Quantum Hall effect models in NCG

Electrons in a crystal:

$\Gamma \subset \mathbb{R}^d$ lattice $\Gamma \simeq \mathbb{Z}^d$ cocompact: $\mathbb{R}^d/\Gamma$ compact

$(d = 2, 3)$

periodic potential (electron-ion interaction)

$$U(x) = \sum_{\gamma \in \Gamma} u(x - \gamma)$$

invariant under translations by Γ

$$T_\gamma U = U \quad \forall \gamma \in \Gamma$$

N-electrons: N-particle Hamiltonian

potential of mutual repulsive force between electrons

$$\sum_{i=1}^{N} (-\Delta_{x_i} + U(x_i)) + \frac{1}{2} \sum_{i \neq j} W(x_i - x_j)$$

Simplify to a single particle problem using "independent electron approximation"

$$\sum_{i=1}^{N} (-\Delta_{x_i} + V(x_i))$$

correct U by an average effect of all other electrons on a given one

→ Usually $U(x)$ unbounded (Coulomb potential well)
<u>but</u> effective potential of independent electron approx $V(x)$ <u>bounded</u> function

(Condensed matter physics)

then wave function

$$\psi(x_1, \dots, x_N) = \det(\psi_i(x_j))$$

$$(-\Delta_{x_i} + V(x_i))\psi_i = E_i \psi_i \qquad \text{spectral problem for energy levels}$$

$$\left\{ \begin{array}{l} \sum_i (-\Delta_{x_i} + V(x_i))\psi_i = E\psi \\ E = \sum_i E_i \end{array} \right. \qquad \text{reduces completely to a single electron problem}$$

(Usually inverse problem of determining V: not known explicitly)

$H = -\Delta + V \qquad T_\gamma$ = translations $\gamma \in \Gamma$ (unitary operators)

on $\mathcal{H} = L^2(\mathbb{R}^d)$

$$T_\gamma H T_\gamma^{-1} = H \qquad \forall \gamma \in \Gamma$$

$\Rightarrow T_\gamma$ commutes w/ H — Simultaneously diagonalize on basis of eigenf. of H (energy states)

$T_\gamma \psi = c(\gamma)\psi \qquad T_{\gamma_1} T_{\gamma_2} = T_{\gamma_1 \gamma_2}$

$\Rightarrow c : \Gamma \to U(1)$ group homom. character of Γ

$c(\gamma) = \exp(i\langle k, \gamma\rangle)$

$k \in \hat{\Gamma}$ = Pontrjagin dual of Γ

$\Gamma \simeq \mathbb{Z}^d \Rightarrow \hat{\Gamma} \cong T^d$ torus

$T^d \simeq \hat{\Gamma} = \mathbb{R}^d / \Gamma^\#$

$\Gamma^\# = \{ k \in \mathbb{R}^d : \langle k, \gamma\rangle \in 2\pi\mathbb{Z}, \ \forall \gamma \in \Gamma \}$

dual lattice (reciprocal lattice)

Brillouin zones of the crystal: fundamental domains of reciprocal lattice $\Gamma^\#$

(identify w/ torus T^d)

Classical Bloch theory of electrons in solids:

$$(*) \begin{cases} (-\Delta + V)\psi = E\psi \\ \psi(x+\gamma) = e^{i\langle k,\gamma\rangle}\psi(x) \end{cases}$$ spectral problems

for given k : eigenvalues $E_1(k), E_2(k), \dots, E_n(k) \dots$

$$E(k) = E(k+u) : u \in \Gamma^\#$$

$$k \mapsto E(k) \qquad k \in \mathbb{R}^d/\Gamma^\#$$

energy-crystal momentum dispersion relation

Discretization of the problem $(*)$:

Replace $\mathbb{R}^d$ by $\mathbb{Z}^d$ lattice

Δ Laplacian replaced by finite difference op. (random walk on a lattice)

$$R\psi(n_1,\dots,n_d) = \sum_{i=1}^{d} \psi(n_1,\dots,n_i+1,\dots,n_d) + \sum_{i=1}^{d} \psi(n_1,\dots,n_i-1,\dots,n_d)$$

$$\Delta^{disc}\psi(n_1,\dots,n_d) = (2d - R)\psi(n_1,\dots,n_d)$$

$(*)$ becomes
$$\begin{cases} \psi \in \ell^2(\Gamma) \text{ satisfying} \\ (R+V)\psi = (\lambda + 2d)\psi \\ R_{\gamma_i}\psi = z_i\psi \end{cases}$$

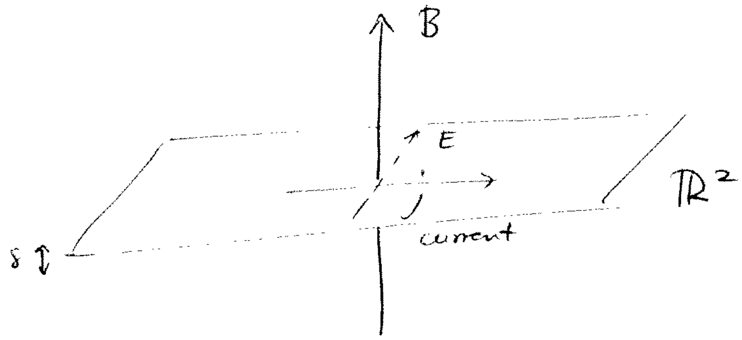
$$(R_{\gamma_i}\psi)(n_1,\dots,n_d) = \psi(n_1,\dots,n_i+a_i,\dots,n_d)$$

$$(R_{\gamma_i}\psi)(\gamma) = \psi(\gamma\gamma_i)$$

$$R\psi = \sum_{i=1}^{2d} R_{\gamma_i}\psi$$

$\langle \gamma_1,\dots,\gamma_d, \gamma_1^{-1},\dots,\gamma_d^{-1}\rangle$ gen. of Γ

This classical theory of electron motion in solids does not work anymore when transversal magnetic field.



Classical Hall effect

j current ; creates electric field E (Hall current)

$$Ne\,\underline{E} + \underline{j} \wedge \underline{B} = 0$$ equation of equilibrium of forces

ratio : intensity of Hall current / intensity ~~electric~~ electric field

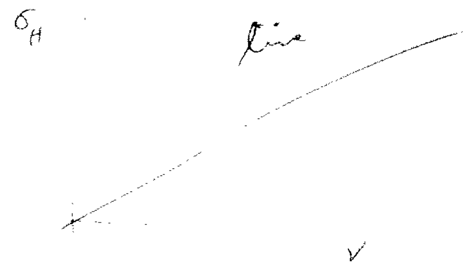
Hall conductance

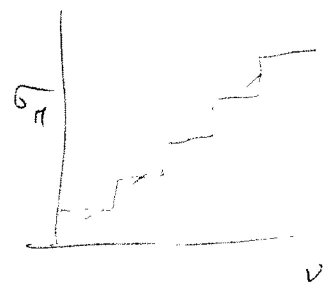
$$\sigma_H = \frac{Ne\,\delta}{B}$$

$$\sigma_H = \frac{\nu}{R_H} \qquad \nu = \frac{\rho h}{eB}$$ filling factor (dimensionless)

density of charges

$$R_H = \frac{h}{e^2}$$ Hall resistance





* Integer Quantum Hall effect :

σ_H has quantized values at integer multiples of $\frac{e^2}{h}$

von Klitzing 1980
Laughlin 1981

* Fractional QHE   Stormer-Tsui 1982 (lower T, stronger B)   certain fractional values also occur

Magnetic field 2-form $\omega = d\eta$

($B = \mathrm{curl}\, A$)

Schrödinger operator $\rightsquigarrow$ magnetic Schrödinger op.

$\Delta^\eta + V$

$$\Delta^\eta = (d - i\eta)^*(d - i\eta)$$

V same indep. of approx electric potential

$\gamma^* \omega = \omega$ translation invariance for 2-form of magnetic field

but $0 = \omega - \gamma^*\omega = d(\eta - \gamma^*\eta)$

does not mean invariant magnetic potential

$d(\eta - \gamma^*\eta) = 0$ only implies

$\eta - \gamma^*\eta = d\phi_\gamma$ (because $\mathbb{R}^2$ no cohom. closed form $\Rightarrow$ exact)

$$\phi_\gamma(x) = \int_{x_0}^{x} (\eta - \gamma^*\eta)$$

$\Rightarrow$ T_γ translations no longer commute with Δ^η

but twisted by phase ϕ again commute

$$T_\gamma^\phi \psi := \exp(i\phi_\gamma)\, T_\gamma \psi$$

$$(d - i\eta)\, T^\phi_\gamma = T^\phi_\gamma\, (d - i\eta)$$ $\Rightarrow$ commutes w/ Δ^η

$\gamma \in \Gamma$:

$$T^\phi_\gamma T^\phi_{\gamma'} = \sigma(\gamma, \gamma')\, T^\phi_{\gamma\gamma'}$$ don't form a commutative algebra anymore

with $\sigma(\gamma, \gamma') = \exp(-i\phi_\gamma(\gamma' x_0))$ cocycle

and $\phi_\gamma(x) + \phi_{\gamma'}(\gamma x) - \phi_{\gamma'\gamma}(x)$ indep. of x

<u>Notice</u> usual T_γ generate $C^*(\Gamma)$ group C^*-alg.

Since $\Gamma \cong \mathbb{Z}^d$ lattice (abelian grp.)

$C^*(\Gamma) = C(\hat{\Gamma})$ Pontrjagin duality

$\hat{\Gamma} = \mathbb{T}^d = \mathbb{R}^d/\Gamma^\#$ $\Rightarrow$ $C^*(\Gamma) = C(\text{Brillouin zone})$

Now with magnetic field T^ϕ_γ generate a $\mathcal{A}$ C^*-algebra noncommutative

replaces Brillouin zone

* In the presence of a magnetic field Brillouin zone becomes noncommutative

Discretized model on lattice $\Gamma = \mathbb{Z}^2$

Harper operator $\longleftrightarrow$ Magnetic Laplacian

(like Random walk operator $\longleftrightarrow$ Laplacian)

$$H_{\alpha_1,\alpha_2}\,\psi(m,n) = e^{-i\alpha_1 n}\,\psi(m+1,n)$$
$$+\, e^{i\alpha_1 n}\,\psi(m-1,n)$$
$$+\, e^{-i\alpha_2 m}\,\psi(m,n+1)$$
$$+\, e^{i\alpha_2 m}\,\psi(m,n-1)$$

Magnetic translations $\qquad \sigma((m',n'),(m,n)) = \exp\left(-i(\alpha_1 m'n + \alpha_2 n m')\right)$

$$U = T^\sigma_{\gamma_1} \qquad V = T^\sigma_{\gamma_2}$$

$\gamma_1 = (0,1) \qquad (U\psi)(m,n) = \psi(m,n+1)\, e^{-i\alpha_2 m}$

$\gamma_2 = (1,0) \qquad (V\psi)(m,n) = \psi(m+1,n)\, e^{-i\alpha_1 n}$

$$H_{\alpha_1,\alpha_2} = U + U^* + V + V^*$$

$$UV = e^{i\theta} VU \qquad \theta = \alpha_2 - \alpha_1$$

$\Rightarrow$ Brillouin zone T^2 replaced by a noncommutative torus A_θ

In general Γ (discrete group)

$\sigma : \Gamma \times \Gamma \to U(1)$ multiplier:

$$\sigma(\gamma_1, \gamma_2)\, \sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\, \sigma(\gamma_2, \gamma_3) \quad \circledast$$

$$\sigma(\gamma, 1) = \sigma(1, \gamma) = 1$$

$\mathcal{H} = \ell^2(\Gamma)$

$$(L^\sigma_\gamma \psi)(\gamma') = \psi(\gamma^{-1}\gamma')\, \sigma(\gamma, \gamma^{-1}\gamma')$$

$$(R^\sigma_\gamma \psi)(\gamma') = \psi(\gamma'\gamma)\, \sigma(\gamma', \gamma)$$

$$L^\sigma_\gamma L^\sigma_{\gamma'} = \sigma(\gamma, \gamma') L^\sigma_{\gamma\gamma'}$$

$$R^\sigma_\gamma R^\sigma_{\gamma'} = \sigma(\gamma, \gamma') R^\sigma_{\gamma\gamma'}$$

L^σ_γ, $R^{\bar\sigma}_\gamma$ commute (using $\circledast$)

conjugate cocycle

$\{\gamma_i\}_{i=1}^r$ set of symmetric generators of Γ (generators & their inverses)

$$R_\sigma = \sum_{i=1}^r R^\sigma_{\gamma_i} \qquad \text{Harper operator}$$

$r - R_\sigma$ discretization of magnetic Laplacian on Γ

Algebra of observables

$\mathbb{C}(\Gamma, \sigma)$ twisted group ring

generated by magnetic translations R_γ^σ

equivalently $f : \Gamma \to \mathbb{C}$ fin. support

$$(f_1 * f_2)(\gamma) = \sum_{\gamma = \gamma_1 \gamma_2} f_1(\gamma_1) f_2(\gamma_2) \, \sigma(\gamma_1, \gamma_2)$$

(cocycle id. $\Rightarrow$ associativity)

$C_r^*(\Gamma, \sigma)$ C^*-completion in rep. on $\ell^2(\Gamma)$

(For $\Gamma = \mathbb{Z}^2$ $C_r^*(\Gamma, \sigma) = \mathcal{A}_\theta$ NC torus)

discrete analog of spectral problem for magnetic Laplacian

$R_\sigma \psi + V\psi = E\psi$ $\quad i \frac{\partial}{\partial t} \psi = R_\sigma \psi + V\psi$ $\quad \psi \in \ell^2(\Gamma)$

Schrödinger eq.

$\mathrm{Spec}(R_\sigma)$ Complement: open sets (band structure)

fin many $\rightsquigarrow$ bands

∞ many $\rightsquigarrow$ Cantor set as spectrum

Hofstadter butterfly : $\theta \in \mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$

Counting gaps in the spectrum

$\Uparrow$

Counting projections in $C_r^*(\Gamma, \sigma)$

$P_E = \chi_{(-\infty, E]}(H_{\sigma, V})$ spectral projections in here if E in a gap

$$P_E = \int_C \frac{d\lambda}{\lambda - H_{\sigma,V}} = \int_C R_\lambda \, d\lambda$$ if $C \supset$ Spec i.e. E not in Spec

$R_\lambda = (\lambda - H_{\sigma,V})^{-1}$ resolvent

$C^*_r(\Gamma,\sigma)$ closed under holomorphic functional calculus
$\Rightarrow P_E \in C^*_r(\Gamma,\sigma)$

Canonical faithful trace

$$\tau : \mathcal{M}(\Gamma,\sigma) \longrightarrow \mathbb{C}$$

von Neumann alg.
closure of $\mathbb{C}(\Gamma,\sigma)$
in $\mathcal{B}(\ell^2(\Gamma))$ weak top.

$\tau(a) = \langle a\delta_1, \delta_1 \rangle_{\ell^2(\Gamma)}$ $\{\delta_\gamma\}$ canonical basis of $\ell^2(\Gamma)$

extended to
~~tr~~ $\tau \otimes \mathrm{Tr} : K_0(C^*_r(\Gamma,\sigma)) \longrightarrow \mathbb{R}$

Range of the trace

e.g. for NC torus $\boxed{\mathbb{Z}\theta + \mathbb{Z}} \subset \mathbb{R}$

So when $\theta \in \mathbb{Q}$ know there are only fin. many gaps

When $\theta \in \mathbb{R}\setminus\mathbb{Q}$ indication that ∞-many but not sure as values could be on other projections

Conjectural

(Smooth subalgebra
$D\,\delta_\gamma = \ell(\gamma)\,\delta_\gamma$ length on generators
$\delta = [D, \cdot\,]$ $\quad R = \bigcap_{k\in\mathbb{N}} \mathrm{Dom}(\delta^k)$)

Conductance cocycle - Kubo formula

$$\sigma_H = \tau\left(P_F [\, \delta_1 P_F , \delta_2 P_F] \right)$$

(from transport theory
current density in X_1 direction
= functional derivative δ_1 of H_0 by A_1-component of magnetic potential

value of current
$tr(P\,\delta_1 H)$ $\quad P$ proj on energy state of system

$$i\,tr(P[\partial_t P, \delta_1 P]) = -i E_2\, tr(P[\delta_2 P, \delta_1 P])$$

$$E = -\frac{\partial A}{\partial t}$$)

will be a cyclic cocycle

conductance cocycle
$$tr_K(f_0, f_1, f_2) := tr\left(f_0 \left(\delta_1(f_1)\delta_2(f_2) - \delta_2(f_1)\delta_1(f_2) \right)\right)$$

for elements
$f_0, f_1, f_2 \in \mathbb{C}(\Gamma, \sigma)$

becomes an index theorem on ordinary torus T^2

$\Rightarrow \mathbb{Z}$-valued

$$\sigma_E = tr_K(P_E, P_E, P_E)$$

Values of conductance: range of this "trace" (index pairing of cyclic cohom. & K-theory)