \mathbf{C}^*$. Since $T(X)$ is locally trivial, given a choice of characters $\chi, \chi' : K \rightarrow \mathbf{C}^*$, the $h_{n,\underline{\varphi}}$ are locally of the form $\chi(f^{(n)}(\underline{x}))\chi'(df^{(n)}(\underline{v}))t^n$, for $\underline{\varphi} \in S^n(\mathcal{L}_1(X))$ corresponding to $\underline{x} \in S^n(X)$ with a tangent vector \underline{v} . The vanishing of the sum follows from the fact that the class $[\mathcal{L}_1(X), f_1]$ can be written in $KExp^c(\mathcal{V})_K$ as a sum of classes

$$[X_i \times V_i, f \circ \pi_{X_i} + \langle df \circ \pi_{X_i}, \pi_{V_i} \rangle],$$

where the second term is the linear form $\langle df_x, \pi_{V_i} \rangle : V_i \rightarrow K$ for all $x \in X(K)$. For any non-trivial linear form $\lambda : V \rightarrow K$, for a finite dimensional K -vector space V , and a character $\chi : K \rightarrow \mathbf{C}^*$, one has

$$\sum_{v \in V} \chi(\lambda(v)) = 0.$$

This shows the vanishing of $\sum_{\underline{v}} \chi'(df_x^{(n)}(\underline{v}))$, hence of the sum of the $h_{n,\underline{\varphi}}$. ■

The argument above can also be rephrased as showing that all the classes $[S^n(\mathcal{L}_1(X), f_1)]$ are in the ideal of $KExp^c(\mathcal{V})_K$ generated by $[\mathbf{A}^1, id]$, hence trivial in the Grothendieck ring with exponentials $KExp(\mathcal{V})_K$ (compare the analogous argument used in Lemma 1.1.11 and Theorem 1.2.9 of [ChamLoe15]).

6.7.2. Proposition. Consider the distribution $\rho = (P_{n,\underline{x}})$ with $n \in \mathbf{N}$ and $\underline{x} \in S^n(X)$ given by

$$P_{n,\underline{x}} = \frac{\chi(f^{(n)}(\underline{x})t^n}{\zeta_\chi((X, f), t)}$$

and an increment $h_{n,\underline{x}}(\underline{v})$ with $\underline{v} \in T_{\underline{x}}S^n(X)$ given by $h_{n,\underline{x}}(\underline{v}) = h_{n,\underline{\varphi}}$ as in Lemma 6.5.1, with $\underline{\varphi} \in S^n(\mathcal{L}_1(X))$ determined by \underline{x} and \underline{v} . One obtains a Fisher–Rao tensor $g = (g_{v,w})$ as in Lemma 6.4.1 given by

$$g_{v,w} = \frac{\zeta_\chi((X, f), t)}{2} \sum_{n,\underline{x}} \chi(f^{(n)}(\underline{x})) \chi'(df_{\underline{x}}^{(n)}(\underline{v})) \chi'(df_{\underline{x}}^{(n)}(\underline{w})) t^n.$$

Proof. As in Lemma 6.6.1 above, the leading order term of $KL(\rho + h || \rho)$, which defines the Fisher–Rao tensor, is given by $\frac{1}{2}Tr(h\rho^{-1}h)$. This gives the Fisher–Rao tensor

$$g_{v,w} := \frac{1}{2} \sum_{n,\underline{x}} P_{n,\underline{x}}^{-1} h_{n,\underline{\varphi}_v} h_{n,\underline{\varphi}_w},$$

where we write φ_v, φ_w for the elements in $\mathcal{L}_1(X)$ corresponding to infinitesimal arcs at the point x with tangent directions $v, w \in T_x(X)$, respectively. More explicitly, on a $X_i \subset X$ that trivializes the tangent bundle,

$$\begin{aligned} g_{v,w} &= \frac{\zeta_\chi((X, f), t)}{2} \sum_{n,\underline{x}} \chi^{-1}(f^{(n)}(\underline{x})) t^{-n} \chi^2(f^{(n)}(\underline{x})) t^{2n} \chi'(df_{\underline{x}}^{(n)}(v)) \chi'(df_{\underline{x}}^{(n)}(w)) \\ &= \frac{\zeta_\chi((X, f), t)}{2} \sum_{n,\underline{x}} \chi(f^{(n)}(\underline{x})) \chi'(df_{\underline{x}}^{(n)}(v)) \chi'(df_{\underline{x}}^{(n)}(w)) t^n, \end{aligned}$$

as stated. ■

The Fisher–Rao tensor obtained in this way is not a Riemannian metric in the usual sense, since it is complex valued and does not define a positive definite quadratic form. However, it retains some of the significance of the information metric, for instance in the sense that we can interpret distributions with the same Fisher–Rao tensor in terms of sufficient statistics.

6.8. Amari-Chentsov tensor In information theory it is customary to use methods from Riemannian geometry, by expressing the distance between two probability distributions on a sample space in terms of the Fisher–Rao information metric ([AmNag07]).

6.8.1. Definition. *A statistical manifold is a datum (M, g, A) of a manifold together with a metric tensor and a totally symmetric 3-tensor A , the Amari-Chentsov tensor,*

$$A_{abc} = A(\partial_a, \partial_b, \partial_c) = \langle \nabla_a \partial_b - \nabla_a^* \partial_b, \partial_c \rangle.$$

The following classes of statistical manifolds are especially interesting ([AmCi10], [Ni20]).

6.8.2. Definition. *A divergence function on a manifold M is a differentiable, non-negative real valued function $D(x||y)$, for $x, y \in M$, that vanishes only when $x = y$ and such that the Hessian in the x -coordinates evaluated at $y = x$ is positive definite.*

A divergence function determines a statistical manifold structure by setting

$$g_{ab} = \partial_{x_a} \partial_{x_b} D(x||y)|_{y=x}$$

$$A_{abc} = (\partial_{x_a} \partial_{x_b} \partial_{y_c} - \partial_{x_c} \partial_{y_a} \partial_{y_b}) D(x||y)|_{y=x}.$$

The Amari-Chentsov tensor A_{abc} obtained in this way vanishes identically if the divergence $D(x||y)$ is symmetric.

6.8.3. Definition. *A statistical manifold is induced by a Bregman generator, if it is determined by a divergence function and there is a potential Φ such that the divergence function is a Bregman divergence, namely it satisfies locally*

$$D(x||y) = \Phi(x) - \Phi(y) - \langle \nabla \Phi(y), x - y \rangle.$$

In particular, one can consider the statistical manifold structure induced by the Shannon entropy on the space of probability distributions on a set, with the Hessian of the Kullback–Leibler divergence given by the Fisher-Rao metric,

$$g_{ab} = \sum_n P_n \partial_a \log P_n \partial_b \log P_n = \sum_n \frac{\partial_a P_n \partial_b P_n}{P_n}$$

$$= - \sum_n P_n \partial_a \partial_b \log P_n = \partial_a \partial_b KL(P||Q)|_{P=Q},$$

and the Amari-Chentsov 3-tensor given by

$$\begin{aligned} A_{abc} &= \sum_n P_n \partial_a \log P_n \partial_b \log P_n \partial_c \log P_n = \sum_n \frac{\partial_a P_n \partial_b P_n \partial_c P_n}{P_n^2} \\ &= (\partial_a \partial_b \partial_{c'} - \partial_c \partial_{a'} \partial_{b'}) KL(P||Q)|_{P=Q}, \end{aligned}$$

where we write a, b, c for the variation indices for P and a', b', c' for Q . This determines an associated flat connection. This connection and the one obtained from it by Legendre transform of the potential determine the mixing and exponential geodesics in information geometry, see Chapter 2 of [AmNag07].

6.8.4. Lemma. *The statistical manifold structure (M, g, A) associated to the Fisher–Rao metric and Amari–Chentsov tensor obtained from the Kullback–Leibler divergence is induced by a Bregman generator.*

Proof. One needs to check that the Kullback–Leibler divergence is a Bregman divergence,

$$KL(P||Q) = \Phi(P) - \Phi(Q) - \langle \nabla \Phi(Q), P - Q \rangle.$$

This is the case for the potential $\Phi(P) = -H(P) = \sum_n P_n \log P_n$, the negative of the Shannon entropy. Indeed, we have $\nabla H(Q) = (1 + \log Q_n)_n$ and

$$\begin{aligned} \Phi(P) - \Phi(Q) - \langle \nabla \Phi(Q), P - Q \rangle &= \sum_n P_n \log P_n - \sum_n Q_n \log Q_n - \sum_n (1 + \log Q_n)(P_n - Q_n) \\ &= \sum_n P_n \log P_n - \sum_n P_n \log Q_n = KL(P||Q). \end{aligned}$$

■

6.8.5. Lemma. *The Fisher–Rao metric is related to the Hessian of the Bregman potential by*

$$\partial_a \partial_b \Phi = g_{ab} + \langle \nabla \Phi, \partial_a \partial_b \Phi \rangle.$$

The Amari–Chentsov tensor is related to the symmetric tensor of the third derivatives of the Bregman potential by

$$\partial_a \partial_b \partial_c \Phi(P) = -A_{abc} + \sum_n (\partial_a \partial_b \partial_c P_n) \log P_n$$

$$+ \sum_n \frac{\partial_a \partial_b P_n \partial_c P_n + \partial_b \partial_c P_n \partial_a P_n + \partial_a \partial_c P_n \partial_b P_n}{P_n}.$$

Proof. One has

$$\partial_a \partial_b \Phi(P) = \sum_n (\partial_a \partial_b P_n) \log P_n + \sum_n \frac{\partial_a P_n \partial_b P_n}{P_n} = \langle \nabla \Phi, \partial_a \partial_b \Phi \rangle + g_{ab},$$

where we used the fact that $\sum_n \partial_a P_n = \sum_n \partial_a \partial_b P_n = 0$. The identity for the Amari–Chentsov tensor follows immediately by applying ∂_c to the previous identity. ■

In the case of the Kullback–Leibler divergence for zeta functions on the Grothendieck ring with exponentials introduced in the previous subsections, we do not impose the positivity of the divergence and the positive definiteness of the Hessian, since these take complex values, but we still have an associated Fisher–Rao 2–tensor (Proposition 6.7.2) and an Amari–Chentsov 3–tensor given as follows.

6.8.6. Lemma. *The Amari–Chentsov 3–tensor associated to the distribution $P_{n,\underline{x}}$ and the increments $h_{n,\underline{x}}(v)$ as in Proposition 6.7.2, is given by*

$$A_{u,v,w} = \zeta((X, f), t) \sum_{n,\underline{x}} \chi(f^{(n)}(\underline{x})) \chi'(df_{\underline{x}}^{(n)}(u)) \chi'(df_{\underline{x}}^{(n)}(v)) \chi'(df_{\underline{x}}^{(n)}(w)) t^n.$$

Proof. Computing as in Proposition 6.7.2 we obtain

$$\begin{aligned} A_{u,v,w} &= \sum_{n,\underline{x}} P_{n,\underline{x}}^{-2} h_{n,\underline{x}}(u) h_{n,\underline{x}}(v) h_{n,\underline{x}}(w) \\ &= \zeta((X, f), t)^2 \sum_{n,\underline{x}} \chi^{-2}(f^{(n)}(\underline{x})) t^{-2n} \chi^3(g^{(n)}(\underline{x})) t^{3n} \chi'(df_{\underline{x}}^{(n)}(u)) \chi'(df_{\underline{x}}^{(n)}(v)) \chi'(df_{\underline{x}}^{(n)}(w)). \end{aligned}$$

■

6.9. Amari–Chentsov tensor and Frobenius manifolds. As we have recalled in the previous sections, a *Frobenius manifold* is a datum (M, g, Φ) of a manifold M with a flat metric g with local flat coordinates $\{x^a\}$, and a potential Φ such that the tensor $A_{abc} = \partial_a \partial_b \partial_c \Phi$ defines an *associative* multiplication

$$\partial_a \circ \partial_b = \sum_c A_{ab}{}^c \partial_c,$$

with the raising and lowering of indices done through the metric tensor, $A_{ab}{}^c = \sum_e A_{abe} e^{ec}$. The associativity condition for the multiplication is expressed as the WDVV nonlinear differential equations in the potential Φ . In terms of the 3-tensor A_{abc} the associativity is simply expressed as the identity

$$A_{bce} g^{ef} A_{fad} = A_{bae} g^{ef} A_{fcd},$$

where g^{ab} is the inverse of the metric tensor. One usually assumes that the manifold M is complex, but we will not necessarily require it here.

The *first structure connection* on a Frobenius manifold is given in the flat coordinate system by

$$\nabla_{\lambda, \partial_a} \partial_b = \lambda \sum_c A_{ab}{}^c \partial_c = \lambda \partial_a \circ \partial_b,$$

for λ a complex parameter. The associativity of the product and the existence of a potential function are equivalent to the connection ∇_λ being flat.

Given a statistical manifold (M, g, A) , as in the previous subsection, with g_{ab} the matrix of the Fisher-Rao metric tensor, with inverse g^{ab} given by the covariance matrix, and A_{abc} the Amari–Chentsov tensor. It is natural to ask whether this structure also gives rise to a Frobenius manifold structure, namely whether the Amari–Chentsov tensor defines an associative multiplication in the tangent bundle of M . This is equivalent to the condition that Amari–Chentsov tensor satisfies the associativity identity

$$A_{bce} g^{ef} A_{fad} = A_{bae} g^{ef} A_{fcd}.$$

In the case of a 3-tensor $A_{abc} = \partial_a \partial_b \partial_c \Phi$, for a potential Φ , this equation becomes the WDVV equation for Φ of Frobenius manifold theory. However, in the case of statistical manifolds, usually the Amari–Chentsov tensor differs from the tensor of third derivatives of the potential as discussed above. The condition that replaces the WDVV equation is then of the following form.

6.9.1. Proposition. *Let (M, g, A) be a statistical manifold that induced by a Bregman generator, as in Definition 6.8.3. Then the Amari–Chentsov tensor defines an associative multiplication on the tangent space TM ,*

$$\partial_a \circ \partial_b = \sum_c A_{ab}{}^c \partial_c,$$

iff the Bregman potential Φ satisfies the identity

$$\begin{aligned} & \langle \partial_e \nabla \Phi(P), \partial_a \partial_b P \rangle g^{ef} \langle \partial_f \nabla \Phi(P), \partial_c \partial_d P \rangle + \langle \partial_e \nabla \Phi(P), \partial_a \partial_b P \rangle g^{ef} \langle \partial_c \partial_d \nabla \Phi(P), \partial_f P \rangle + \\ & \langle \partial_a \partial_b \nabla \Phi(P), \partial_e P \rangle g^{ef} \langle \partial_f \nabla \Phi(P), \partial_c \partial_d P \rangle + \langle \partial_a \partial_b \nabla \Phi(P), \partial_e P \rangle g^{ef} \langle \partial_c \partial_d \nabla \Phi(P), \partial_f P \rangle = \\ & \langle \partial_e \nabla \Phi(P), \partial_a \partial_c P \rangle g^{ef} \langle \partial_f \nabla \Phi(P), \partial_b \partial_d P \rangle + \langle \partial_e \nabla \Phi(P), \partial_a \partial_c P \rangle g^{ef} \langle \partial_b \partial_d \nabla \Phi(P), \partial_f P \rangle + \\ & \langle \partial_a \partial_c \nabla \Phi(P), \partial_e P \rangle g^{ef} \langle \partial_f \nabla \Phi(P), \partial_b \partial_d P \rangle + \langle \partial_a \partial_c \nabla \Phi(P), \partial_e P \rangle g^{ef} \langle \partial_b \partial_d \nabla \Phi(P), \partial_f P \rangle. \end{aligned}$$

Proof. This follows by a direct computation from the dependence of the divergence function on the Bregman potential,

$$D(P||Q) = \Phi(P) - \Phi(Q) - \langle \nabla \Phi(Q), Q - P \rangle,$$

and the expression of the Amari-Chenstov tensor as a function of the divergence function

$$A_{abc} = (\partial_a \partial_b \partial_{c'} - \partial_c \partial_{a'} \partial_{b'}) D(P||Q)|_{P=Q},$$

which gives

$$A_{abc} = \langle \partial_c \nabla \Phi(P), \partial_a \partial_b P \rangle + \langle \partial_a \partial_b \nabla \Phi(P), \partial_c P \rangle.$$

Replacing these expressions in the associativity identity gives the condition stated above. ■

We also obtain the following special case from the previous proposition together with Lemma 6.8.5.

6.9.2. Corollary. *If the dependence of P on the deformation parameters is linear, so that $\partial_a \partial_b P = 0$, then the associativity condition reduces to*

$$\langle \partial_a \partial_b \nabla \Phi(P), \partial_e P \rangle g^{ef} \langle \partial_c \partial_d \nabla \Phi(P), \partial_f P \rangle = \langle \partial_a \partial_c \nabla \Phi(P), \partial_e P \rangle g^{ef} \langle \partial_b \partial_d \nabla \Phi(P), \partial_f P \rangle.$$

In this case the Amari-Chenstov tensor is the tensor of third derivatives of the Bregman potential, so the equation above is an equivalent formulation of the WDVV equation for the potential.

References

[AmCi10] S. Amari, A. Chichoki, *Information Geometry derived of divergence functions*, Bull. Polish Acad. Sci. Tech. Ser., Vol.58 (2010), No. 1, 183–195.

- [AmNag07] S. Amari, H. Nagaoka, *Methods of Information Geometry*, American Mathematical Society, 2007.
- [Ber02] W. Bertram, *Generalized projective geometries: general theory and equivalence with Jordan structures*, *Adv. Geom.* 2 (2002), 329–369.
- [Ber03] W. Bertram, *The geometry of null systems, Jordan algebras and von Staudt’s Theorem*, *Ann. Inst. Fourier* 53:1 (2003), 1, 193–225.
- [Bilu18] M. Bilu, *Motivic Euler products and motivic height zeta functions*, arXiv:1802.06836.
- [BoCo95] J.B. Bost, A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, *Selecta Math. (N.S.)* 1 (1995), no. 3, 411–457.
- [BrK66] H. Braun, M. Koecher, *Jordan-Algebren*. Die Grundlehren der math. Wiss., Bd. 128, Springer-Verlag, Berlin-Heidelberg-New York (1966), xvi+357 pp.
- [CCM07] A. Connes, C. Consani, M. Marcolli, *Noncommutative geometry and motives: the thermodynamics of endomotives*. *Adv. Math.* 214 (2007), no. 2, 761–831
- [ChamLoe15] A. Chambert-Loir, F. Loeser, *Motivic height zeta function*, *Amer. J. Math.* 138 (2016), no. 1, 1–59. arXiv:1302.2077v4.
- [CluLoe10] R. Cluckers, F. Loeser, *Constructible exponential functions, motivic Fourier transform and transfer principle*, *Ann. of Math. (2)* 171 (2010), no. 2, 1011–1065.
- [CoMa20] N. C. Combe, Yu. Manin, *F-manifolds and geometry of information*. *Bull. LMS*, 5:2 (2020) . arXiv:2004.08808v2. 21 pp.
- [Den91] C. Deninger, *On the Γ -factors attached to motives*, *Invent. Math.* 104 (1991), no. 2, 245–261.
- [DenLoe01] J. Denef, F. Loeser, *Geometry on arc spaces of algebraic varieties*, in “European Congress of Mathematics, Vol. I (Barcelona, 2000)”, 327–348, *Progr. Math.*, Vol. 201, Birkhäuser, Basel, 2001.
- [Howe19] S. Howe, *Motivic random variables and representation stability II: Hypersurface sections*, *Adv. Math.* 350 (2019), 1267–1313.
- [Howe20] S. Howe, *Motivic random variables and representation stability, I: Configuration spaces*, *Algebr. Geom. Topol.* 20 (2020), no. 6, 3013–3045.
- [HruKaz09] E. Hrushovski, D. Kazhdan, *Motivic Poisson summation*. arXiv:0902.0845.

- [Ju90] B. Julia, *Statistical Theory of Numbers*. Springer Proceedings in Physics, Vol. 47 (1990), pp. 276–293.
- [Kap00] M. Kapranov, *The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups*, arXiv:math/0001005.
- [KS06] M. Kashivara, P. Schapira, *Categories and Sheaves*. Springer–Verlag Berlin Heidelberg (2006), X+497 pp.
- [Lo74] O. Loos. *A structure theory of Jordan pairs*. Bull. AMS, 80 (1974), pp. 65–71.
- [Ma95] Yu.I. Manin, *Lectures on zeta functions and motives (according to Deninger and Kurokawa)*. Columbia University Number Theory Seminar (New York, 1992). Astérisque No. 228 (1995), 4, 121–163.
- [Ma99] Yu. Manin, *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*. AMS Colloquium Publication, vol. 47 (1989), xiii+303 pp.
- [Ma17] Yu. Manin, *Grothendieck–Verdier duality patterns in quantum algebra*. Izvestiya: Mathematics, 81:4 (2017). arXiv:1701.01261. 14 pp.
- [Ma88] Yu. Manin, *Quantum groups and non-commutative geometry*. CRM, Montréal, (88. 91 pp.
- [MaMar21] Yu.I. Manin, M. Marcolli, *Homotopy Spectra and Diophantine Equations*, arXiv:2101.00197.
- [Mar19] M. Marcolli, *Gamma spaces and information*. Journ. of Geom. and Physics, 140 (2019), pp. 26–55.
- [Mar19b] M. Marcolli, *Motivic information*, Boll. Unione Mat. Ital. 12 (2019), no. 1-2, 19–41.
- [MaVa20] Yu. Manin, B. Vallette, *Monoidal structures on the categories of quadratic data*. Documenta Mathematica, 25 (2020), pp. 77–102. arXiv:1902.0378.
- [MoChen76] E. A. Morozova, N. N. Chentsov, *Structures of family of quantum Markov Chain stationary states*. Preprint IPM, Moscow, **130** (1976).
- [MoChen89] E. A. Morozova, N. N. Chentsov, *Markov invariant geometry on state manifolds*. Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Nov. Dostizh., VINITI, Moscow, 36 1(989), 69–102; J. Soviet Math., 56:5 (1991), 2648–2669.
- [MoChen92] E. A. Morozova, N. N. Chentsov, *Projective Euclidean geometry and noncommutative probability theory*. Discrete geometry and topology, dedicated to the 100th anniversary of the birth of Boris Nikolaevich Delone, Trudy Mat. Inst.

Steklov., Nauka, Moscow, 196 (1991), 105–113; Proc. Steklov Inst. Math., 196 (1992), 117–127

[Ni20] F. Nielsen, *An Elementary Introduction to Information Geometry*, Entropy, 2020, 22, 1100, 61 pages

[Ram15] N. Ramachandran, *Zeta functions, Grothendieck groups, and the Witt ring*, Bull. Sci. Math. Soc. Math. Fr. 139 (2015) N.6, 599–627. arXiv:1407.1813.

[RamTab15] N. Ramachandran, G. Tabuada, *Exponentiable motivic measures*. J. Ramanujan Math. Soc. 30 (2015), no. 4, 349–360. arXiv:1412.1795.

[Se70] J.-P. Serre, *Facteurs locaux des fonctions zêta des variétés algébriques*. Séminaire Delange–Pisot–Poitou. Théorie des nombres, tome 11, no. 2 (1969–1970), exp. no 19, pp. 1–15.

[Sh16] Yunyi Shen, *The GL_n -Connes–Marcolli Systems*. arXiv:1609.08727.

[TruBie82] P. Truini, L. C. Biedenharn, *An $\mathcal{E}_6 \otimes U(1)$ invariant quantum mechanics for a Jordan pair*. J. Math. Phys. **23:7** (1982), 1327–1345.

[Var68] V. S. Varadarajan, *Geometry of Quantum Theory*. Springer (1968) ISBN 978-0-387-49385-5

[Vi63] E. B. Vinberg, *The theory of homogeneous convex cones*. Trans. Mosc. Math. Soc. 12 (1963), pp. 340–403.

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