Sharkovsky's Ordering and Chaos
Introduction to Fractal Geometry and Chaos

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M 1-2 and T 10-12 BA6180

References for this lecture:

- K. Burns and B. Hasselblatt, *The Sharkovsky theorem: a natural direct proof*, Amer. Math. Monthly 118 (2011), no. 3, 229–244
- T.Y.Li and A.Yorke, *Period three implies chaos*, The American Mathematical Monthly, Vol. 82, No. 10. (Dec., 1975) 985–992
- O.M. Sharkovsky, *Co-existence of cycles of a continuous mapping of the line into itself*, Ukrain. Mat. Z. 16 (1964) 61–71
- B. Luque, L. Lacasa, F.J. Ballesteros, A. Robledo, *Feigenbaum Graphs: A Complex Network Perspective of Chaos*, PLoS ONE 6(9): e2241
- L. Alsedà, J. Llibre, M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, World Scientific, 2000

Period 3 implies Chaos

• T.Y.Li and A.Yorke, *Period three implies chaos*, The American Mathematical Monthly, Vol. 82, No. 10. (Dec., 1975) 985–992

- considered the historical origin of Chaos Theory
- if a continuous function $f : \mathcal{I} \to \mathcal{I}$ of an interval $\mathcal{I} \subset \mathbb{R}$ has a periodic point of period 3, then it has periodic points of any order $n \in \mathbb{N}$
- there is also an uncountable subset of $\mathcal{I}$ of points that are not even "asymptotically periodic"
- this is a typical situation of *chaotic dynamics*
- *chaos* = sensitive dependence on the initial conditions (starting points that are very close have very different behavior under iterates of the function)

Sharkovsky's theorem

• in fact the result of Li and Yorke is a special case of a previous much more general theorem of Sharkovsky

• Oleksandr M. Sharkovsky, *Co-existence of cycles of a continuous mapping of the line into itself*, Ukrainian Math. J. 16 (1964) 61–71

- Sharkovsky ordering: there is an ordering of the natural numbers, different from usual ordering such that if a continuous function $f : \mathcal{I} \to \mathcal{I}$ has a periodic point of period n, then it has also periodic points of period k for all natural numbers k that follow n in the Sharkovsky ordering
- the Sharkovsky ordering starts with 3 (hence Li–Yorke follows)

Sharkovsky Ordering

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2{\cdot}3 \rhd 2{\cdot}5 \rhd 2{\cdot}7 \rhd \cdots \rhd 2^2{\cdot}3 \rhd 2^2{\cdot}5 \rhd 2^2{\cdot}7 \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

- first all odd numbers in ascending order
- then all numbers $2 \cdot n$ with n odd in ascending order
- then all $2^2 \cdot n$ with n odd
- etc. for increasing powers $2^N \cdot n$ with ascending odd n
- then all powers of 2 in *descending* order
- last in the ordering is 1

$$\text{odds}, 2 \cdot \text{odds}, 2^2 \cdot \text{odds}, 2^3 \cdot \text{odds}, \ldots, 2^3 \cdot 1, 2^2 \cdot 1, 2 \cdot 1, 1.$$

Some notation

- an interval $\mathcal{I}$ covers an interval $\mathcal{J}$ (notation $\mathcal{I} \to \mathcal{J}$) with respect to continuous map f if $\mathcal{J} \subset f(\mathcal{I})$
- by intermediate value theorem $\mathcal{I} \to \mathcal{J}$ when f maps endpoints of $\mathcal{I}$ to opposite sides of $\mathcal{J}$
- notation $\mathcal{I} \rightarrowtail \mathcal{J}$ if $\mathcal{J} = f(\mathcal{I})$
- $\mathcal{O}$-interval: interval $\mathcal{I}$ whose endpoints are part of a cycle (periodic orbit) $\mathcal{O}$ of f
- basic $\mathcal{O}$-interval: $\mathcal{O}$-interval containing no other point of cycle $\mathcal{O}$ besides endpoints (endpoints called adjacent in this case)
- knowing a cycle $\mathcal{O}$ of f says how $\mathcal{O}$-intervals moved by the map (intermediate value theorem)
- in turn knowing how intervals are mapped gives other information about cycles

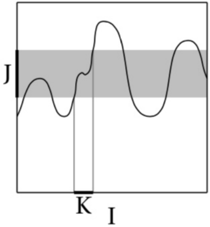
Intervals and Cycles

- if $[a_1, a_2] \to [a_1, a_2]$ (that is, $[a_1, a_2] \subseteq f([a_1, a_2])$) then f has a fixed point in $[a_1, a_2]$
  - since $[a_1, a_2] \to [a_1, a_2]$ there are $b_1, b_2 \in [a_1, a_2]$ with $f(b_1) = a_1$ and $f(b_2) = a_2$
  - so $f(b_1) - b_1 \leq 0 \leq f(b_2) - b_2$
  - intermediate value theorem: $f(x) - x = 0$ is satisfied somewhere between b_1 and b_2

- points that follow a loop
  - a loop of intervals $\mathcal{J}_k$ for $k = 0, \ldots, n-1$ with $\mathcal{J}_k \to \mathcal{J}_{k+1}$ and $\mathcal{J}_{n-1} \to \mathcal{J}_0$
  - a point x follows the loop if $f^n(x) = x$ and $f^k(x) \in \mathcal{J}_k$ for all $k = 0, \ldots, n-1$

$$J_0 \rightarrow \cdots \rightarrow J_{n-1} \quad (J_{n-1} \rightarrow J_0)$$

- existence of points that follow loops: every loop has a point that follows it
  - if $\mathcal{I} \to \mathcal{J}$ there is a subinterval $\mathcal{K} \subseteq \mathcal{I}$ with $\mathcal{K} \rightarrowtail \mathcal{J}$



  - thus have $\mathcal{K}_0 \rightarrowtail \mathcal{K}_1 \rightarrowtail \cdots \mathcal{K}_{n-1} \rightarrowtail \mathcal{J}_0$ with $\mathcal{K}_k \subseteq \mathcal{J}_k$
  - $f^{\circ n}$ maps $\mathcal{K}_0 \subseteq \mathcal{J}_0$ to $\mathcal{J}_0 = f^{\circ n}(\mathcal{K}_0)$, so by previous fixed point statement there is $x \in \mathcal{K}_0$ with $f^{\circ n}(x) = x$
  - this point x follows the loop

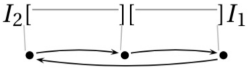
- elementary loop of n intervals: every point that follows it has period n (not a divisor of n)

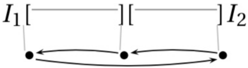
- in particular existence of an elementary loop implies existence of a period n point

- what ensures existence of an elementary loop?
  - suppose loop is made of $\mathcal{O}$-intervals, when is it elementary?
  - points that follow loop are not in the cycle $\mathcal{O}$ and the interior of $\mathcal{J}_0$ is disjoint from the other intervals $\mathcal{J}_1, \ldots, \mathcal{J}_{n-1}$
  - there is a point x that follows the cycle, but it is not in $\mathcal{O}$ so it must me in the interior of $\mathcal{J}_0$, but then $f^{\circ k}(x) \in \mathcal{J}_k$ must be $f^{\circ k}(x) \neq x$ because the interior of $\mathcal{J}_0$ disjoint from $\mathcal{J}_k$ so period cannot be shorter than n

Proof of Li-Yorke period three implies chaos

- period three intervals



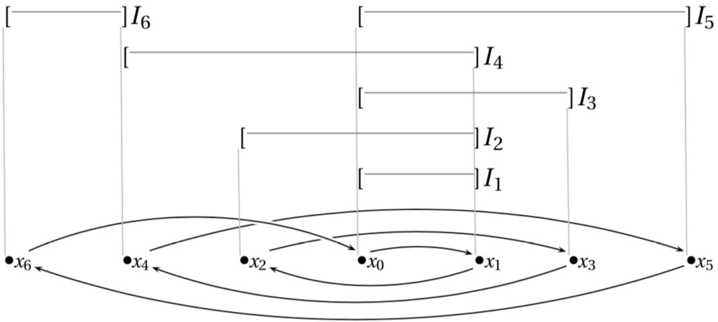


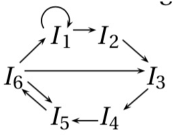
$$\circlearrowleft I_1 \rightleftarrows I_2$$

- $I_1 \to I_1$ implies I_1 contains a fixed point
- by existence of points that follow the cycle there is a point of period 2 (endpoints are period three so they don't follow the cycle and interior of I_2 disjoint from I_1)
- consider a loop $I_2 \to I_1 \to I_1 \cdots \to I_1 \to I_2$ with $\ell - 1$ copies of I_1 with $\ell > 3$
- because endopoints are order 3 cycle $\mathcal{O}$ they cannot stay in I_1 for more than two iterations of f
- so this is an elementary cycle of $\mathcal{O}$-intervals
- so it contains a periodic point of order exactly ℓ (and no divisor of ℓ) that follows the cycle

Another example: Period 7 implies periods of all following numbers in the Sharkovsky order

- just look at one possibility for $\mathcal{O}$-cycle and $\mathcal{O}$-intervals



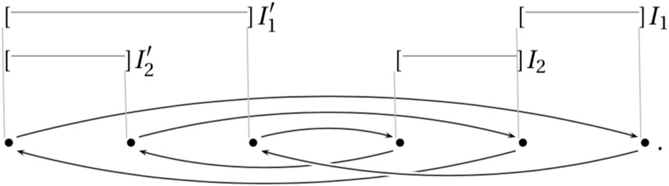


- loop $I_1 \to I_1$ implies existence of a fixed point
- other loops that are elementary because of previous argument

$I_6 \to I_5 \to I_6$,
$I_6 \to I_3 \to I_4 \to I_5 \to I_6$,
$I_6 \to I_1 \to I_2 \to I_3 \to I_4 \to I_5 \to I_6$,
$I_6 \to I_1 \to I_1 \to \cdots \to I_1 \to I_2 \to I_3 \to I_4 \to I_5 \to I_6$ with 3 or more copies of I_1.

- these give existence of points of period 2, 4, 6, and then of any number that follows 7 in the Sharkovsky ordering

Another example (slightly different): Period 6 implies periods of all following numbers in Sharkovsky order



• new aspect here: a symmetry that maps the three points on the left to the three points on the right

• can use this to obtain information on the fixed points of f from the fixed points of f^2

- f^2 has covering relations $I_1 \to I_1$, $I_1 \to I_2$ and $I_2 \to I_1$
- same as period three case, so f^2 has periodic points of all orders
- obtain periodic points for f from those of f^2
    - each occurrence of $I_1 \to$ in loops for f^2 replaced by $I_1 \to I_1' \to$ for f
    - each occurrence of $I_2 \to$ for f^2 replaced by $I_2 \to I_2' \to$ for f
    - so a k-loop for f^2 becomes a $2k$-loop for f
    - these $2k$ loops for f are still elementary if corresponding k-loop for f^2 was, because if a point has period exactly k for f^2 and follows the loop, under f the point keeps alternating between the two sides of figure, so period is $2k$
    - obtain that f has all even periods (and a fixed point)
    - these are all numbers following 6 in the Sharkovsky ordering

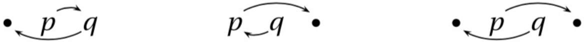
Proof of Sharkovsky's Theorem

- assume $\mathcal{O}$ is a cycle for f of order m
- show that f has periodic points of periods all numbers following m in Sharkovsky ordering

- First Step: points that switch sides
  - given non-trivial cycle $\mathcal{O}$, pick rightmost point p of $\mathcal{O}$ such that $f(p) > p$; let q be next point of $\mathcal{O}$ to the right of p, then $f(p) \geq q$ and $f(q) \leq p$
  - take $\mathcal{I} = [p, q]$, this interval satisfies the covering relation $\mathcal{I} \to \mathcal{I}$
  - pick a point c in the interior of $\mathcal{I}$
  - a point x switches sides (with respect to c) if x and $f(x)$ are on opposite sides of c
  - if all points of $\mathcal{O}$ switch sides then $\mathcal{O}$ is even (m is even)

• Second Step: assume that not all points of $\mathcal{O}$ switch sides and construct a sequence $x_0, \dots, x_k$ of points in $\mathcal{O}$ that "spiral out as fast as possible"

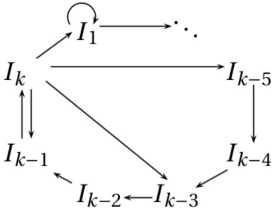
- start with x_0, x_1 the endpoints of $\mathcal{I}$ with $f(x_1) \neq x_0$ (possible since $\mathcal{O} \neq \{x_0, x_1\}$ other wise all points of $\mathcal{O}$ switch sides): possibilities



$x_1 = q$ $\quad$ $x_1 = p$ $\quad$ $x_1 = p$ or q

- continue inductively: if all points of $\mathcal{O}$ in the interval of endpoints x_i and c switch sides then x_{i+1} is point of $\mathcal{O}$ in image of this interval furthest from c; otherwise stop at x_i
- consecutive terms in this sequence on opposite sides of c
- also can see from the construction that x_{i+1} is further away from c than x_i
- all points in this sequence are distinct and the sequence terminates at some x_k with $k < m$ (otherwise orbit closes up)

- Third Step: assume that not all points of $\mathcal{O}$ switch sides and show in this case all numbers ℓ that follow m in Sharkovsky order are periods Sketch of proof:
    - use the previous sequence of points $x_0, \ldots, x_k$ to construct a sequence of intervals $I_1, \ldots, I_k$ satisfying the covering relations



    - check also using previous argument that all these loops are elementary

$I_1 \to I_1$;
$I_k \to I_{k-(l-1)} \to I_{k-(l-2)} \to \cdots \to I_{k-2} \to I_{k-1} \to I_k$ for even $l \leq k$;
$I_k \to I_1 \to I_1 \to \cdots \to I_1 \to I_2 \to \cdots \to I_{k-1} \to I_k$ with j occurrences of I_1.

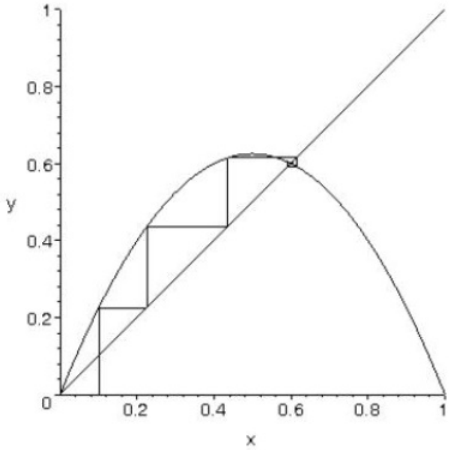
- the elementary loop $I_1 \to I_1$ implies existence of a fixed point
- the second family of elementary loops implies the existence of periodic points of all *even* orders $\ell \leq k$
- third family of elementary loops imply existence of periodic points of order any $\ell \geq k$ (except possibly m)
- note that these are all the numbers that follow m in the Sharkovsky ordering

- Fourth Step: the case where all points of $\mathcal{O}$ switch sides
  - use an induction argument on the length m of $\mathcal{O}$
  - suppose proved for all numbers less than m (in the usual ordering of $\mathbb{N}$)
  - consider an m-cycle $\mathcal{O}$: if there is a point that does not switch sides then result established by previous argument
  - if all points switch sides then case similar to the period 6 examples seen earlier: all points switch sides and m has to be even
  - the second iterate f^2 has an $m/2$-cycle for which the result is proved (induction hypothesis)
  - so f^2 has periods all numbers following $m/2$ in Sharkovsky ordering
  - use the same argument as in the period 6 example to show f then has elementary $2k$-loop for each elementary k-loop of f^2 so get result for m

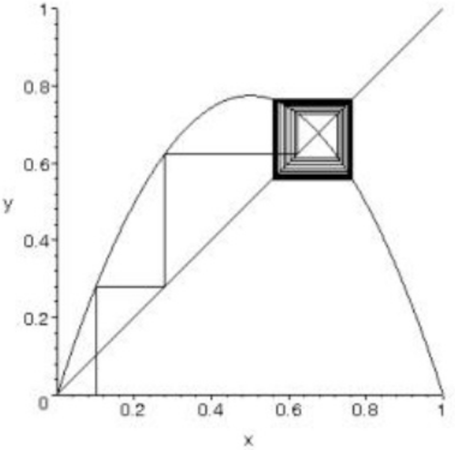
Logistic Map

$$f(x) = \mu\, x(1-x)$$

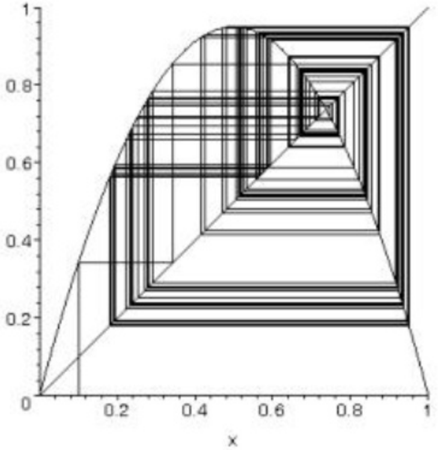
- orbits $f(x_0), f(f(x_0)), \ldots, f^{\circ n}(x_0), \ldots$
- fixed points and periodic points (depending on the value of the parameter $\mu > 0$)
- increase in complexity of periodic points structure then transition to chaos
- starts with a period doubling cascade (climbing the Sharkovsky ordering from the end)
- then reach a limit point of the parameter μ where transition to chaos (all periods)
- for higher values of μ substructures that reproduce the cascade



an orbit of the logistic map for $\mu = 2.5$: attractive fixed point

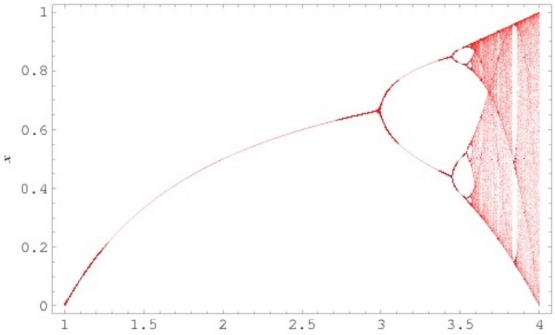


an orbit of the logistic map for $\mu = 3.1$: structure of attractor set, periodic orbits



an orbit of the logistic map for $\mu = 3.8$: chaotic dynamics

Logistic Equation Bifurcation Diagram



Plot as a function of the parameter μ the attractor set of the orbits $f^{\circ n}(x_0)$: for small μ single fixed point, then period 2 attractor appears, then period 4, ... cascade of period doubling and transition to chaotic region with attractor in bands rather than points and periodic points of all orders; above transition to chaos regions with splitting of bands into 2^n bands and orbits visiting different bands (like periodic orbits)

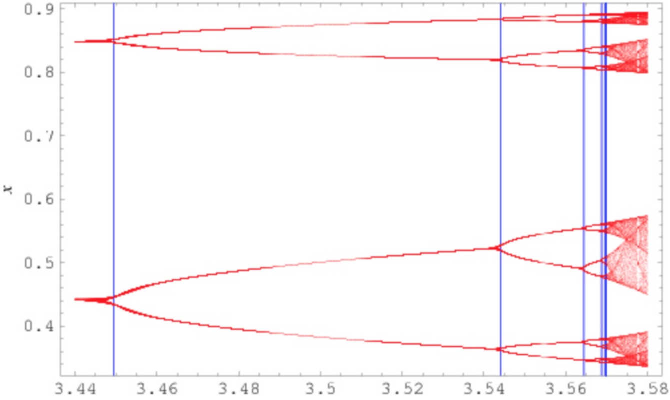
Compute where new periodic orbits appear as μ grows

- Example: $f(f(x)) = x$ gives an equation for the period two points

$$x_{\pm} = \frac{1}{2}\left((1+\mu^{-1}) \pm \mu^{-1}\sqrt{(\mu-3)(\mu+1)}\right)$$

these $x_{\pm}$ are real numbers only for $\mu \geq 3$ so occurrence of a period two orbit only possible for $\mu > 3$

- similarly can see the sequence μ_n of values of the parameter where orbits of period 2^n first appear



the sequence μ_n where period doubling occurs (blue lines)

Occurrence of period three

- to identify where there can be a non-trivial three-cycle (eliminating the trivial case of a fixed point)

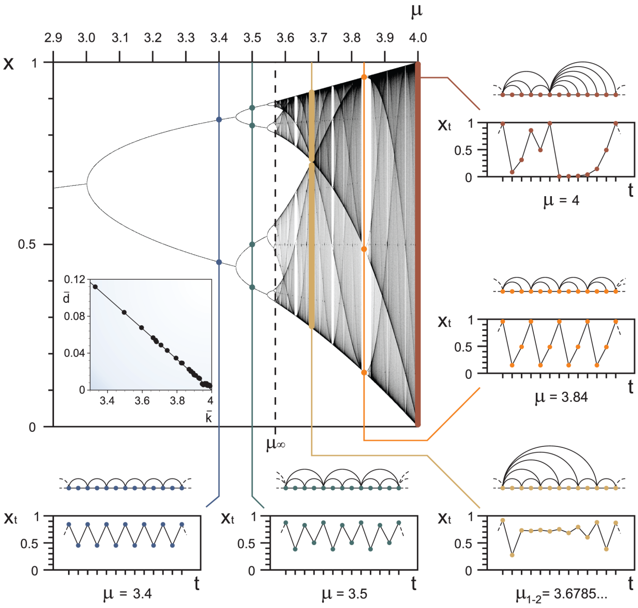
$$\frac{f^{\circ 3}(x) - x}{f(x) - x} = 0$$

- get polynomial equation in x with μ-dependent coefficients
- check when solutions are real numbers: for $\mu \leq \mu_F = 1 + 2\sqrt{2} = 3.828427\ldots$ imaginary roots
- for $\mu = \mu_F$ two of the roots become real
- μ_F solution to a discriminant equation $(\mu^2 - 5\mu + 7)^2(\mu^2 - 2\mu - 7)^3(1 + \mu + \mu^2)^2 = 0$
- at $\mu = \mu_F$ two roots coincide that then separate $\Rightarrow$ three-cycle starts at μ_F
- a simpler derivation of the three-cycle onset in P. Saha, S.H. Strogatz, *The Birth of Period Three*, Math. Mag. 68 (1995) 42–47

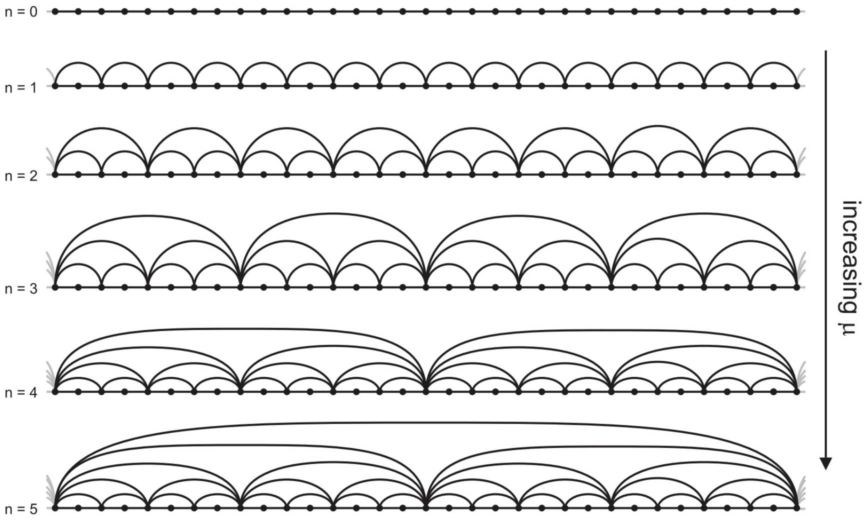
Feigenbaum Graphs

- given a sequence $\{x_k\}$ or a time series $\{x_t\}$
- one vertex for each datum k
- edge between vertices i and j if $x_i, x_j > x_n$ for all $i < n < j$
- for a fixed value of the parameter μ consider iterations of the logistic maps
- initial transient phase then approach the attractor
- form corresponding Feigenbaum graph with successive values of iterates

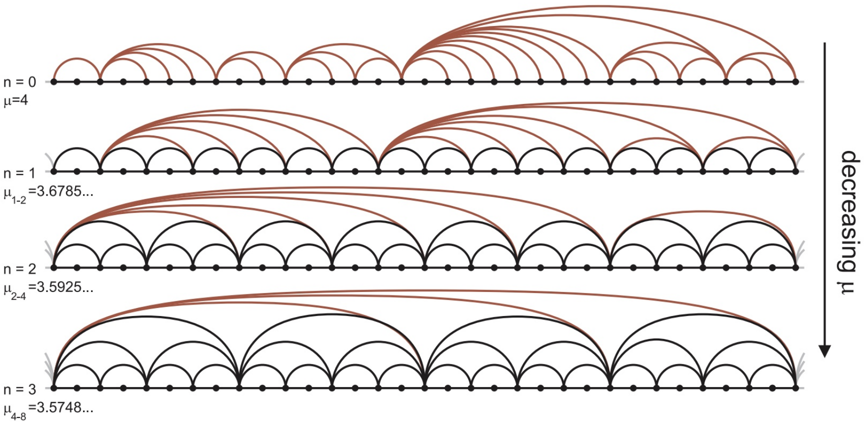
Feigenbaum Graphs of the Logistic Map



Period Doubling Cascade and Feigenbaum Graphs



Aperiodic Feigenbaum Graphs above the Transition to Chaos



For $n > 0$ phase space partitioned into 2^n disconnected chaotic bands; the orbit visits each chaotic band in the same order as in the periodic region; order of visits gives structure in the Feigenbaum graphs (black edges) similar to the period-doubling cascade case

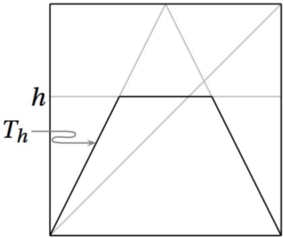
Tent Map realization of Sharkovsky's Theorem

• Sharkovsky's Realization Theorem: every tail of the Sharkovsky Ordering is the set of periods for a continuous map $f : \mathcal{I} \to \mathcal{I}$

• L. Alsedà, J. Llibre, M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, World Scientific, 2000

- explicitly exhibiting a family of maps that realizes tails of the Sharkovsky Ordering as set of periods
- truncated tent maps $T_h : [0,1] \to [0,1]$

$$T_h(x) = \min\{h, 1 - 2|x - \frac{1}{2}|\}, \quad \text{for } 0 \le h \le 1$$



sketch of proof of realization by truncated tent maps

- T_0 has a single fixed point at $x = 0$
- T_1 has a 3-cycle $\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$ hence all Sharkovsky ordering
- if $h \leq h'$ a cycle $\mathcal{O}$ of T_h with $\mathcal{O} \subset [0, h)$ is a cycle of $T_{h'}$
- if $h \leq h'$ a cycle $\mathcal{O}$ of $T_{h'}$ with $\mathcal{O} \subset [0, h]$ is a cycle of T_h
- $\max \mathcal{O} =$ largest x in the cycle $\mathcal{O}$

$$h(m) := \min\{\max \mathcal{O} \mid \mathcal{O} \text{ and } m\text{-cycle of } T_1\}$$

- T_h has an ℓ-cycle $\mathcal{O} \subset [0, h)$ iff $h(\ell) < h$ (previous property with $h' = 1$)
- orbit of $h(m)$ is an m-cycle for $T_{h(m)}$ and all other cycles for $T_{h(m)}$ contained in $[0, h(m))$
- $T_{h(m)}$ has an ℓ-cycle in $[0, h(m))$ for every ℓ following m in the Sharkovsky ordering and $h(\ell) < h(m)$
- $h(\ell) < h(m)$ iff ℓ follows m in Sharkovsky ordering
- for any $m \in \mathbb{N}$ the set of periods of $T_{h(m)}$ is the tail of the Sharkovsky ordering starting with m