Thursday Feb 18

$K_0(A)$ description via idempotents

$e \in M_k(A)$ (proj. module eA^k)

- equivalent $e_1 \sim e_2$ if $\exists\, u \in Gl(k, A)$

  $e_1 = u e_2 u^{-1}$

- stably equiv. if

$M_k(A) \hookrightarrow M_{k+1}(A) \qquad a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

$M(A) = \bigcup_k M_k(A) \qquad GL(A) = \varinjlim_{n} GL(k, A)$

$e_1 \overset{s}{\sim} e_2$ iff related by $e_1 = u e_2 u^{-1}$ in $M(A)$ w/ $u \in GL(A)$

equiv. classes monoid under

$(e, f) \mapsto e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \quad \rightsquigarrow K_0$ - Groth group

<u>Note</u>: $\mathcal{E}_1 = e_1 A^k \quad \mathcal{E}_2 = e_2 A^k$ isomorphic iff $e_1 \overset{s}{\sim} e_2$

$HC^{2n}(A) \otimes K_0(A) \longrightarrow \mathbb{C}$ bilinear map

$\quad\; \psi \qquad\qquad\; \psi$

$\quad [\varphi] \qquad\quad\; [e]$

$\tilde{\varphi}(m_0 \otimes a_0, \ldots, m_{2n} \otimes a_{2n})$

$= tr(m_0 \cdots m_{2n})\, \varphi(a_0, \ldots, a_{2n})$

$\langle [\varphi], [e] \rangle = \frac{1}{n!} \tilde{\varphi}(e, \ldots, e)$

well defined on equiv. classes: if by

$b\psi(e, \ldots, e) = \psi(ee, e, \ldots, e) - \psi(e, ee, \ldots, e) + \cdots$

$e^2 = e \Rightarrow \quad = 0$

(because ψ cyclic so $\psi(e, \ldots, e) = 0$)

In terms of (b,B)-bicomplex under quasi-isomorphism

$$\langle [\varphi], [e] \rangle = \sum_{k=1}^{n} (-1)^k \frac{k!}{(2k)!} \varphi_{2k}(e-\tfrac{1}{2}, e, \ldots, e)$$

$\varphi = (\varphi_0, \varphi_2, \ldots, \varphi_{2n})$

Well defined pairing on $HP^0(A)$!

$$\langle [\varphi], [e] \rangle = \langle S[\varphi], [e] \rangle$$

$$S[\varphi] = (2n+1)(2n+2)\left[b(1-\lambda)^{-1} b' N^{-1} \varphi \right]$$

$N^{-1}\varphi = \frac{1}{2n+1}\varphi$ since φ cyclic

$$(1-\lambda)^{-1} b' \varphi = \frac{-1}{2n+2} (1 + 2\lambda + 3\lambda^2 + \cdots + (2n+2)\lambda^{2n+1}) b' \varphi$$

$$\Rightarrow S\varphi(e, \ldots, e) = -b(1+2\lambda+3\lambda^2+\cdots) b' \varphi(e, \ldots, e)$$
$$= (n+1) b' \varphi(e, \ldots, e) = (n+1)\varphi(e, \ldots, e)$$

So $\langle S[\varphi], [e] \rangle = \frac{1}{(n+1)!}(S\varphi)(e, \ldots, e) = \frac{1}{n!}\varphi(e, \ldots, e) = \langle [\varphi], [e] \rangle$

Similarly invariance under $[e]$ in class:

$\langle [\varphi], [ueu^{-1}] \rangle = \langle [\varphi], [e] \rangle$ enough check for $k=1$;

u = inner automorphism $u \in \mathcal{U}(A)$

$\Rightarrow$ identity on Hochschild & cyclic

$$\varphi \overset{U}{\longmapsto} u^*(\varphi)(a_0, \ldots, a_n) = \varphi(ua_0u^{-1}, \ldots, ua_nu^{-1})$$

htopy $h_i(a_0 \otimes \cdots \otimes a_n) = a_0u^{-1} \otimes ua_1u^{-1} \otimes \cdots \otimes \overset{(i)}{u} \otimes a_{i+1} \otimes \cdots \otimes a_n$

$h = \sum (-1)^i h_i$ htpy between id and $\mathcal{U} : C^n(A) \to C^n(A)$

odd case :

$$K_1(A) = Gl(A) \Big/ \overline{[Gl(A), Gl(A)]}$$

(topological K_1) closure of commutator subgroup

for $u \in M_k(A)$ representative of an element $[u] \in K_1(A)$

$$\Rightarrow \langle [\varphi], [u] \rangle = \frac{2^{-(2n+1)}}{(n-\frac{1}{2})\cdots\frac{1}{2}} \tilde{\varphi}(u^{-1}-1, u-1, \dots u^{-1}-1, u-1)$$

$\tilde{\varphi}$ as above

again $\langle [\varphi], [u] \rangle = \langle S[\varphi], [u] \rangle$

Chern characters (Connes)

$$Ch_0^{2n} : K_0(A) \longrightarrow HC_{2n}(A)$$

$$Ch_1^{2n+1} : K_1(A) \longrightarrow HC_{2n+1}(A)$$

$$Ch_0^{2n}(e) = \frac{1}{n!} Tr(\underbrace{e \otimes \cdots \otimes e}_{2n+1}) = \sum_{i_0, i_1, \dots, i_{2n}} e_{i_0 i_1} \otimes e_{i_1 i_2} \otimes \cdots \otimes e_{i_{2n} i_0}$$

$$Ch_0^0(e) = \sum_{i=1}^{k} e_{ii}$$

$$Ch_0^2(e) = \sum_{i_0, i_1, i_2} e_{i_0 i_1} \otimes e_{i_1 i_2} \otimes e_{i_2 i_0}$$

$$b\, Ch_0^{2n}(e) = \frac{1}{2}(1-\lambda)\, Tr(\underbrace{e \otimes \cdots \otimes e}_{2n}) \quad \Rightarrow \text{cycle}$$

$$Ch_1^{2n+1}(u) = Tr(\underbrace{(u^{-1}-1) \otimes (u-1) \otimes (u^{-1}-1) \otimes \cdots \otimes u-1}_{2n+2})$$

e.g. $\langle [\varphi],[u] \rangle = \varphi(u,u^{-1}) = \int_{S^1} u\,du^{-1} = -2\pi i$

$\varphi(f_0,f_1) = \int_{S^1} f^0 df_1$ $\quad \varphi(1,f) = \varphi(f,1) = 0$

K-homology and Chern-Characters

K-homol. dual to topl. K-theory (Atiyah)

"abstract elliptic operators" (index theorem)

$(\mathcal{H},F)$ $\quad H = H^+ \oplus H^-$ (even case)

$\pi : A \to B(\mathcal{H})$ $\quad \pi(a) = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{pmatrix}$

$F : H \to H$ $\quad F = \begin{pmatrix} 0 & F^+ \\ F^- & 0 \end{pmatrix}$ bounded self-adjoint $F^* = F$ or $F^2 = Id$

$F^2 - Id \in K(\mathcal{H})$ compact

and

$[F, \pi(a)] \in K(\mathcal{H})$

p. summable if $\in \mathcal{L}^p(\mathcal{H})$

$\langle (\mathcal{H},F), [e] \rangle = \text{index}(F_e^+)$

$F_e^+ = eFe : e\mathcal{H}^+ \to e\mathcal{H}^-$ $\quad$ Fredholm operator:

$\dim \operatorname{Ker} F_e^+ < \infty$

$\dim \operatorname{CoKer} F_e^+ < \infty$

$\text{index}(F_e^+) = \dim \operatorname{Ker} F_e^+ - \dim \operatorname{CoKer} F_e^+ \in \mathbb{Z}$

Odd case: (H,F) p-summable odd Fredholm module

$$2n \geq p$$

$$\varphi^{(H,F)}_{2n-1}(a_0,\dots,a_{2n-1}) := \mathrm{Tr}(F[F,\pi(a_0)]\cdots[F,\pi(a_{2n-1})])$$

$$[F,a] \in \mathcal{L}^p \qquad (\mathcal{L}^p \text{ two sided ideal})$$

$$A_i \in \mathcal{L}^p \Rightarrow A_1\cdots A_p \in \mathcal{L}^1 = \text{Trace class}$$

cyclic $(2n-1)$-cocycle

$$(b\varphi_{2n-1})(a_0,\dots,a_{2n}) = \mathrm{Tr}\left(\sum (-1)^i F\,da_0\cdots d(a_i\,da_{i+1})\cdots da_{2n}\right) + (-1)^{2n+1}\,\mathrm{Tr}(F\,d(a_{2n}a_0)\,da_1\cdots da_{2n})$$

for $da = [F,a]$ satisfying $d(ab) = a\,db + da\cdot b$

$\Rightarrow$ Leibnitz rule gives $b\varphi_{2n-1} = 0$

as in case of $\int_X f^0 df^1 \wedge \cdots \wedge df^n$

Also φ_{2n-1} is cyclic (from cyclic property of trace) and $F\,da = -da\,F$

$(-1)^m\, 2(m-\tfrac{1}{2})\cdots \tfrac{1}{2}$

$$S\varphi^{(H,F)}_{2m-1} = -(m+\tfrac{1}{2})\,\varphi^{(H,F)}_{2m+1}$$

$$Ch^{2m-1}(H,F) = \varphi^{(H,F)}_{2m-1}\cdot(\;\;) \qquad S\,Ch^{2m-1} = Ch^{2m+1}$$

$$Ch^{2m+1} : K^{odd}(A) \longrightarrow HC^{2m+1}(A)$$

in fact defines element in $HP^{odd}(A)$

Even case $\quad 2m \geq p$

$$\varphi_{2m}^{(H,F)}(a_0, a_1, \dots a_{2m}) = Tr(\gamma F [F, a_0] \cdots [F, a_{2m}])$$

cyclic $2m$-cocycle $\qquad$ use also $(\gamma da = -da\, \gamma)$

$$S \varphi_{2m}^{(H,F)} = -(m+1) \varphi_{2m+2}^{(H,F)}$$

$$Ch^{2m}(H,F) = \frac{(-1)^m m!}{2} \varphi_{2m}^{(H,F)}$$

$$S(Ch^{2m}) = Ch^{2m+2}$$

$$Ch^{2m} : K^{even}(A) \longrightarrow HC^{2m}(A)$$

in $HP^{even}(A)$

Pairing of K-homol. & K-theory

$$\begin{cases} \langle (H,F,\gamma), [e] \rangle = \mathrm{Index}(F_e^+) \\ \langle (H,F), [u] \rangle = \mathrm{Index}(PUP) \qquad P = \frac{F+1}{2} \text{ projector} \end{cases}$$

Thm (Connes):

- $\langle (H,F,\gamma), [e] \rangle = \langle Ch^{2n}(H,F,\gamma), Ch_{2n}[e] \rangle$
- $\langle (H,F), [u] \rangle = \langle Ch^{2n-1}(H,F), Ch_{2n-1}[u] \rangle$

$$\begin{array}{ccccc} K^*(A) & \times & K_*(A) & \longrightarrow & \mathbb{Z} \\ {\scriptstyle Ch^*}\downarrow & & \downarrow{\scriptstyle Ch_*} & & \downarrow\int \\ HC^*(A) & \times & HC_*(A) & \longrightarrow & \mathbb{C} \end{array}$$

<u>check</u>:

$$\mathrm{Index}(F_e^+) = \frac{(-1)^n}{2}\varphi_{2n}^{(H,F)}(e,e,\dots,e)$$
$$= \frac{(-1)^n}{2}\mathrm{Tr}(\gamma F[F,e]\cdots[F,e])$$

$\mathrm{Index}(F_e^+) = \mathrm{Tr}(\gamma(e-(eFe)^2)^{n+1})$ since

generally $P': H' \to H''$ Fredholm & $Q': H'' \to H'$ s.t. $1-P'Q' \in \mathcal{L}^{n+1}(H'')$ and $1-Q'P' \in \mathcal{L}^{n+1}(H')$ some n

$\Rightarrow$ $\mathrm{index}(P') = \mathrm{Tr}(1-Q'P')^{n+1} - \mathrm{Tr}((1-P'Q')^{n+1})$
$= \mathrm{Tr}(\gamma'(1-(F')^2)^{n+1})$

$F' = \begin{pmatrix} 0 & Q' \\ P' & 0 \end{pmatrix}$ $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$(H' = e_0 H \quad H'' = e_1 H \quad H' \oplus H'' = eH)$

Then use $e-(eFe)^2 = eF(Fe-eF)e = (eF-Fe+Fe)(Fe-eF)e$
$= -e\,de\,de$

Similarly need to check that

$$\text{Index}(uPu) = \frac{(-1)^n}{2^{2n}} \varphi_{2n-1}^{(H,F)}(u^{-1}, u, \ldots, u^{-1}, u)$$

with $\varphi_{2n-1}^{(H,F)}(a_0, \ldots, a_{2n-1}) = \text{Tr}(F[F,a_0] \cdots [F, a_{2n-1}])$

$P = \frac{F+1}{2}$ $\quad H' = PH \quad P' = PuP : H' \to H'$

$Q' = Pu^{-1}P : H' \to H'$

$du = [F, u]$

$$\begin{aligned} 1 - Q'P' &= 1 - Pu^{-1}PPuP \\ &= -\frac{1}{2} Pu^{-1}du\, P \\ &= -\frac{1}{4} P\, du^{-1} du \end{aligned}$$

(using $P\, du^{-1} P = 0$)

$[P, u^{-1}] = P[P, u^{-1}] + [P, u^{-1}]P$

$P^2 = P$

similarly get $1 - P'Q' = -\frac{1}{4} P du du^{-1}$

$$\begin{aligned} \text{Index}(PuP) &= \text{Tr}((1 - Q'P')^n) - \text{Tr}((1 - P'Q')^n) \\ &= \frac{(-1)^n}{2^{2n}} \text{Tr}\left(\frac{1+F}{2}(du^{-1}du)^n\right) \\ &\quad - \frac{(-1)^n}{2^{2n}} \text{Tr}\left(\frac{1+F}{2}(du\, du^{-1})^n\right) \\ &= \frac{(-1)^n}{2^{2n}} \text{Tr}(F(du^{-1}du)^n) \end{aligned}$$

Spectral triples

* Refinement of Fredholm modules (H, F) or (H, F, γ)
(local index formula)

* NCG analog of Riemannian geometry
(Spin$_c$ manifolds)

(A, H, D) (in fact w/ additional data (A, H, D, J, γ))

$\pi : A \longrightarrow B(\mathcal{H})$

"case of compact manifold"

$D^* = D$ $\quad (D - \lambda)^{-1}$ compact $\quad \lambda \notin \mathbb{R}$

compact resolvent

$a \in \mathcal{A} \Rightarrow f(a) \in \mathcal{A}$

stable under holom. funct. calculus

$f(a) = \frac{1}{2\pi i} \oint_\gamma f(z)(z-a)^{-1}$

f holom around $Sp(a)$

$[D, \pi(a)] \in B(\mathcal{H})$ for all $a \in$ dense subalg. of A

e.g. $\mathcal{A} = \bigcap_{k=1}^{\infty} \mathrm{Dom}(\delta^k)$

δ = derivation w/ dense domain

$(D = F|D| \rightsquigarrow (\mathcal{H}, F)$ Fredholm module$)$

even:

$\gamma^2 = 1$ $\quad [a, \gamma] = 0 \quad D\gamma + \gamma D = 0 \quad D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}$

$\gamma^* = \gamma$ $\quad \pi(a) = \begin{pmatrix} \pi^+(a) & 0 \\ 0 & \pi^-(a) \end{pmatrix}$

* Real structure J (additional)

Real structure J:

antilinear isometry $J: \mathcal{H} \to \mathcal{H}$

with
$$\left.\begin{aligned} J^2 &= \pm 1 \\ JD &= \pm DJ \\ J\gamma &= \pm \gamma J \end{aligned}\right\} \text{ 3 signs}$$

$\pi^\circ(a) = J\,\pi(a^*)\,J^{-1}$ gives another representation of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H})$

Require that
$$[\pi(a), \pi^\circ(b)] = 0 \qquad \forall a, b \in \mathcal{A}$$

i.e. a real structure J makes $\mathcal{H}$ into a bimodule over $\mathcal{A}$ with π, π° left and right actions

Also require order-one condition on Dirac operator:

$$[[D, \pi(a)], \pi^\circ(b)] = 0 \qquad \forall a, b \in \mathcal{A}$$

by Jacobi identity:
$$[a, [D, b^\circ]] = [[a, D], b^\circ] + [D, [a, b^\circ]] = -[[D, a], b^\circ] = 0$$

In commutative case:
$$[[\not{D}, f], h] = -i\,[c(df), h] = 0$$
Clifford multipl. by df

while second order Laplacian satisfies
$$[[\Delta, f], h] = 2\, g^{-1}(df, dh) \neq 0$$
that's why order one condition

For simplicity assume D has trivial kernel
(use $D+\varepsilon$ otherwise)

$$ds := |D|^{-1} \quad \text{"infinitesimal length element"}$$

compact operator (discrete spectrum λ_n)

"Metric dimension of the NC space $(\mathcal{A}, \mathcal{H}, D)$"
rate of growth of eigenvalues of D i.e.

$$\lambda_n = O(k^{-\alpha}) \quad : \; ds \text{ is an "infinitesimal of order } \alpha\text{"}$$

if $\lambda_n = O(\frac{1}{n})$ then $\sigma_N(D^{-1}) = \sum_{k<N} \lambda_k = O(\log N)$
"infinitesimal order 1" logarithmically divergent series

Metric dimension $= n$ if D^{-n} is infinitesimal of order one

$\Rightarrow$ eigenv. of $|D|^{-n}$ form log divergent series

Can compute Dixmier trace $Tr_\omega(|D|^{-n}) =: \int\!\!\!\!\!- |D|^{-n}$
$= \text{coefficient of log divergence} = \lim_{N\to\infty} \frac{\sigma_N(D^{-n})}{\log N}$

This is analog in NCG of integrating on a mfld
using the volume form

$$\int_X f \, dvol. \quad \leftsquigarrow \quad \int\!\!\!\!\!- a|D|^{-n} = Tr_\omega(a|D|^{-n})$$

<u>Note</u>: this can also be expressed as a residue

$$\zeta_D(s) = Tr(|D|^{-s})$$

if $|D|^{-n}$ has eigenv. that form a log divergent series $\lambda_n \sim \frac{1}{n}$
then $Tr(|D|^{-s})$ has a pole at $s=n$

$$\operatorname{Res}_{s=n} \zeta_D(s) = \int\!\!\!\!\!- |D|^{-n}$$

More refined notion of dimension:

Dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$

$\Sigma \subset \mathbb{C}$ — not one number but a set of complex numbers each a "dimension"

Σ = set of poles of $\zeta_{a,D}(s) = \mathrm{Tr}(a\,|D|^{-s})$

$$\operatorname*{Res}_{s \in \Sigma} \zeta_{a,D}(s) = \int\!\!\!\!\!\!- \; a\,|D|^{-s}$$

an integration theory in that "dimension" w/ a different "volume form"

A third notion of dimension: KO-dimension

defined modulo 8 coming from algebraic properties of the real structure J

n even

n mod 8	0	2	4	6
$J^2 = \pm 1$	+	−	−	+
$JD = \pm DJ$	+	+	+	+
$J\gamma = \pm \gamma J$	+	−	+	−

n odd

n mod 8	1	3	5	7
$J^2 = \pm 1$	+	−	−	+
$JD = \pm DJ$	−	+	−	+

These signs come from the commutative case where they distinguish the real Clifford algebra representations (real Bott periodicity mod 8)

Metric and KO dim (mod 8) agree in commutative case; not necessarily in NC case (example of standard model)