1

$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$

Schwartz rapid decay functions | L^2 (Lebesgue) | tempered distributions

each dense in the following

can interpret $\varphi \in \mathcal{S}(\mathbb{R})$ as a funct in L^2

and pairing

$\langle \varphi, \psi \rangle$ in L^2-inner prod

$$\langle \varphi, \psi \rangle = \int_{\mathbb{R}} \bar{\varphi}(x) \psi(x)\, dx$$

or can interpret ψ as distribution in $\mathcal{S}'(\mathbb{R})$

and interpret

$\langle \varphi, \psi \rangle$ as pairing of distribution ψ acting on test function $\bar{\varphi}$

$$\langle \varphi, \psi \rangle = \Lambda_{\psi}(\bar{\varphi}) = \int_{\mathbb{R}} \psi(x)\, \bar{\varphi}(x)\, dx$$

<u>for $\varphi, \psi \in L^2(\mathbb{R})$</u> these two readings of $\langle \varphi, \psi \rangle$ same

but this continues to make sense when one <u>of them is in larger space $\mathcal{S}'(\mathbb{R})$</u> provided <u>other one is in smaller $\mathcal{S}(\mathbb{R})$</u>

2

Suppose given a linear (unbounded) operator on $\mathcal{H} = L^2(\mathbb{R})$ densely defined with

$$\mathrm{Dom}(T) \supset \mathcal{S}(\mathbb{R})$$

adjoint T^* on $\mathcal{H}$ satisfying

$$\langle T^*\psi, \varphi \rangle = \langle \psi, T\varphi \rangle \quad \text{for } \varphi \in \mathrm{Dom}(T)$$

$$\mathrm{Dom}(T^*) = \{ \psi \in \mathcal{H} : T^*\psi \in \mathcal{H} \}$$

in general if define $T^*\psi$ by

$$\langle T^*\psi, \varphi \rangle = \langle \psi, T\varphi \rangle \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R})$$

and interpret pairing between $\mathcal{S}(\mathbb{R})$ & $\mathcal{S}'(\mathbb{R})$
instead of " " $L^2(\mathbb{R})$ & $L^2(\mathbb{R})$

then $T^*\psi$ makes sense on any
$\psi \in L^2(\mathbb{R})$ (not just in $\mathrm{Dom}(T^*)$)
with $T^*\psi \in \mathcal{S}'(\mathbb{R})$

the distribution defined by

$$\Lambda_{T^*\psi}(\varphi) := \langle \psi, T\varphi \rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R})$$

3 Spectral theorem for self-adjoint (unbounded) operator on a Hilbert space

$$A = \int_{\mathbb{R}} \lambda \, dE(\lambda)$$

spectral projections E_λ

for <u>discrete spectrum</u> reduces to expansion in eigenfunctions

o.n. system

$\psi_k \qquad A\psi_k = \lambda_k \psi_k$

for <u>continuum spectrum</u> eigenvalues/eigenf. don't make sense

but can sometimes still express the spectral theorem as an expansion in "generalized eigenfunctions"

that are <u>weak solutions</u> of the eigenfunction/eigenvalue equation

and make sense as distributions

4

Schrödinger equation on a line

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + U(x)\psi$$

$\psi \in \mathcal{H} = L^2(\mathbb{R})$ $\qquad H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)$

case of free particle $U(x) \equiv 0$

$H\psi = \lambda\psi$ $\qquad$ equation of stationary state (eigenvalue equation

up to constant $\frac{\hbar^2}{2m}$ eigenv. of 1D Laplacian

$$\psi_\pm = A_\pm \exp\left(\frac{\pm i}{\hbar}\sqrt{2m\lambda}\, x\right)$$

if $\lambda < 0$ exp growth at $+\infty$ or $-\infty$
(so not even in $\mathcal{S}'(\mathbb{R})$)

for $\lambda > 0$ $\qquad p := \pm\sqrt{2m\lambda}$

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar} p x\right) \qquad \begin{cases} p \in \mathbb{R} \\ \lambda = \frac{p^2}{2m} \end{cases}$$

$\psi_p \notin \mathcal{S}(\mathbb{R})$
$\psi_p \notin L^2(\mathbb{R})$ but $\underline{\psi_p \in \mathcal{S}'(\mathbb{R})}$ $\qquad$ <u>weak sol'n of the stationary eq.</u>

5

what about orthogonality ?

cannot pair elements in $\mathcal{S}'(\mathbb{R})$

but can pair $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$

so instead of $\langle \psi_{p'}, \psi_p \rangle$ consider

$\psi_{p,l} \in \mathcal{S}(\mathbb{R})$ $\psi_{p,l} \underset{l\to\infty}{\longrightarrow} \psi_p$ (density of $\mathcal{S}(\mathbb{R})$)

or $\psi_{p,l} \in \mathcal{H}$ s.t. pairing still possible

e.g. $\psi_{p,l}$ w/ compact support

$$\lim_{l\to\infty} \langle \psi_{p'}, \chi_{[-l,l]}, \psi_p \rangle = \frac{1}{2\pi\hbar} \lim_{l\to\infty} \int_{-l}^{l} e^{-\frac{i}{\hbar}p'x} e^{\frac{i}{\hbar}px} dx$$

$$= \lim_{l\to\infty} \frac{1}{\pi} \frac{\sin\left(\frac{p-p'}{\hbar} l\right)}{p-p'} = \delta(p-p')$$

limit in $\mathcal{S}'(\mathbb{R})$

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \qquad i\hbar \frac{\partial c(p,t)}{\partial t} = c(p,t)\frac{p^2}{2m}$$

$$\psi = \int_{\mathbb{R}} c(p)\, \psi_p \, dp \qquad i\hbar \frac{\partial \psi}{\partial t} = \int_{\mathbb{R}} i\hbar \frac{\partial c}{\partial t} \psi_p \, dp$$

$$H\psi = \int_{\mathbb{R}} c(p) \frac{p^2}{2m} \psi_p \, dp$$

$\boxed{6}$

$$c(p,t) = \exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}t\right) c(p)$$

with $c(p)$ determined by the initial condition

$$\psi_0(x) = \psi(x,t)\Big|_{t=t_0}$$

if in $\mathcal{S}(\mathbb{R})$

$$c(p) = \langle \psi_0, \psi_p \rangle$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} c(p) \exp\left(\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)\right) dp$$
