X spin-manifold compact   Tue Feb 23   1

Riemannian mfld. s.t. $SO(n)$-bundle of frames lifts to $Spin(n)$-bundle $Spin(n) \to SO(n)$ univ. cov. $\mathbb{Z}/2\mathbb{Z}$

Spinor bundle S repres. $\mathbb{C}^{2^{[n/2]}}$ of Clifford algebra
$Spin(n) \subset Cl(n)$

$L^2(X, S)$ = square integrable sections of the spinor bundle

$\mathcal{A} = C^\infty(X)$ acting by multiplication operators (bounded in sup norm since X compact) on $L^2(X, S)$

$D = \not{\partial}$ Dirac operator for the Levi-Civita connection of the metric g

$$g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}$$

η^{ab} flat metric

e^μ_a = tetrad vierbein

Coeff's of Levi-Civita connection

$$\nabla_\mu e_a = \omega_{\mu a}^{\ \ b} e_b$$

i.e. $(\omega_{\mu a}^{\ \ b})$ solutions of

$$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu - \omega_{\mu b}^{\ \ a} e^b_\nu + \omega_{\nu b}^{\ \ a} e^b_\mu = 0$$

Clifford algebra action on $L^2(X, S)$

$$c(dx^\mu) = \gamma^a e^\mu_a = \gamma^\mu(x)$$

γ-matrices $\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab}$

$$\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = -2g(dx^\mu, dx^\nu) = -2g^{\mu\nu}$$

∇^S = Levi Civita connection on spinor bundle

(2)

$$\nabla^S_\mu = \partial_\mu + \omega^S_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b$$

$$\not{D} = \gamma \circ \nabla^S = \gamma(dx^\mu)\nabla^S_\mu$$

$$= \gamma^\mu(x)(\partial_\mu + \omega^S_\mu) = \gamma^a e^\mu_a(\partial_\mu + \omega^S_\mu)$$

$$\not{\partial}^2 = \Delta^S + \frac{1}{4}R$$ ← scalar curvature of metric

↑ Laplacian lifted to spinors

$$\Delta^S = -g^{\mu\nu}(\nabla^S_\mu\nabla^S_\nu - \Gamma^\rho_{\mu\nu}\nabla^S_\rho)$$

Christoffel symbols

$\mathbb{Z}/2\mathbb{Z}$ - grading : when n even

$\gamma = i^{n/2}\gamma^1 \cdots \gamma^n$ anticommutes w/ $\not{\partial}$

$$\gamma\not{\partial} + \not{\partial}\gamma = 0$$

$$\gamma^2 = id \quad \gamma^* = \gamma$$

Recovering metric from $\not{\partial}$ Dirac operator :

$$dist(x,y) = \sup_{f \in A \,:\, \|[\not{\partial}, f]\| \le 1} |f(x) - f(y)|$$

$$[\not{\partial}, f]\psi = (\gamma^\mu \partial_\mu f)\psi$$

$[\not{\partial}, f] = \gamma^\mu \partial_\mu f = c(df)$ Clifford multipl

(bounded op. on a compact manifold)

$$\|[\not{\partial}, f]\| = \sup_{x \in X} |(\gamma^\mu \partial_\mu f)(\gamma^\nu \partial_\nu f)^*|^{1/2}$$

Lipschitz norm of f

$$\|f\|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{dist_\gamma(x,y)}$$

geod. distance

$$dist_\gamma(x,y) = \inf \text{ length of paths in } X \text{ between } x, y$$

$$\sup | \gamma^{\mu\nu} \partial_\mu f \, \partial_\nu f^* |^{1/2}$$

from $\| [D,f] \| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_\gamma(x,y)} \leq 1$

$$\Rightarrow \sup_{\|[D,f]\| \leq 1} |f(x) - f(y)| \leq d_\gamma(x,y)$$

reverse

$$f_{\gamma,y}(x) := d_\gamma(x,y)$$

$$\Rightarrow \| [D, f_{\gamma,y}] \| \leq 1$$

$$\Rightarrow \sup_{\|[D,f]\| \leq 1} |f(x) - f(y)| \geq | f_{\gamma,y}(x) - f_{\gamma,y}(y) | = d_\gamma(x,y)$$

Note: $[\not{D}, [\not{D}, f]] = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu f$

$$+ (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)(\partial_\nu f) \partial_\mu$$

not a bounded operator!

but $|\not{D}|$ works instead

$[|\not{D}|, [|\not{D}|, f]]$ bounded

explicit expr. in terms of principal symbol

NC case

$(\mathcal{A}, \mathcal{H}, D)$

$\phi, \psi : \mathcal{A} \to \mathbb{C}$ states

$$\mathrm{dist}(\phi, \psi) = \sup_{\substack{a \in \mathcal{A} \\ \|[D,a]\| \leq 1}} |\phi(a) - \psi(a)|$$

distance function on "points"

($X_1 \times X_2$ spaces)

Product of spectral triples

$(\mathcal{A}_1, \mathcal{H}_1, D_1)$ $(\mathcal{A}_2, \mathcal{H}_2, D_2)$

$\downarrow$

suppose even γ_1

$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$D = D_1 \otimes 1 + \gamma_1 \otimes D_2$

if both even have also choice of Dirac

$D_1 \otimes \gamma_2 + 1 \otimes D_2$

these two choices are unitarily equivalent

Unitary equivalence

$U : \mathcal{H} \to \mathcal{H}'$ unitary equiv.

$\pi'(a) = U \pi(a) U^*$

$D' = U D U^*$ $\quad \gamma' = U \gamma U^*$ $\quad J' = U J U^*$

Differential forms (gauge potentials)

$\Omega_D A \qquad a_0[D,a_1]\cdots[D,a_p]$ ⊛

$[D,a]^* = -[D,a^*]$

Universal differential forms

$\Omega^p A = \Omega^1(A)\otimes_A \cdots \otimes_A \Omega^1(A)$

$\Omega^1 A = \mathrm{Ker}(m : A\otimes_{\mathbb{C}} A \to A)$

$\delta a \longleftrightarrow 1\otimes_{\mathbb{C}} a - a\otimes_{\mathbb{C}} 1$

$\sum a_i \delta b_i \longleftrightarrow \sum a_i(1\otimes_{\mathbb{C}} b_i - b_i\otimes_{\mathbb{C}} 1)$

$\delta : A \to \Omega^1 A$

$a \mapsto 1\otimes_{\mathbb{C}} a - a\otimes_{\mathbb{C}} 1$

($\sum a_i b_i = 0$
$\Downarrow$
$\sum a_i \otimes_{\mathbb{C}} b_i = \sum a_i(-b_i\otimes_{\mathbb{C}} 1 + 1\otimes_{\mathbb{C}} b_i)$)

submod of $A\otimes_{\mathbb{C}} A$ (bimod.)

$\pi: \quad a_0\delta a_1\cdots\delta a_n \longmapsto a_0[D,a_1]\cdots[D,a_n]$

$\Omega_D^*(A) = \Omega(A)/J_D \qquad J_D = \{\omega + \delta\omega \, ; \, \pi(\omega)=0\}$, $\omega \in \Omega(A)$

Note: probl. w/ just ⊛ as def. of p-forms
can have $\omega\in\Omega(A)$ w/ $\pi(\omega)=0$ but
$\pi(\delta\omega)\neq 0$
need to mod these out

then

$d: \Omega_D^p(A) \longrightarrow \Omega_D^{p+1}(A)$

$d[\omega] = [\delta\omega] = [\pi(\delta\omega)]$

e.g. 2-forms are

$\sum_j a_0^j[D,a_1^j][D,a_2^j]$ modulo those

$\sum [D,b_0^j][D,b_1^j]$ s.t. $\sum b_0^j[D,b_1^j] = 0$

Spectral triples on the NC torus

- $\mathcal{H} = L^2(\mathcal{A}_\theta, \tau)$ $\quad\quad \tau$ = trace on $\mathcal{A}_\theta$

 = GNS representation Hilbert space of the state (= trace) τ

$$\|a\|_2 = \sqrt{\tau(a^* a)} \quad \text{norm}$$

(no ker since τ faithful)
$\mathcal{A}_\theta \hookrightarrow \mathcal{H}$ inject.

- action of $\mathcal{A}_\theta$ on $\mathcal{H}$ through GNS rep.

- $J_0(\xi_a) = \xi_{a^*}$ $\quad \xi_a \in \mathcal{H}$ $\quad a \in \mathcal{A}_\theta$

$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ two copies of same $L^2(\mathcal{A}_\theta, \tau)$
w/ $\pi(a) = \begin{pmatrix} \pi^+(a) & 0 \\ 0 & \pi^-(a) \end{pmatrix}$ diag. action

$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ grading

$$J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}$$

$$D = -i \begin{pmatrix} 0 & \partial_\tau^* \\ \partial_\tau & 0 \end{pmatrix} \quad \text{where}$$

$$\partial_\tau = \delta_1 + \tau \delta_2 \quad\quad \tau \in \mathbb{C}$$

(can assume $\mathrm{Im}(\tau) \geq 0$ since can replace by $\tau^{-1}\partial_\tau$ otherwise)

$$\partial_\tau^* = -\delta_1 - \bar{\tau}\delta_2$$

Will have to exclude $\tau \in \mathbb{R}$ (see later)

$\therefore$ — $$[D, \pi(a)] = -i\begin{pmatrix} 0 & -\partial_\tau^*(a) \\ \partial_\tau(a) & 0 \end{pmatrix}$$

bounded

— $[J_0 \partial_\tau J_0, a^*] = (\partial_\tau a)^* = -\partial_\tau^* a^* = -[\partial_\tau^*, a^*]$

$$\Rightarrow \quad J_0 \partial_\tau J_0 = -\partial_\tau^* \quad \Rightarrow J D J^* = D$$

— first order cond. satisfied (Tomita's thm)

$[D, \pi(a)] \in \pi^o(A_\theta)'$ commutant

left action — right action

— $|D_\tau|^{-1}$ infinitesimal of order 2
(metrically 2-dim geometry)

$$\partial_\tau^* \partial_\tau (U^m V^n) = \partial_\tau \partial_\tau^* (U^m V^n)$$
$$= -(\delta_1 + \tau \delta_2)(\delta_1 + \bar{\tau}\delta_2)(U^m V^n)$$
$$= 4\pi^2 |m + \tau n|^2 U^m V^n$$

$$D_\tau^2 = \partial_\tau^* \partial_\tau \oplus \partial_\tau \partial_\tau^*$$

$$\sum_{(m,n) \neq (0,0)} |m + n\tau|^{-2} \quad \text{diverges logarithmically}$$

property of Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k}$$

for $\underline{\Im(\tau) > 0}$

Volume (area) of NC torus

$$⨍ |D_\tau|^{-2} = \frac{2}{4\pi^2} \lim_{R\to\infty} \frac{1}{2\log R} \sum_{\substack{m^2+n^2\le R^2 \\ (m,n)\neq(0,0)}} |m+n\tau|^{-2}$$

$$= \frac{1}{4\pi^2} \lim_{R\to\infty} \frac{1}{\log R} \int_1^R \frac{r\,dr}{r^2} \int_{-\pi}^{\pi} \frac{d\theta}{(\cos\theta + s\sin\theta)^2 + t^2\sin^2\theta}$$

$\tau = s + it$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \frac{d\theta}{(\cos\theta + s\sin\theta)^2 + t^2\sin^2\theta}$$

$$= \frac{1}{4\pi^2}\left(\frac{2\pi}{t}\right) = \frac{i}{\pi(\tau - \bar\tau)}$$

Note: when τ approaches real axis
Volume $(A_\theta, D_\tau) \longrightarrow \infty$

$$2\pi ⨍ |D_\tau|^{-2} = \frac{1}{\mathrm{Im}(\tau)}$$

Isospectral deformations (Connes – Landi)

$$(C^\infty(X),\ \underset{\mathcal{H}}{\underset{\parallel}{L^2(X,S)}},\ \not{D}\ )$$ commut. sp. triple

Suppose X has T^2-action by isometries

$\mathcal{A}_{\theta,X}$ NC algebra obtained:

$$\pi(f) \in B(\mathcal{H})$$

$$\pi(f) = \sum_{n,m\in\mathbb{Z}} \pi(f_{n,m}) \quad \text{with}$$

$$\alpha_\tau(\pi(f_{n,m})) = e^{2\pi i(n\tau_1 + m\tau_2)}\, \pi(f_{n,m})$$

$$\tau = (\tau_1,\tau_2) \in T^2 = S^1 \times S^1$$

where

$$\alpha_\tau(T) = U(\tau)\, T\, U(\tau)^*$$

$U(\tau)$ unitary in $B(\mathcal{H})$

$$U(\tau)\,\psi(x) = \psi(\tau^{-1}(x))$$ implementing action of T^2 on $\mathcal{H} = L^2(X,S)$

$$U(\tau) = \exp(2\pi i\, \tau L) = \exp(2\pi i(\tau_1 L_1 + \tau_2 L_2))$$

L: infinitesimal generators of the action

Algebra generated by (in $B(\mathcal{H})$)

$$\pi_{\xi_1,\xi_2}(f) = \sum_{n,m} \pi(f_{n,m})\, e^{-2\pi i(\xi_1 n L_2 + \xi_2 m L_1)}$$

operator prod becomes

$$\pi(f_{n,m}) *_{\xi_1,\xi_2} \pi(h_{k,r}) = e^{-2\pi i(\xi_1 n r + \xi_2 m k)}\, \pi(f_{n,m})\, \pi(h_{k,r})$$

$U(\tau) D U(\tau)^* = D$ since isometries

$$[\, D, \pi_{\xi_1, \xi_2}(f)\,] = \sum_{n,m} [D, \pi(f)]_{n,m}\, e^{-2\pi i(\xi_1 n l_2 + \xi_2 m l_1)}$$

bounded

$(-\xi_1 = +\xi_2 = \theta/2)$ gives back A_θ when $X = T^2$