Thursday Feb 25

(1)

Local index formula (Connes-Moscovici)

$(\mathcal{A}, \mathcal{H}, D)$ finitely summable spectral triple
simple poles in Dim Spec.

Regular : a and $[D,a]$ satisfy
$t \mapsto \sigma_t(a) = e^{it|D|} a e^{-it|D|}$
$\sigma_t([D,a])$ ......
smooth in t

even: γ

then consider:

$$\varphi_n(a^0, \dots, a^n) = \sum_{k=(k_1,\dots,k_n)} c_{n,k} \, ⨍ \gamma a^0 [D,a^1]^{(k_1)} \cdots [D,a^n]^{(k_n)} |D|^{-n-2|k|}$$

for $|k| = k_1 + \dots + k_n$ and

$$T^{(k_i)} = \nabla^{k_i}(T) \qquad \nabla(T) = D^2T - TD^2 = [D^2, T]$$

- Only finitely many nonzero terms in sum above
- $(\varphi_n)_{n=0,2,4,\dots}$ defines a cocycle in (b,B)-complex for $\mathcal{A}$
- Pairing of this cocycle with $K_0(\mathcal{A})$ gives
  $HC^{2n}(\mathcal{A})$
  Index of D
  (twisted w/ connection from $e\mathcal{A}^N = \mathcal{E}$ in $K_0(\mathcal{A})$)

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i.e. realizes index pairing using
Residues $⨍$ instead of Traces Tr

Morita equivalences and inner fluctuations of the metric for spectral triples

$\mathcal{A}$, B Morita equivalent via bimodule ${}_B\mathcal{E}_{\mathcal{A}}$:

$(\mathcal{A}, \mathcal{H}, D)$ spectral triple

$\pi : \mathcal{A} \to B(\mathcal{H})$

Take $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ have $\pi' : B \to B(\mathcal{H}')$

Dirac operator: note

$D' = 1 \otimes D$ does not suffice

because $[D, a] \neq 0$ so not compatible w/ $\otimes_{\mathcal{A}}$ defining $\mathcal{H}'$

$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ $\quad \xi a \otimes \psi = \xi \otimes \pi(a) \xi$

have $(1 \otimes D)(\xi a \otimes \psi) = \xi a \otimes D\psi = \xi \otimes \pi(a) D\psi$

while $(1 \otimes D)(\xi \otimes \pi(a)\psi) = \xi \otimes D\pi(a)\psi$

but $\pi(a) D\psi \neq D\pi(a)\psi$ so $1 \otimes D$ not well defined as operator on $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$

To make it well defined need to correct by <u>difference</u>

a <u>connection</u> on the bimodule $\mathcal{E}$

$\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$

$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ satisfying Leibnitz rule

$$\nabla(\xi a) = (\nabla \xi) a + \xi \otimes da \qquad da = [D, a]$$

produces the term that compensate for $1 \otimes D$

$$\Omega^1_D(\mathcal{A}) = \{ \sum a_i [D, b_i] \}$$

Then define new Dirac operator on $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$
by setting $D'(\xi \otimes \eta) = \xi \otimes D\eta + (\nabla \xi)\eta$

($D' = 1 \otimes D + \nabla \otimes 1$)

Self-Morita equivalences

trivial one with $\mathcal{E} = \mathcal{A}$

(some algebras also have non-trivial self Morita equivalences)

then $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \cong \mathcal{H}$ here

and only change is in Dirac operator

with connection: gauge potential on $\mathcal{A}$

$$D \longmapsto D + A \qquad A = \sum_i a_i [D, b_i]$$

a self adjoint element of $\Omega^1_D(\mathcal{A})$

$$A^* = A$$

(so new Dirac op. still self-adj.)

If have also a real structure J on $(\mathcal{A}, \mathcal{H}, D)$

$$D \longmapsto D + A + \varepsilon' JAJ^{-1}$$

sign $\pm$ of

$$JD = \varepsilon' DJ$$

So that new Dirac still same commutation relation

Action of unitaries on gauge potentials

$D + A + \varepsilon' JAJ^{-1}$

$u \in \mathcal{U}(\mathcal{A})$ unitary

$$\mathrm{Ad}(u)\,(D + A + \varepsilon' JAJ^{-1})\,\mathrm{Ad}(u^*) = D + \gamma_u(A) + \varepsilon' J\gamma_u(A)J^{-1}$$

$\mathrm{Ad}(u) = u\, JuJ^{-1}$

(right & left action)

$$\gamma_u(A) = u[D, u^*] + uAu^*$$

$U = uv = u\,JuJ^{-1}$

$UDU^* = u(vDv^*)u^* = u(D + v[D,v^*])u^*$

$= uDu^* + v[D,v^*] = D + u[D,u^*] + v[D,v^*]$

& $v[D,v^*] = JuJ^{-1}[D, Ju^*J^{-1}] = \varepsilon' Ju[D,u^*]J^{-1}$

Note: Inner fluctuations NOT an equivalence relation e.g. fin dim example, can fluctuate D so that $D + A = 0$ but then no $a[D', b] \neq 0$ if $D' = 0$ so cannot fluctuate back to D : not symmetric

Spectral action functional for finitely summable spectral triples

$(\mathcal{A}, \mathcal{H}, D)$ possibly with additional γ, J

$\mathrm{Tr}\left(f\left(\frac{D}{\Lambda}\right)\right)$ D = Dirac operator
$\Lambda \in \mathbb{R}_+^*$ energy scale so $\frac{D}{\Lambda}$ dimensionless

f positive real function (even)

Assume that the Dirac operator has heat Kernel expansion

$$\mathrm{Tr}\left(e^{-tD^2}\right) \sim \sum a_\alpha t^\alpha \qquad t \to 0 \qquad (*)$$

if points of dim spectrum are simple
(only simple poles of ζ function)
then indeed no $\log(t)$ terms in expansions

$$\zeta_D(s) = \mathrm{Tr}\left(|D|^{-s}\right)$$

(1) • A non-zero term in $(*)$ $a_\alpha \neq 0$ with $\alpha < 0$
$\Rightarrow$ pole of ζ_D at $s = -2\alpha$

$$\mathrm{Res}_{s=-2\alpha}\, \zeta_D(s) = \frac{2a_\alpha}{\Gamma(-\alpha)}$$

(2) • Since no $\log(t)$ terms in $(*)$ $\Rightarrow$ ζ_D regular at $s=0$

$$\zeta_D(0) + (\dim \mathrm{Ker}\, D) = a_0$$

assume $\mathrm{Ker}(D) = 0$

Pf: $|D|^{-s} = \Delta^{-s/2} = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty e^{-t\Delta}\, t^{\frac{s}{2}-1}\, dt$ (6)

$\Delta = D^2$

Mellin transform

(1) $\int_0^1 t^{\alpha + \frac{s}{2} - 1}\, dt = (\alpha + \frac{s}{2})^{-1}$

(2) $\frac{1}{\Gamma(\frac{s}{2})} \sim \frac{s}{2}$ $s \to 0$

$\Rightarrow$ pole part at $s=0$ of

$\int_0^\infty Tr(e^{-t\Delta})\, t^{\frac{s}{2}-1}\, dt$ given by

$a_0 \int_0^1 t^{\frac{s}{2}-1} dt = a_0 \frac{2}{s}$

gives $\zeta_D(0)$

Asymptotic expansion of the spectral action

$$Tr\left(f\left(\frac{D}{\Lambda}\right)\right) \sim \sum_{\beta \in \underbrace{Sp(\mathcal{A},H,D)}_{\text{(poles of } \zeta_D) \geq 0}} f_\beta \Lambda^\beta ⨍ |D|^{-\beta} + f(0)\zeta_D(0)$$

$+ \cdots$ neg. terms $\Lambda^{-\beta}$

DimSp ≤ 0

$$f_\beta = \int_0^\infty f(v)\, v^{\beta-1}\, dv$$

$2n$

Following terms are

$f_{-2k}\, a_{2k}\, \Lambda^{-2k}$

$-2k : 4+2k'$

$$f_{-2k} = (-1)^k \frac{(2+k)!}{(4+2k)!} f^{(4+2k)}(0)$$ (dim 4)

a_{2k} again residues

Suppressed for $\Lambda \to \infty$