Tuesday Feb 2

$\theta = \frac{p}{q} \in \mathbb{Q}$ $\quad \mathcal{A}_\theta$ NC torus $\quad$ (say $(p,q)=1$ & $q>0$)

$\exists$ rank $q$ vector bundle $E$ on $T^2$-torus s.t.

$\mathcal{A}_\theta \cong \Gamma(T^2, \mathrm{End}(E))$ $\quad C^*$-alg. of sections of $\mathrm{End}(E)$

Pf: $T^2 \times \mathbb{C}^q$ trivial bundle quotient by free action of two $\mathbb{Z}/q\mathbb{Z}$ $u, v$

~~$G = \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$~~

Constructed as follows:

$$u = \begin{pmatrix} 1 & & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda^{q-1} \end{pmatrix} \qquad V = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots 0 & 0 \\ 0 & 1 & \cdots 0 & 0 \\ \vdots & & & \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

satisfy $vu = \lambda uv$

$u^q = v^q = 1 \qquad \lambda = \exp(2\pi i\theta)$

~~$T_1 : (z_1, z_2, \xi) \in T^2 \times \mathbb{C}^q \longmapsto (\lambda z_1, z_2, u\xi)$~~

~~$T_2 : (z_1, z_2, \xi) \longmapsto (z_1, \lambda z_2, v\xi)$~~

~~$(\lambda z_1, \lambda z_2, uv\xi)$~~

~~$T_1 T_2 = \lambda T_2 T_1$~~ ~~$(\lambda z_1, \lambda z_2, vu\xi)$~~

Define $E$ as vector bundle s.t. $\Gamma(T^2, \mathrm{End}(E))$
$= \{ \sum_{i,j} f_{ij}(z_1^q, z_2^q)\, u^i v^j \}$

$\mathrm{End}(T^2 \times \mathbb{C}^q) = M_q(\mathbb{C})$

$\mathrm{End}(E) = M_q(\mathbb{C})^G$

$u^i v^j$ basis of $M_q(\mathbb{C})$

$\gamma \in \mathrm{End}(T^2 \times \mathbb{C}^q)$

$\gamma = \sum f_{ij}(z_1, z_2)\, u^i v^j$

G-invariant if $f_{ij}(z_1, z_2) = f_{ij}(z_1^q, z_2^q)$

In $\mathcal{A}_{\frac{p}{q}}$ $\qquad U^q V = V U^q \qquad V^q U = U V^q$

$\Rightarrow U^q, V^q$ in center of $\mathcal{A}_{\frac{p}{q}}$

in fact generate center: $C(T^2)$ $\qquad i,j = 1, \ldots, q$ basis of $\mathrm{End}(E)$

$\mathcal{A}_{\frac{p}{q}} \ni a = \sum_{i,j=1}^{q} f_{ij}(U^q, V^q)\, U^i V^j \in \Gamma(T^2, \mathrm{End}(E))$

②

$A = C(X) \qquad B = \Gamma(X, \mathrm{End}(E))$ $\quad \begin{matrix} E \\ \downarrow \\ X \end{matrix}$ complex vector bundle <u>hermitian</u>

bimodule giving Morita equivalence

is $\Gamma(X, \mathrm{End}(E))$ completed w/ inner prod.

$$\langle s_1, s_2 \rangle_{C(X)} = \langle s_1(x), s_2(x) \rangle$$

with actions of

$\Gamma(X, \mathrm{End}(E))$ and $C(X)$ on left and right

$\Rightarrow \theta = \frac{p}{q} \quad A_\theta \underset{M.e.}{\simeq} C(T^2)$ Commutative up to Morita eq.

in terms of elliptic curves

$\mathbb{C}^* / \lambda^{\mathbb{Z}}$ for $\lambda = \exp(2\pi i \frac{p}{q})$ Commutative (good quotient)

---

$A \underset{M.e.}{\simeq} B \Rightarrow Z(A) \underset{\text{isom.}}{\cong} Z(B)$ Morita equiv. algebras have isomorphic centers

(A categorical argument:

$\mathcal{C} \rightsquigarrow \mathrm{Fun}(\mathcal{C})$ self functors

$Z(\mathcal{C}) := \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C})}(id, id)$ natural transformations of identity functor

Center of category

$\mathcal{M}_A$ = cat of (right) modules $\qquad Z(\mathcal{M}_A) = Z(A)$

$a \in Z(A) \mapsto R_a \qquad R_a(m) = ma$
$m \in M \quad M \in \mathrm{Obj}(\mathcal{M}_A)$

Equivalent categories same center)

---

Morita equiv. of $C^*$-alg's and stable isomorphism

$A \underset{M.eq.}{\simeq} B \quad C^*$-algebras

$\Leftrightarrow \quad A \otimes K(\mathcal{H}) \underset{\text{isom.}}{\cong} B \otimes K(\mathcal{H})$

$$A \underset{M.e.}{\simeq} M_n(A)$$

$A^n = \mathcal{E}$ ~~span of columns of matrix~~

$$\langle x,y\rangle_A = \sum_i (x_i^*)^t y_i$$

$$_{M_n(A)}\langle x,y\rangle = x(y^t)^* = (x_i(y_j^*))_{ij}$$

<u>note</u>: by

$$_A\langle \xi_1,\xi_2\rangle \xi_3 = \xi_1\langle \xi_2,\xi_3\rangle_B$$

$$\Downarrow$$

if $\langle \xi_1,\xi_2\rangle_B$ conjugat on left var. then $_A\langle \xi_1,\xi_2\rangle$ conjugates on right

---

other way of seeing $SL_2(\mathbb{Z})$ Morita equiv. for NC tori $\theta \in \mathbb{R}\setminus\mathbb{Q}$

check $SL_2(\mathbb{Z})$ generators

$\theta \longmapsto \frac{1}{\theta}$ and $\theta \mapsto \theta+1$ for this isomorphism

$\uparrow$ This Morita equiv.

$$\mathcal{E} = \mathcal{S}(\mathbb{R})$$

$$(\xi U)(t) = \xi(t+\theta)$$
$$(\xi V)(t) = e^{2\pi i t}\xi(t)$$

$U, V \in \mathcal{A}_\theta$

$$(U'\xi)(t) = \xi(t+1)$$
$$(V'\xi)(t) = e^{-\frac{2\pi i t}{\theta}}\xi(t)$$

$U', V' \in \mathcal{A}_{\frac{1}{\theta}}$

$$_{\mathcal{A}_{\frac{1}{\theta}}}\langle \xi_1,\xi_2\rangle = \sum_{n,m}\sum_k \xi_1(n-k)\,\overline{\xi_2(n-k-m\theta)}\,U'^m V'^n$$

$$\langle \xi_1,\xi_2\rangle_{\mathcal{A}_\theta} = \sum_{n,m}\sum_k \overline{\xi_1(n-k\theta)}\,\xi_2(n-m-k\theta)\,U^m V^n$$

Quotients : Good quotients and Morita equivalence

④

$G$ acting on $X$ freely and properly (G discrete)

$X/G$ loc. comp. Hausdorff still

$G \times X \to X \times X$
proper map

$$C_0(X/G) \underset{\text{M.equiv}}{\simeq} C_0(X) \rtimes G$$

$\mathcal{E} = C_c(X)$ compactly supp. functions

$C_0(X/G)$ acts by pointwise multipl. (functions on $X$ invariant under $G$)

$$\langle f_1, f_2 \rangle_{C_0(X/G)} = \text{[illegible]}$$

[illegible]

$$= \sum_{g \in G} \overline{f_1(\alpha_g(x))}\, f_2(\alpha_g(x))$$

Action of $C_0(X) \rtimes G$ on left

$$\left(\sum f_g \delta_g\right) f = \sum_g f_g \tilde{\alpha}_g(f)$$

$$\tilde{\alpha}_g(f)(x) = f(\alpha_g(x))$$

for discrete group enough else
new factor related to
how measure $d\mu$ on $G$
scales $d\mu(g^{-1}h)$ ---

$$\langle f_1, f_2 \rangle_{C_0(X) \rtimes G} = f_1(x) \sum_{g \in G} \overline{f_2(\alpha_g(x))}\, \delta_g$$

So that one gets:

$$\langle f_1, f_2 \rangle_{C_0(X) \rtimes G}\, f_3 = f_1 \langle f_2, f_3 \rangle_{C_0(X/G)}$$

$$\left(f_1(x) \sum_g \overline{f_2(\alpha_g(x))}\, \delta_g\right) f_3 = f_1(x) \cdot \sum_g \overline{f_2}(\alpha_g(x))\, f_3(\alpha_g(x))$$

# Noncommutative tori and range of trace on projections

$K_0(A)$ = Grothendieck group of
fin proj. modules $\mathcal{E}$ with $\oplus$

(like $K^0(X)$ = Groth. group of Vector bundles $\mathcal{E}$ with $\oplus$)
so that $K_0(C(X)) = K^0(X)$ )

In terms of projections $p \in M_n(A)$ $p^2 = p^* = p$

Groth. group of semigroup $\mathcal{P}(A) = \{ p \in \bigcup_{n \geq 1} M_n(A) ; p^2 = p^* = p \}$
with $\oplus$ direct sum
up to equivalence

$S$ semigroup $+$ (cancellation semigroup), $0$

$G(S)$ Groth. group
"formal differences"
$s - t$

$s_1 - t_1 \simeq s_2 - t_2$ iff $s_1 + t_2 = s_2 + t_1$ in $S$
(cancellative property $\Rightarrow$ this is equiv. rel.)
$s_1 + t = s_2 + t \Rightarrow s_1 = s_2$

$K_0^+(A) \subset K_0(A)$ semigroup of fin proj. mod's
(like vector bundles as opposed to virtual vector bundles)

---

$\theta \in \mathbb{R} \setminus \mathbb{Q}$

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

continued fraction expansion

$a_i \geq 1$
non-terminating
if $\theta$ irrational

$\theta = [a_0; a_1, a_2, a_3, \ldots, a_n \ldots]$

$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$ rational approximations

$$\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1} & q_{n-1} \\ p_{n-2} & q_{n-2} \end{pmatrix}$$

$p_0 = a_0$ $q_0 = 1$
$p_1 = a_0 a_1 + 1$ $q_1 = a_1$

⑥

$$p_n \cdot q_{n-1} - p_{n-1} \cdot q_n = \det\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = (-1)^{n-1}$$

AF - algebra $\mathcal{F}_\theta = \overline{\bigcup_{n \geq 1} \mathcal{F}_{\theta, n}}$

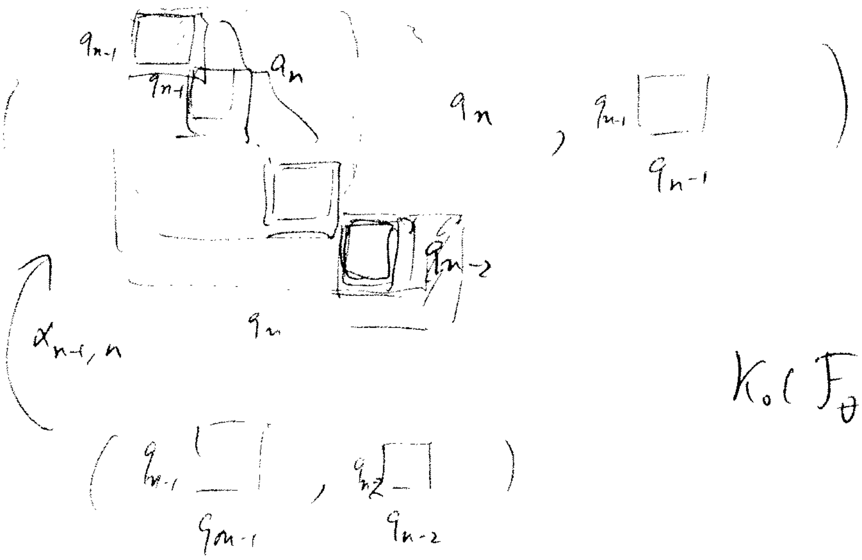
$\mathcal{F}_{\theta, n} = \mathcal{M}_{q_n}(\mathbb{C}) \oplus \mathcal{M}_{q_{n-1}}(\mathbb{C})$ sum of matrix algebras

embeddings $\mathcal{F}_{\theta, n-1} \xhookrightarrow{\alpha_{n+1, n}} \mathcal{F}_{\theta, n}$

by multiplicities $\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q_{n-1} \\ q_{n-2} \end{pmatrix} = \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix}$$

$a_n q_{n-1} + q_{n-2}$



$$K_0(\mathcal{F}_\theta) = \varinjlim_n K_0(\mathcal{F}_{\theta, n})$$

$K_0(M_n(\mathbb{C})) = K_0(\mathbb{C}) = \mathbb{Z}$

Morita equiv.
invariance of $K_0$
or else just def.
$K_0(A)$ proj's in $M_n(A)$
stable equiv.
$\Rightarrow M_{nk}(A)$ gives same

$K_0(M_{q_n}(\mathbb{C}) \oplus M_{q_{n-1}}(\mathbb{C})) = \mathbb{Z}^2$

~~embeddings~~ induced maps

$\mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$

$(\alpha_{n+1,n})_*$

$\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{Z}^2 \xrightarrow[\text{isom.}]{\simeq} \mathbb{Z}^2$

since on projections

$(P, Q) \longmapsto \left(\begin{pmatrix} P & & \\ & P & \\ & & Q \end{pmatrix}^{a_n}, P\right)$

$\Rightarrow \varinjlim_n K_0(\mathcal{F}_{\theta,n}) = \mathbb{Z}^2$

$K_0^+(F_\theta)$ pos cone $\{(n, m) : \theta n + m \geq 0\}$

$\lim_{n \to 0} \underbrace{K_0^+(F_{n,\theta})}$ $(n, m)$ $n \geq 0$ $m \geq 0$

Cone spanned by ~~vectors~~ ~~action of~~ action of $\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix}$ on $\mathbb{Z}_+^2$ $\Big/$ $\overset{\geq 0}{\frac{p_n}{q_n} n + m} \to \overset{\geq 0}{\theta n + m}$

$\frac{p_{n-1}}{q_{n-1}} n + m \geq 0$

To embed $\mathcal{A}_\theta$ in $\mathcal{F}_\Theta$

$U_n, V_n$ $\quad U_n e_k^{(n)} = e_{k+1}^{(n)} \quad V_n e_k^{(n)} = e^{2\pi i k p_n/q_n} e_k^{(n)}$

on $L^2(\mathbb{Z}/q_n\mathbb{Z})$ basis $e_k^{(n)}$ $k = 1, \dots, q_n$

$$U_n V_n = e^{2\pi i p_n/q_n} V_n U_n$$

pairs $(U_n \oplus U_{n-1}, V_n \oplus V_{n-1}) \in M_{q_n}(\mathbb{C}) \oplus M_{q_{n-1}}(\mathbb{C}) = \mathcal{F}_{n,\theta}$

Need to show that under the embedding $\mathcal{F}_{n-1,\theta} \xhookrightarrow[\alpha_{n-1,n}]{} \mathcal{F}_{n,\theta}$

the sequences $U_n \oplus U_{n-1}$ and $V_n \oplus V_{n-1}$

form a Cauchy sequence

i.e. that shift operator $U_n$ is approximated well by image of shift operators of lower order

$U_n \oplus U_{n-1}$ & $\alpha_{n-1,n}(U_{n-1} \oplus U_{n-2})$

(Berg's technique)

Once know this
$$\begin{tikzcd} U_n \oplus U_{n-1}, \arrow[d] & V_n \oplus V_{n-1} \arrow[dl] \\ U, & V \end{tikzcd}$$
in $\mathcal{F}_\theta$

with $UV = e^{2\pi i\theta} VU$

hence realize a copy of $\mathcal{A}_\theta$ inside $\mathcal{F}_\theta$

<u>Note</u> AF algebra isom. if $(K_0, K_0^+, \text{trace})$ isom.

$\Rightarrow \mathcal{F}_\theta \underset{\text{isom.}}{\cong} \mathcal{F}_{\theta'}$ iff $\theta = \pm\theta' \bmod \mathbb{Z}$

$\Rightarrow \mathcal{A}_\theta \underset{\text{isom}}{\cong} \mathcal{A}_{\theta'}$ iff $\theta = \pm\theta' \bmod \mathbb{Z}$

Also if already know that $\exists\ \mathcal{E}_{n,m}$ fin gen proj. modules s.t. $\tau(\mathcal{E}_{n,m}) = n\theta + m$

know $\mathbb{Z}\theta + \mathbb{Z} = \tau(K_0(\mathcal{A}_\theta))$ because nothing more $K_1(\mathcal{A}_\theta) \subseteq K_0(\mathcal{F}_\theta) = \mathbb{Z}\theta + \mathbb{Z}$

Existence of $\mathcal{E}_{n,m}$ fin. proj. modules ⑧

s.t. $\tau([\mathcal{E}_{n,m}]) = n\theta + m$ :

See construction outlined earlier of fundam. modules :

$$\mathcal{E}_{n,m} = p\mathcal{A}_\theta^{\infty} \qquad \tau(p) = |\theta n + m|$$

---

<u>Next</u>: models of quantum Hall effect based on noncommutative geometry