An Introduction to Fuzzy Propositional Calculus Using Proofs from Assumptions

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Abstract. The subject of this paper is fuzzy propositional calculus. The proposed approach is related to the basic fuzzy propositional logics, i.e. to each of the following three most important such systems (in short: BL): Łukasiewicz's, Gödel's, and product logic. The logical calculi considered here are based on a system of rules that define the methods used in proofs from assumptions.. To simplify the considered proofs some set of laws called also 'primitive rules' is next introduced. It was shown that any fuzzy propositional formula provable under Hájek's axioms of the logic BL is also provable under the above-proposed approach.

1 Introduction

In general, the following two main directions in fuzzy logic can be distinguished: fuzzy logic in the broad sense and fuzzy logic in the narrow sense, i.e. in senso stricto. The *broad sense fuzzy logic* has become a common buzzword in machine control. The *basic strict fuzzy logic* (in short: BL) is a strict fuzzy logic system using the logic of continuous t-norms. This system was developed in [5]. The most of the studies in this area focus attention on methodological problems, e.g: compactness, consistency, decidability or satisfiability of t-tautologies [1,5,7,11], on various proving techniques [2,8] or also introducing some new t-norms [11]. A survey of different such systems was given in [9].

The subject of this paper is fuzzy propositional calculus. Without loss of generality, the proposed approach is related to the basic fuzzy propositional logics, i.e. to each of the following three most important such systems: Łukasiewicz's, Gödel's, and product logic (in short: Ł-, G-, and π-BL systems). There exist two classical approaches in constructing of the propositional calculus: the *axiomatic approach* and the *approach from assumptions*. In general, the actual research methodology and extensions have been related to the Hájek's axiomatic approach. A new system based on assumptions is presented below. An inductive definition of the notion of a fuzzy propositional formula is first introduced. The proposed method is next presented. To simplify the considered proofs, as in the classical approach, some set of laws called also 'primitive rules' is next introduced. And finally, it was shown that any fuzzy propositional formula provable under Hájek's axioms of the logic BL is also provable under the above-proposed approach.

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2 Basic Notions

Let $\otimes$ be a binary operation over $[0,1]$ which is commutative, associative, monotonic, and has 1 as unit element. Any such operation is called to be a *t-norm.* The t-norm operator provides the characterisation of the AND operator. The dual *t-conorm* $\oplus$ (called also: *s-norm*), characterising the OR operator, is defined in a similar way having 0 as unit element (a more formal treatment is omitted, see : [12]).

The following t-norms are assumed below (for any $a,b \in [0,1]$):

$a \,\&\, b =_{df} \max\{0, a + b - 1\}$	the Łukasiewicz's strong conjunction,
$a \wedge b =_{df} \min\{a,b\}$	the logical operation minimum, and
$a \cdot b =_{df} ab$	the usual arithmetic product

with the corresponding dual t-conorms:

$a \underline{\vee} b =_{df} \min\{1, a + b\}$	the Łukasiewicz's strong disjunction,
$a \vee b =_{df} \max\{a,b\}$	the logical operation maximum, and
$a \overline{\vee} b =_{df} a + b - ab$	the algebraic sum.

Next we shall say that $\otimes$ and $\oplus$ are *t- conjunction* and *t- disjunction*, respectively. The *fuzzy implication* connective is sometimes disregarded but is of fundamental importance for fuzzy logic in the narrow sense. The *implication* and *negation connectives* are assumed under [3,4,5,6].

The used names for the primitive and/or derived rules given below are in accordance with the Łukasiewicz´s symbols of negation, conjunction, disjunction, implication, and equivalence denoted by N, K, A, C, and E, respectively. The following (generalised *primitive) rules* are considered below: – C (*rule of detachment for implication* or *omitting an implication*), ± K (*rules of joining/ omitting a t-conjunction*), ± A (*rules of joining/omitting a t-disjunction*), and ± E (*rules of joining/omitting an equivalence*). The *rule of substitution for equivalence* is denoted by SR. Some additional rules are also used, such as: ± N (rules of *joining / omitting double negation*), CR (*implication rule*), CC (the *law of transposition or contraposition of implication*), NA (rule of *negating a t-disjunction*), NK (rule of *negating a t-conjunction*), NC (rule of *negating an implication*), Toll (*rule modus tollendo tollens*), TC (the law of *transitivity for implication*), MC (the law of *multiplication of consequents* of two or more implications having the same antecedents). Some additional inference rules of the first-order predicate logic calculus are also used below, such as: the rule of *negating an universal quantifier* (denoted by N∀), the rules of *omitting an universal* and *an existential quantifiers* (denoted by: $-\forall$ and $-\exists$, respectively). The following abbreviations are also introduced below: a, aip, ada, and contr., denoting: *assumption*(*s*), *assumption*(*s*) *of indirect proof*, *additional assumption of a proof*, and *contradiction,* respectively.

2 The Approach from Assumptions

In the fuzzy propositional calculus any formula is constructed by using the following three kinds of symbols: (i) propositional variables (denoted by $p, q, r, s, \ldots, p_1, p_2, \ldots$),

(ii) some fuzzy connectives (depending on the used system), e.g. such as: the Łukasiewicz's (strong) fuzzy conjunction, disjunction, implication, logical equivalence, and negation, denoted by: $\&$, $\underline{\vee}$, $\Rightarrow$, $\Leftrightarrow$, and $\sim$, respectively or the Gödel's (weak) fuzzy conjunction, disjunction, implication, logical equivalence, and negation, denoted by: $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, and $\sim$, respectively or also the product logic's fuzzy conjunction, disjunction, implication, logical equivalence, and negation, denoted by: $\cdot$, $\overline{\vee}$, $\Rightarrow$, $\Leftrightarrow$, and $\sim$, respectively and also (iii) parentheses (left: '(' and right: ')'). The following two truth and falsity constants are used below: $\underline{1}$ and $\underline{0}$ (denoted also by $\top$ and $\bot$ respectively, i.e. the truth degree 1 and 0, corresponding to 'T' and 'F' in the classical case). To minimise the number of used parentheses in an expression, some priorities for logical connectives can be introduced. The following convention is assumed below [13]: $\sim$, $\otimes$, $\oplus$, $\Rightarrow$, $\Leftrightarrow$ (i.e. the symbol of negation binds more strongly than the symbol of t-conjunction, the last binds more strongly than the symbol of t-disjunction, etc.), where $\otimes \in \{\&, \wedge, \cdot\}$ and $\oplus \in \{\underline{\vee}, \vee, \overline{\vee}\}$ are depending on the used system (Ł-, G- or π-BL). The set of *fuzzy propositional formulae* (called equivalently *fuzzy propositional expressions*, in short: *expressions* or also *sentential formulae*) of this propositional calculus can be considered as the smallest set which includes propositional variables, and which is closed under the operations of forming the negation, conjunction, disjunction, implication and equivalence. Hence, any propositional variable can be considered as an expression and also the compound formulae formed from them by means of the corresponding logical functors. More formally, the following inductive definition can be used (a generalisation of the classical one [13]).

Definition 1

A *fuzzy propositional formula* is:

1. Any propositional variable,
2. If φ and ψ are some fuzzy propositional formulae, then such formulae are also: $\sim(\varphi)$, $(\varphi)\otimes(\psi)$, $(\varphi)\oplus(\psi)$, $(\varphi)\Rightarrow(\psi)$, and $(\varphi)\Leftrightarrow(\psi)$,
3. Every fuzzy propositional formula in this propositional calculus either is a propositional variable or is formed from propositional variables by a single or multiple application of rule (2). And this should be in accordance with the used definitions of fuzzy connectives, depending on the considered system, where $\otimes \in \{\&, \wedge, \cdot\}$ and $\oplus \in \{\underline{\vee}, \vee, \overline{\vee}\}$.

Any evaluation of fuzzy propositional variables can be considered as a mapping v assigning to each fuzzy propositional variable p its truth-value in [0,1]. This extends to each fuzzy propositional formula φ as an evaluation of propositional variables in φ by truth degrees in [0,1] [5,6]. Below by $v_{\otimes}(\varphi) \in [0,1]$ (in short: $\varphi \in [0,1]$, e.g. $\varphi = a \in [0,1]$) we shall denote the *logical value of the fuzzy propositional formula* φ with respect to $\otimes$ (in short: wrt $\otimes$). In a similar way, e.g. by $\varphi \le \psi$ we shall denote: $\varphi = a$, $\psi = b$, and $a \le b$ $(a,b \in [0,1])$.

Definition 2 [4]

Let $\otimes$ be a given continuous t-norm and $v_{\otimes}(\varphi) \in [0,1]$ be the *logical value of the fuzzy propositional formula* φ *wrt* $\otimes$. So, we shall say φ is *t-tautology*, *t-thesis* or also *standard BL-tautology* of that calculus if $v_{\otimes}(\varphi) = 1$ for each evaluation of propositional variables in φ by truth degrees in $[0,1]$ and each continuous t-norm.

Let φ be a fuzzy propositional formula obtained under Definition 1. Hence, as in the classical case, the main task is to verify if φ is a t-thesis. Unfortunately, the usefulness of Definition 2 seems to be limited considering arbitrary t-norms. Next we shall assume only t-norms related to the basic fuzzy propositional logics. Any such t-thesis is said to be a *strong t-thesis* (or equivalently: *strong t-tautology*, *strong standard BL-tautology*). The last definition can be modified assuming "t-norm dependence", i.e. the following definition can be introduced.

Definition 3

Let $\otimes$ be a given continuous t-norm and $v_{\otimes}(\varphi) \in [0,1]$, or $v(\varphi)$ if $\otimes$ is understood, be the *logical value of the fuzzy propositional formula* φ *wrt* $\otimes$. We shall say φ a *weak t-thesis* if $v_{\otimes}(\varphi) = 1$ for each evaluation of propositional variables in φ by truth degrees in $[0,1]$.

The *proof* in the fuzzy propositional calculus can be interpreted as a process of joining new lines by using some primitive or derived rules and/or other theses in accordance with the used assumptions. The proposed approach is an extension of the classical one [13]. An illustration is given in the next example.

Example 1

Consider the following well-known classical law (of addition an arbitrary proposition to the antecedent and consequent of a given implication):

$$(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$$

This law can be proved both using a direct or also an indirect proof. In general, the indirect proof is a more universal approach, but corresponding to more proof lines than the direct one (if it exists). The following indirect proof can be obtained.

Proof:

(1)	$p \Rightarrow q$	$\{1{,}2 / a\}$
(2)	$p \vee r$	
(3)	$\sim (q \vee r)$	$\{aip\}$
(4)	$\sim q$	$\{4{,}5 / NA: 3\}$
(5)	$\sim r$	
(6)	p	$\{- A : 2{,}5\}$
(7)	q	$\{- C : 1{,}6\}$
	contr. □	$\{4{,}7\}$

Since any $\varphi \in \{a\} \cup \{aip\}$ is assumed to be a true formula (i.e. true in any interpretation), the following proof technique can be equivalently introduced.

(1)	$\forall\, p, q \in \{0,1\}\, (p \Rightarrow q = 1)$	$\{1{,}2 / a\}$
(2)	$\forall\, p, r \in \{0,1\}\, (p \vee r = 1)$	
(3)	$\sim\forall\, q, r \in \{0,1\}\, (q \vee r = 1)$	$\{aip\}$

(4) $\exists\, q, r \in \{0,1\}\,(q \vee r \neq 1)$ {N∀: 3}

(5) $q_0 \vee r_0 = 0$ {–∃: 4}

(6) $(q_0 = 0) \wedge (r_0 = 0)$ {df.'∨': 5}

(7) $q_0 = 0$ {7,8 / – K: 6}

(8) $r_0 = 0$

(9) $p_0 \Rightarrow q_0 = 1$ {–∀: 1}

(10) $p_0 \vee r_0 = 1$ {–∀: 2}

(11) $p_0 \leq q_0$ {df.'⇒': 9}

(12) $(p_0 = 1) \vee (r_0 = 1)$ {df.'∨': 10}

(13) $p_0 = 0$ {7,11}

(14) $r_0 = 1$ {– A: 12,13}

contr. □ {8,14}

The above proof technique can be easily extended to the whole interval [0,1]. Hence, the following implication is satisfied.

Thesis 1(*law of addition an arbitrary fuzzy proposition to the antecedent and consequent of a given implication*)

$$(p \Rightarrow q) \Rightarrow (p \oplus r \Rightarrow q \oplus r)$$

Proof (e.g. Ł-BL: $\oplus =_{df} \veebar$):

(1) $\forall\, p, q \in [0,1]\,(p \Rightarrow q = 1)$ {1,2 / a}

(2) $\forall\, p, r \in [0,1]\,(p \veebar r = 1)$

(3) $\sim\forall\, q, r \in [0,1]\,(q \veebar r = 1)$ {aip}

(4) $\exists\, q, r \in [0,1]\,(q \veebar r \neq 1)$ {N∀: 3}

(5) $q_0 \veebar r_0 \neq 1$ {–∃: 4}

(6) $q_0 + r_0 < 1$ {df.' ⊻': 5}

(7) $p_0 \Rightarrow q_0 = 1$ {–∀: 1}

(8) $p_0 \veebar r_0 = 1$ {–∀: 2}

(9) $p_0 \leq q_0$ {df.'Ł-⇒': 7}

(10) $p_0 + r_0 \geq 1$ {df.' ⊻': 8}

(11) $p_0 + r_0 \leq q_0 + r_0$ {+ r_0: 9}

(12) $q_0 + r_0 \geq 1$ {10,11}

contr. □ {6,12}

In accordance with our considerations, T1 is a strong t-thesis. Also, the following example strong t-theses are satisfied (the corresponding proofs are omitted here).

Thesis 2 (*law of compound constructive dilemma*)

$$(p \Rightarrow q) \otimes (r \Rightarrow s) \otimes (p \oplus r) \Rightarrow q \oplus s. \square$$

Thesis 3 (*law of compound destructive dilemma*)

$(p \Rightarrow q) \otimes (r \Rightarrow s) \otimes \sim (q \oplus s) \Rightarrow \sim (p \oplus r)$. □

Thesis 4 (De Morgan's law of *negating a t-disjunction*)

$\sim (p \oplus q) \Leftrightarrow \sim p \otimes \sim q$. □

Thesis 5 (De Morgan's law of *negating a t-conjunction*)

$\sim (p \otimes q) \Leftrightarrow \sim p \oplus \sim q$. □

Thesis 6 (*rule modus tollendo tollens*)

$(p \Rightarrow q) \otimes \sim q \Rightarrow \sim p$. □

Thesis 7 (*law of transitivity for implication*)

$(p \Rightarrow q) \otimes (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$. □

Thesis 8 (*laws of exportation and importation*)

$p \otimes q \Rightarrow r \Leftrightarrow p \Rightarrow (q \Rightarrow r)$. □

Thesis 9 (*law of reduction ad absurdum*)

$(p \Rightarrow q \otimes \sim q) \Rightarrow \sim p$. □

Thesis 10 (*law of transposition* or *contraposition of implication*)

$p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p$. □

Thesis 11 (*law of the hypothetical, called also conditional, syllogism*)

$(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$. □

Thesis 12 (A4 [4,5,6])

$p \otimes (p \Rightarrow q) \Rightarrow q \otimes (q \Rightarrow p)$. □

Thesis 13 (Hauber's *law of converting implications*)

$(p \Rightarrow q) \otimes (r \Rightarrow s) \otimes (p \oplus r) \otimes \sim (q \otimes s) \Rightarrow (q \Rightarrow p) \otimes (s \Rightarrow r)$. □

Thesis 14

$(p \oplus q \Rightarrow r) \Rightarrow (p \Rightarrow r) \otimes (q \Rightarrow r)$. □

Thesis 15 (*law of multiplication of consequents*)

$(p \Rightarrow q) \otimes (p \Rightarrow r) \Leftrightarrow (p \Rightarrow q \otimes r)$. □

The following example weak t-theses are satisfied (Ł-BL only): $\sim \sim p \Leftrightarrow p$ (the law of double negation, and hence the rules ± N), $p \Rightarrow q \Leftrightarrow \sim p \veebar q$ (the law of implication, i.e. the rule CR), $\sim (p \Rightarrow q) \Leftrightarrow p \,\&\, \sim q$ (the law of negating an implication, i.e. NC), $p \Rightarrow p \wedge p$ (idempotence of t-conjunction: G-BL only, the opposite implication is strong t-thesis related to – K), the axiom [4]: $\sim\sim p \Rightarrow ((p \Rightarrow p \otimes q) \Rightarrow q \otimes \sim\sim q)$ is not satisfied for G-BL, the well-known absorptive and distributive axioms are satisfied only in G-BL, the law of addition of antecedents $(p \Rightarrow r) \otimes (q \Rightarrow r) \Leftrightarrow p \oplus q \Rightarrow r$ is satisfied only in G- and π-BL (strong t-thesis is the opposite implication: see T14), etc. The corresponding proofs are omitted. The following strong t-theses are considered as a generalisation or extension of the classical primitive and derived rules. In general, any strong t-thesis can be considered as a new derived rule.

$$-\mathrm{C}: \ \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \varphi\end{array}}{\psi}, \qquad +\mathrm{K}: \ \frac{\begin{array}{c}\varphi \\ \psi\end{array}}{\varphi \otimes \psi}, \qquad -\mathrm{K}: \ \frac{\varphi \otimes \psi}{\varphi \setminus \psi \setminus \begin{smallmatrix}\varphi \\ \psi\end{smallmatrix}}, \qquad +\mathrm{A}: \ \frac{\varphi}{\varphi \oplus \psi},$$

$$-\mathrm{A}: \ \frac{\begin{array}{c}\varphi \oplus \psi \\ \sim \varphi\end{array}}{\psi}, \qquad +\mathrm{E}: \ \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \psi \Rightarrow \varphi\end{array}}{\varphi \Leftrightarrow \psi}, \qquad -\mathrm{E}: \ \frac{\varphi \Leftrightarrow \psi}{\varphi \Rightarrow \psi \setminus \psi \Rightarrow \varphi \setminus \begin{smallmatrix}\varphi \Rightarrow \psi \\ \psi \Rightarrow \varphi\end{smallmatrix}},$$

$$\mathrm{Toll}: \ \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \sim \psi\end{array}}{\sim \varphi}, \quad \mathrm{CC}: \ \frac{\varphi \Rightarrow \psi}{\sim \psi \Rightarrow \sim \varphi}, \quad \mathrm{NA}: \ \frac{\sim (\varphi \oplus \psi)}{\sim \varphi \setminus \sim \psi \setminus \begin{smallmatrix}\sim \varphi \\ \sim \psi\end{smallmatrix}}, \quad \mathrm{NK}: \ \frac{\sim (\varphi \otimes \psi)}{\sim \varphi \oplus \sim \psi},$$

$$\mathrm{SR}: \ \frac{\varphi \Leftrightarrow \psi}{\chi \Leftrightarrow \chi(\varphi // \psi)}, \quad \mathrm{TC}: \ \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \psi \Rightarrow \chi\end{array}}{\varphi \Rightarrow \chi}, \quad \mathrm{MC}: \ \frac{\begin{array}{c}\varphi \Rightarrow \psi \\ \varphi \Rightarrow \chi\end{array}}{\varphi \Rightarrow \psi \otimes \chi}.$$

Example 2

Consider the proof of T1 under the above-introduced primitive rules. This proof can be realised as follows.

(1)	$p \Rightarrow q$	{1,2 / a}
(2)	$p \veebar r$	
(3)	$\sim (q \veebar r)$	{aip}
(4)	$\sim q$	{4,5 / NA, NK: 3}
(5)	$\sim r$	
(6)	p	{– A : 2,5}
(7)	q	{– C : 1,6}
	contr. □	{4,7}

The following proposition was shown (using the notion of t-consequence).

Proposition 1 (*BL-provability*)

Any t-thesis provable under the Hájek's axiomatic approach of the logic BL is also provable under the above-proposed approach from assumptions. □

4 Conclusions

The above-considered approach in constructing of the fuzzy propositional calculus can be considered as an extension of the classical one. The presented technique of proofs from assumptions can be simplified by using some rules, i.e. a set of strong t-theses, as a generalisation or extension of the classical primitive and derived rules. Since some generalised classical rules are satisfied only as weak t-theses, the general study of consistency and completeness, i.e. the provability by the assumed rules may be a difficult problem (whether every t-thesis provable by the rules of the assumptional system of this calculus is a true formula). On the other hand, the provability properties of the Hájek's axiomatic approach of the logic BL can be preserved by using the classical first-order predicate logic calculus. Finally, the presented approach from assumptions seems to be more attractive, more simpler and natural than the axiomatic one wrt the practical use, e.g. in the case of approximative reasoning or automated theorem proving (using Gentzen's sequents or also Robinson's resolution), etc. And so, essentially most remains to be done there.

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