Hochschild cohomology Tue Feb 16

Generalization of group cohomology to algebras
(helps classif. of algebra extensions; deformation theory of assoc. algebras)

A alg. over $\mathbb{C}$ M A-bimodule

Complex: $C^0(A,M) \xrightarrow{\delta} C^1(A,M) \xrightarrow{\delta} C^2(A,M) \xrightarrow{\delta} \cdots$

$\otimes = \otimes_{\mathbb{C}}$:
Hom = $\mathrm{Hom}_{\mathbb{C}}$

$C^0(A,M) = M$ $C^n(A,M) = \mathrm{Hom}(A^{\otimes n}, M)$ $n \geq 1$

$\delta : C^n(A,M) \to C^{n+1}(A,M)$ given by

$(\delta m)(a) = ma - am$ (difference of left & right action)

$$(\delta f)(a_1,\dots,a_{n+1}) = a_1 f(a_2,\dots,a_{n+1}) + \sum_{i=1}^{n} (-1)^{i+1} f(a_1,\dots,a_i a_{i+1},\dots a_{n+1}) + (-1)^{n+1} f(a_1,\dots,a_n)\, a_{n+1}$$

lemma It is a differential

$\boxed{\delta^2 = 0}$

Hochschild cohom.

$$H^n(A,M) = H^n(C^*(A,M), \delta)$$

Particular cases: $M = A$ Gerstenhaber complex
used to study deformation theory of algebras

$\longrightarrow$ $M = A^* = \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$ linear dual of A (algebraic)
bimodule with $afb\,(c) := f(bca)$

$$\mathrm{Hom}(A^{\otimes n}, A^*) \cong \mathrm{Hom}(A^{\otimes(n+1)}, \mathbb{C})$$
$$f \longmapsto \varphi$$

$$\varphi(a_0, a_1,\dots,a_n) = f(a_1,\dots,a_n)(a_0)$$

δ differential rewritten in these terms as
b-differential

$$(b\varphi)(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots a_{n+1})$$
$$+ (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \ldots, a_n)$$

in particular

$$(b\varphi)(a_0, a_1) = \varphi(a_0 a_1) - \varphi(a_1 a_0)$$

$$(b\varphi)(a_0, a_1, a_2) = \varphi(a_0 a_1, a_2) - \varphi(a_0, a_1 a_2) + \varphi(a_2 a_0, a_1)$$

$$(b\varphi)(a_0, a_1, a_2, a_3) = \varphi(a_0 a_1, a_2, a_3) - \varphi(a_0, a_1 a_2, a_3)$$
$$+ \varphi(a_0, a_1, a_2 a_3) - \varphi(a_3 a_0, a_1, a_2)$$

notation:

$C^*(A) = C^*(A, A^*)$

$HH^*(A) = H^*(A, A^*)$

(*) $HH^0(A) =$ traces on A

(*) $f \in C^1(A, M)$ 1-cocycle $=$ derivation $f : A \to M$

$f(ab) = a f(b) + f(a) b$ Leibnitz rule

coboundary $=$ inner derivation

$\exists\, m \in M$ s.t. $f(a) = ma - am$

$$H^1(A, M) = \frac{\text{derivations}}{\text{inner derivations}} = \text{outer derivations}$$

(*) $H^2(A, M)$ classifies abelian extensions of A by M

$$0 \to M \to B \xrightarrow{\pi} A \to 0$$

trivially made into an algebra by zero multipl $M^2 = 0$

$s : A \to B$ linear splitting of (proj) π

$f : A \otimes A \to M$

$f(a,b) = s(ab) - s(a)s(b)$ how much it fails to be an alg. homom

$\Rightarrow f$ Hochschild 2-cocycle (class indep of choice of s) differ by coboundary

Or conversely given 2-cocycle $f : A \otimes A \to M$

$\circledast$ $(a,m)(a',m') = (aa', am' + ma' + f(a,a'))$

associative product (using 2-cocycle property to give associativity)

$\Rightarrow$ extension $0 \to M \to A \oplus M \to A \to 0$ w/ product structure $\circledast$

$HH^0(\mathbb{C}) = \mathbb{C}$ $HH^n(\mathbb{C}) = 0$ $\forall n \geq 1$

$A = C^\infty(\cancel{X})$ X manifold

$$\varphi(f^0, \ldots, f^n) = \int_X f^0 df^1 \wedge \ldots \wedge df^n$$

$$\varphi : A^{\otimes n} \longrightarrow \mathbb{C}$$

$$(b\varphi)(f^0, \ldots, f^{n+1}) = \sum_{i=0}^{n} (-1)^i \int_X f^0 df^1 \wedge \ldots \wedge d(f^i f^{i+1}) \wedge \ldots \wedge df^{n+1}$$

$$+ (-1)^{n+1} \int_X f^{n+1} f^0 \, df^1 \wedge \ldots \wedge df^n = 0$$

by Leibnitz rule of d de Rham diff.

topological dual of de Rham forms

C m-current

$\Rightarrow$ $\varphi_C(f^0, f^1, \ldots, f^m) = \langle C, f^0 df^1 \wedge \ldots \wedge df^m \rangle$ Hochschild cocycle

$$\Omega_m(X) \hookrightarrow HH^m_{cont}(C^\infty(M))$$

de Rham currents

Connes ('81): isomorphism $HH^m_{cont}(C^\infty(M)) \cong \Omega_m(X)$

compatible differentials

Note: continuous Hochschild cocycles when A Banach algebras

tend to get trivial $H^*(A,M)=0$ $*\geq 1$

because like derivations: have dense domains

only trivial extend to full Banach space

(unbounded operators)

for C^*-alg.: work w/ dense subalg. when doing Hochschild cohom. & cyclic cohomol.

Group cohomology as Hochschild cohomology:

G group M repres. (left $\mathbb{C}G$-linear action)

$$M \xrightarrow{\delta} C^1(G,M) \xrightarrow{\delta} C^2(G,M) \xrightarrow{\delta} \cdots$$

$$C^n(G,M) = \{ f : G^n \to M \}$$

$$(\delta m)(g) = gm - m$$

$$(\delta f)(g_1,\dots,g_{n+1}) = g_1 f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1,\dots,g_i g_{i+1},\dots,g_{n+1})$$
$$+ (-1)^{n+1} f(g_1,g_2,\dots,g_n)$$

is Hochschild for $A = \mathbb{C}[G]$ M bimodule w/

$\delta_g \cdot m = g(m)$ $m \cdot \delta_g = m$ trivial on other side

$$H^n(\mathbb{C}[G],M) \simeq H^n(G,M)$$

Result of Burghelea: direct prod. over conjugacy classes

$$HH^*(\mathbb{C}[G]) \cong \prod_{\langle G \rangle} H^*(C_g)$$

centralizer of conj class

group cohomology $H^*(C_g,\mathbb{C})$

Cyclic cohomology (Connes ~1981)

Start from Hochschild complex

$C^n(A) = \mathrm{Hom}(A^{\otimes n+1}, \mathbb{C})$ with b-differential

n-cochain is cyclic if

$$f(a_n, a_0, \dots, a_{n-1}) = (-1)^n f(a_0, a_1, \dots, a_n)$$

$C^n_\lambda(A)$ = cyclic n-cochains in $C^n(A)$

b-differential restricts to cyclic cochains,

$$(\lambda f)(a_0, \dots, a_n) = (-1)^n f(a_n, a_0, \dots, a_{n-1})$$

$$(b' f)(a_0, \dots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots a_{n+1})$$

(like b but without last term)

$$\Rightarrow \quad (1-\lambda) b = b'(1-\lambda)$$

$$C^n_\lambda(A) = \mathrm{Ker}(1-\lambda)$$

~~a b descends to b~~

$\Rightarrow$ b preserves $\mathrm{Ker}(1-\lambda)$

$$C^0_\lambda(A) \xrightarrow{b} C^1_\lambda(A) \xrightarrow{b} C^2_\lambda(A) \xrightarrow{b} \cdots$$

cyclic complex of A

$HC^n(A) = H^n(C^*_\lambda(A), b)$ cyclic cohomology of A

$A = C^\infty(X)$ $\quad \varphi(f^0, f^1, \dots, f^n) = \int_X f^0 df^1 \wedge \dots \wedge df^n$ is cyclic

i.e. $\varphi(f^n, f^0, \dots, f^{n-1}) = (-1)^n \varphi(f^0, f^1, \dots, f^n)$ Stokes' thm

$$\int_X f^n df^0 \wedge \dots \wedge df^{n-1} - (-1)^n \int_X f^0 df^1 \wedge \dots \wedge df^n = \int_X d(f^n f^0 df^1 \wedge \dots \wedge df^{n-1})$$

if $\partial X = \emptyset$ $\quad \int_X d\omega = \int_{\partial X} \omega = 0$

This gives map

$$Z_m(X,\mathbb{C}) \longrightarrow HC^m(C^\infty(X))$$

C with $\partial C = 0$ cycle $= \mathrm{Ker}(\partial)$

$$\varphi_C(f^0, f^1, \dots, f^m) = \int_C f^0 df^1 \wedge \dots \wedge df^m$$

Note m-current C closed if $\langle C, d\omega\rangle = 0$ $\omega \in \Omega^{m-1}(X)$

if C closed $\partial C = 0$

$\Rightarrow$ φ_C is cyclic (Stokes' theorem applies as before)

So Hochschild cocycle φ_C is cyclic

<u>Example</u> : NC torus A_θ

$\delta_1, \delta_2 : A_\theta \to A_\theta$ derivations $\frac{1}{2\pi i} U \frac{\partial}{\partial U}$; $\frac{1}{2\pi i} V \frac{\partial}{\partial V}$

$$\varphi(a_0, a_1, a_2) = \tau\left(a_0\left(\delta_1(a_1)\,\delta_2(a_2) - \delta_2(a_1)\,\delta_1(a_2)\right)\right)$$

is a cyclic 2-cocycle

<u>Example</u> : $A = \mathbb{C}[G]$ G discrete group

$c : G^n \to \mathbb{C}$ group n-cocycle

$$g_1 c(g_2, \dots, g_{n+1}) - c(g_1 g_2, \dots, g_{n+1}) + \dots + (-1)^{n+1} c(g_1, \dots, g_n) = 0$$

and normalized $c(g_1, \dots, g_n) = 0$ if some $g_i = e$ or $g_1 \cdots g_n = e$

$$\varphi_c(\delta_{g_0}, \dots, \delta_{g_n}) = \begin{cases} c(g_1, \dots, g_n) & \text{if } g_0 g_1 \cdots g_n = e \\ 0 & \text{otherwise} \end{cases}$$

is a cyclic n-cocycle on $A = \mathbb{C}[G]$

Connes' exact sequence of Hochschild and cyclic cohomologies:

$$0 \to C_\lambda \to C \to C/C_\lambda \to 0 \quad \text{exact seq. of complexes}$$

$\Rightarrow$ assoc. long exact seq. in cohomology

$$\cdots \to HC^n(A) \to HH^n(A) \to H^n(C/C_\lambda) \xrightarrow{\delta} HC^{n+1}(A) \to \cdots$$

describe this

another exact seq:

$$0 \to C/C_\lambda \xrightarrow{1-\lambda} (C, b') \xrightarrow{N} C_\lambda \to 0 \qquad (*)$$

$$N = 1 + \lambda + \lambda^2 + \cdots + \lambda^n : C^n \to C^n$$

$$N(1-\lambda) = (1-\lambda)N = 0 \qquad bN = Nb'$$

i.e. $(1-\lambda)$ & N are morphisms of complexes

to show exactness of $(*)$ note that $\mathrm{Ker}(N) \subset \mathrm{Im}(1-\lambda)$ because

$$(1-\lambda)(1 + 2\lambda + 3\lambda^2 + \cdots + (n+1)\lambda^n) = N - (n+1)\,\mathrm{id}$$

for A unital (C, b') complex is exact

contracting homotopy $\quad s : C^n \to C^{n-1}$

$$(s\varphi)(a_0, \ldots, a_{n-1}) = (-1)^{n-1} \varphi(a_0, \ldots, a_{n-1}, 1)$$

$$b's + sb' = \mathrm{id}$$

$$\Rightarrow \cdots \to H^n(C/C_\lambda) \to \underset{\underset{0}{\|}}{H^n(C, b')} \to HC^n(A) \to H^{n+1}(C/C_\lambda) \to \underset{\underset{0}{\|}}{H^{n+1}(C, b')} \to \cdots$$

$$\Rightarrow H^n(C/C_\lambda) \cong HC^{n-1}(A)$$

Get exact sequence

$$\cdots \to HC^n(A) \xrightarrow{I} HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \to \cdots$$

where $B : HH^n(A) \xrightarrow{\pi} H^n(C/C_\lambda) \underset{\cong}{\xrightarrow{\delta^{-1}}} HC^{n-1}(A) \qquad B = (1-\lambda)^{-1} b' N^{-1}$

on cochains $B = Ns(1-\lambda)$

(B, b) – bicomplex (Connes ~1981)

$$B = Ns(1-\lambda) = NB_0 \qquad B_0 : C^n \to C^{n-1}$$

$$(B_0\varphi)(a_0, \dots, a_{n-1}) = \varphi(1, a_0, \dots, a_{n-1}) - (-1)^n \varphi(a_0, \dots, a_{n-1}, 1)$$

$$\Rightarrow \boxed{bB + Bb = 0 \quad \text{and} \quad B^2 = 0}$$

from $(1-\lambda) b = b'(1-\lambda)$ and $(1-\lambda)N = N(1-\lambda) = 0$

and $bN = Nb'$ and $sb' + b's = 1$

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
b \uparrow & & & & \\
C^2(A) & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) \\
b \uparrow & & b \uparrow & & \\
C^1(A) & \xrightarrow{B} & C^0(A) & & \\
b \uparrow & & & & \\
C^0(A) & & & &
\end{array}$$

total complex

$$(C^{*,*}, b, B)$$

$$C^n_{tot} = \bigoplus_{p+q=n} C^{p,q} \qquad d = b + B$$

map $\varphi \mapsto (0, 0, \dots, \varphi)$ quasi isomorphism of complexes

$$(C^*_\lambda(A), b) \xrightarrow{\simeq} (C^*_{tot}(A), b + B)$$ both compute cyclic cohomol.

Periodic cyclic cohomology!

$$HP^i(A) = \varinjlim_S HC^{2n+i}(A) \qquad i = 0, 1$$

(even/odd graded) stabilizing over map S

Example: $A = C^\infty(M)$ bicomplex:

$$\begin{tikzcd}
\cdots & \Omega^2(M) \arrow[l] \arrow[d, "0"'] & \Omega^1(M) \arrow[l, "d"'] \arrow[d, "0"] & \Omega^0(M) \arrow[l, "d"'] \\
\cdots & \Omega^1(M) \arrow[l] \arrow[d, "0"'] & \Omega^0(M) \arrow[l, "d"'] & \\
 & \Omega^0(M) & &
\end{tikzcd}$$

i.e. under $\mu(f_0 \otimes \cdots \otimes f_n) = \frac{1}{n!} f^0 df_1 \wedge \cdots \wedge df_n$

$$\begin{tikzcd}
C_n(A) \arrow[r, "\mu"] \arrow[d, "B"'] & \Omega^n(M) \arrow[d, "d"] \\
C_{n+1}(A) \arrow[r, "\mu"'] & \Omega^{n+1}(M)
\end{tikzcd}$$

$$HC_n(C^\infty(M)) = \frac{\Omega^n M}{\operatorname{Im} d} \oplus H^{n-2}_{dR}(M) \oplus \cdots \oplus H^{0 \text{ or } 1}_{dR}(M)$$

$$HP_n(C^\infty(M)) = \bigoplus_i H^{2i+n}_{dR}(M)$$

Same for $A = \mathcal{O}(X)$ ring of affine alg. variety (smooth)
Hochschild-Kostant-Rosenberg
More complicated for singular X
but periodic cyclic same $HP_k(\mathcal{O}(X)) \cong \bigoplus_i H^{2i+k}(X_{top}, \mathbb{C})$ sing cohoml.
(Feigin-Tsygan)

Noncommutative torus: (Connes '80s)

$\theta \notin \mathbb{Q}$

$HH^0(A_\theta) = \mathbb{C}$

$HH^i(A_\theta) = \begin{cases} \mathbb{C}^2 & i=1 \\ \mathbb{C} & i=2 \end{cases}$ if θ satisfies Diophantine condition $|1-\lambda^n|^{-1} = O(n^k)$ some integer k

$HH^i(A_\theta)$ infinite dim non-Hausdorff spaces if θ not satisfies diophantine condition

in all cases

$HP^0(A_\theta) = \mathbb{C}^2 \qquad HP^1(A_\theta) = \mathbb{C}^2$

basis: 1-cocycles: $\varphi_1(a_0, a_1) = \tau(a_0\delta_1(a_1)) \qquad \varphi_2(a_0, a_1) = \tau(a_0\delta_2(a_1))$

2-cocycles: $\{ \varphi_1(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))$

where $\delta\tau(a_1,a_2,a_3) = \tau(a_1 a_2 a_3)$

δ_1, δ_2 usual canon. derivations on A_θ $\quad U\frac{\partial}{\partial U}, \; V\frac{\partial}{\partial V}$

Note : algebraic HH groups fin. dim in all cases of θ
even though HH_{cont} ∞-dim
for subalg. gen. algebraically by U, V (polyn.)

Connes' result uses complex :

$$A_\theta \xleftarrow{\varepsilon} B_\theta \otimes \Omega_0 \xleftarrow{b_1} B_\theta \otimes \Omega_1 \xleftarrow{b_2} B_\theta \otimes \Omega_2 \leftarrow 0$$

$\Omega_i = \Lambda^i \mathbb{C}^2 \quad i = 0,1,2 \qquad U_1, U_2$ gen. of A_θ

$B_\theta = A_\theta \hat{\otimes} A_\theta^{op}$

$b_1(1 \otimes e_j) = 1 \otimes U_j - U_j \otimes 1 \qquad j = 1,2$

$b_2(1 \otimes (e_1 \wedge e_2)) = (U_2 \otimes 1 - \lambda \otimes U_2) \otimes e_1 - (\lambda U_1 \otimes 1 - 1 \otimes U_1) \otimes e_2$

$\varepsilon(a \otimes b) = ab$

Index theorem in NCG : pairing of HP^i and K_i

First recall Chern-Weyl theory of characteristic classes in commutative cases

(gauge potentials, field strenghts, charges)

X smooth manifold

$$\begin{array}{c} E \\ \downarrow \\ X \end{array} \quad \text{complex vector bundle}$$

Connection $\quad \nabla : \Gamma^\infty(X, E) \longrightarrow \Gamma^\infty(X, E) \otimes_{C^\infty(X)} \Omega^1(X)$

satisfying Leibnitz rule ($\mathbb{C}$-lin but not $C^\infty(X)$-lin.)

$$\nabla(f\xi) = f\,\nabla(\xi) + \xi \otimes df$$

can extend uniquely to

$$\hat{\nabla} : \Gamma^\infty(X,E) \otimes_{C^\infty(X)} \Omega^\bullet(X) \to \Gamma^\infty(X,E) \otimes_{C^\infty(X)} \Omega^{\bullet+1}(X) \quad \text{by}$$

$$\hat{\nabla}(\xi\omega) = \hat{\nabla}(\xi)\omega + (-1)^{\deg \xi}\, \xi\, d\omega$$

Curvature of connection

$$\hat{\nabla}^2 \in \mathrm{End}_{\Omega^\cdot(X)}\left(\Gamma(X,E) \otimes_{C^\infty(X)} \Omega^\cdot(X) \right) = \Gamma^\infty(\mathrm{End}(E)) \otimes \Omega^\cdot(X)$$

$\Omega^\cdot(X)$-linear

Restriction to $\Gamma^\infty(X,E)$ curvature form ∇^2

$$R \in \Gamma^\infty(\mathrm{End}(E)) \otimes \Omega^2(X)$$

Using $\mathrm{Tr} : \Gamma^\infty(\mathrm{End}(E)) \otimes_{C^\infty(M)} \Omega^{even}(X) \to \Omega^{even}(X)$

$$\mathrm{Ch}(E) := \mathrm{Tr}(e^R) = \mathrm{Tr}\left(\sum_{n \geq 0} \frac{R^n}{n!} \right)$$

closed differential form
cohom. class indep. of choice of connection

Note $\quad \begin{matrix} E \\ \downarrow \\ X \end{matrix}$ corresponds to a fin. proj. mod $\mathcal{E}$ on $\mathcal{A} = C^\infty(X)$

$\Rightarrow \exists\, e = e^2 = e^* \in M_n(\mathcal{A})$ idempotent

$\mathcal{E} = e\mathcal{A}^n$

associated connection (Levi-Civita connection)

$$\nabla(\xi) = e\, d\xi$$

$$R(\xi) = e\, d(e\, d\xi)$$

since $e\xi = \xi$ have $d\xi = (de)\xi + e\, d\xi$

since $e^2 = e$ have $e\, de\, e = 0$.

$$\Rightarrow R(\xi) = e\, de\, de\, \xi$$

$\Rightarrow$ curvature 2-form is $e\, de\, de$

using powers

$$R^n = (e\, de\, de)^n \overset{\checkmark}{=} e\, \underbrace{de \ldots de}_{2n \text{ times}}$$

$$Ch(E) = Tr\left(\sum_{n\geq 0} \frac{R^n}{n!}\right)$$

$$Tr\left(\frac{R^n}{n!}\right) = \frac{1}{n!} Tr(e\,\underbrace{de \ldots de}_{2n}) \in \Omega^{2n}(X)$$

Note: this is image of cyclic cocycle

$$Ch_0^{2n}(e) := \frac{1}{n!} Tr(\underbrace{e\otimes \cdots \otimes e}_{2n+1}) \qquad HC_{2n}(A) \longrightarrow H^{2n}_{dR}(A)$$

$$a_0 \otimes \cdots \otimes a_{2n} \longmapsto a_0\, da_1 \cdots da_{2n}$$

Note: Dual homologies

$$C_n(A) = A^{\otimes(n+1)}$$

$$b : C_n(A) \to C_{n-1}(A) \qquad b' : C_n(A) \to C_{n-1}(A)$$

$$\lambda : C_n(A) \to C_n(A)$$

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$
$$+ (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

$$\lambda(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

$$(1-\lambda)\, b' = b\,(1-\lambda)$$

$$C_n^\lambda(A) = C_n(A) / \mathrm{Im}(1-\lambda)$$

$$HC_n(A) = H_n(C_*^\lambda(A), b)$$

$$HC_0(A) = A/[A,A] \qquad [A,A] = \text{subspace spanned by } [a,b]$$

(Commutative: $HC_0(A) = A$)

Index theory diagram

$$\begin{tikzcd} K^*(A) \times K_*(A) \arrow[r] \arrow[d, "ch^*"', "ch_*"] & \mathbb{Z} \arrow[d] \\ HP^*(A) \times HP_*(A) \arrow[r] & \mathbb{C} \end{tikzcd}$$

$K^*(A) =$ K-homol. $=$ Fredholm modules

$K_*(A) =$ K-theory