ENDOMOTIVES OF TORIC VARIETIES

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Abstract. We construct endomotives associated to toric varieties, in terms of the decomposition of a toric variety into torus orbits and the action of a semigroup of toric morphisms. We show that the endomotives can be endowed with time evolutions and we discuss the resulting quantum statistical mechanical systems. We show that in particular, one can construct a time evolution related to the logarithmic height function. We discuss relations to $\mathbb{F}_1$-geometry.

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1. Introduction

The notion of endomotive was introduced in [8] as a way to describe, in terms of arithmetic data, the construction of quantum statistical mechanical systems associated to number theory, starting with the prototype Bost–Connes system [6]. Endomotives are algebras obtained from projective limits of Artin motives (zero dimensional algebraic varieties) endowed with semigroup actions. Out of these algebraic data one obtains noncommutative spaces, in the form of semigroup crossed product C^*-algebras.

There are natural time evolutions on the C^*-algebra of an endomotive, and the associated Hamiltonian, partition function and KMS equilibrium states relate to properties of L-functions in cases arising from number theory, see [8], [11], [23]. In particular, endomotives were recently used by Bora Yalkinoglu to construct arithmetic subalgebras for all the quantum statistical mechanics associated to number fields, [23], in the context of the noncommutative geometry approach to the explicit class field theory problem. It is also known, see [5], [9], [17], [19], [23], that the notion of endomotive is closely related to the notion of Λ-rings studied by Borger, in his approach to geometry over the "field with one element" $\mathbb{F}_1$.

An interesting question in the theory of endomotives is whether the construction extends to more general algebro-geometric objects, besides the cases underlying the construction of quantum statistical mechanical systems of number fields, that was the main focus in [8], [11], [23]. Toric varieties are a natural choice of a class of varieties for which this question can be addressed. In fact, toric varieties constitute an important class of algebraic varieties, which is sufficiently concrete and well understood ([12], [13]) to provide a good testing ground for various constructions. Moreover, toric varieties play an important role in the theory of $\mathbb{F}_1$-geometry. A first step in the direction of the construction of associated endomotives already existed, in the form of the multivariable Bost–Connes systems introduced in [19], which we will interpret here as the simplest case of our construction for toric varieties, corresponding to the case where the variety is just a torus $\mathbb{T}^n$.

In this paper, we construct endomotives for abstract toric varieties, generalizing the existing constructions of the Bost-Connes system and its generalizations.

In the rest of this section, we recall the basic notion of an endomotive, as defined in [8] and the main properties of the associated quantum statistical mechanical systems. We also recall the torus-cone correspondence for toric varieties, which will be the basis of our construction. We then identify a semigroup of toric morphisms which we will use in the endomotive data.

In Section 2 we describe the construction of the endomotives of abstract toric varieties. We give two variants of the construction, which we refer to as the additive and multiplicative case, which correspond, respectively, to the abelian part of the endomotive algebra being a direct sum or a tensor product of contributions from the single orbits. The direct sum choice is more closely tied up to the geometry of the variety, as it corresponds to the decomposition into a union of orbits, with the abelian part of the endomotive given by the algebra of functions on a set of algebraic points on the toric variety obtained as a projective limit over the action of the semigroup. The tensor product case corresponds instead to regarding the torus orbits as defining independent quantum mechanical systems, so that the resulting partition function will decompose as a product over orbits. In both cases we obtain Hilbert space representations of the abelian algebras and of the semigroup, in such a way that they determine a representation of the semigroup crossed product algebra. We describe the explicit generators and relations of the crossed product. We then describe a general procedure to construct time evolutions on the algebra with the corresponding Hamiltonians that are the infinitesimal generators in the given representation. We describe the group of symmetries and the partition function. In some especially nice cases, from the point of view of symmetries of the fan defining the toric variety, we give a more concrete description of the Hamiltonian and the partition function.

In Section 3, we relate the endomotives of toric varieties to $\mathbb{F}_1$-geometry, both in the sense of Borger, [5], via the notion of Λ-ring, and in the sense of Soulé, [21]. We also show that a weaker form of the endomotive construction can be extended from toric varieties to torified spaces in the sense of the approach to $\mathbb{F}_1$-geometry of Lorscheid and López-Peña, [15].

In Section 4, we focus on the case of projective toric varieties, and in particular on the concrete example of projective spaces. We replace a set of distinguished points in the torus orbits used in the abstract construction of Section 2 with the set of $\bar{\mathbb{Q}}$-algebraic points of the variety with bounded height and degree over $\mathbb{Q}$, and we describe a time evolution and covariant representations of the resulting C^*-dynamical system related to the logarithmic height function. We show a variant of the construction for the case of affine spaces.

Finally, in Section 5, we discuss briefly the Gibbs equilibrium states fort the quantum statistical mechanical systems of endomotives of abstract toric varieties.

1.1. **The notion of endomotive.** We recall here briefly the notion of endomotive from [8] and the main properties we will be discussing in the rest of the paper.

The data of an endomotive consist of a projective system X_α of zero dimensional algebraic varieties over a field $\mathbb{K}$, where $X_\alpha = \mathrm{Spec}(A_\alpha)$, together with an action by endomorphisms of a semigroup S on the limit $X = \varprojlim_\alpha X_\alpha$. In [8] the field $\mathbb{K}$ is assumed to be a number field.

In [8] the semigroup S is assumed to be countably generated and *abelian*, while more general situations with S not necessarily abelian were discussed, for instance, in [19] and will also be considered here.

At the algebraic level, one associates to the data (X, S) of an endomotive the algebraic semigroup crossed product algebra $A \rtimes S$, while at the analytic level, one considers the C^*-algebra $C(X(\bar{\mathbb{K}})) \rtimes S$.

At the algebraic level, one considers the algebraic semigroup crossed product $\mathbb{K}$-algebra $A \rtimes S$, where $A = \varinjlim_\alpha A_\alpha$ and $X = \mathrm{Spec}(A)$. This is generated algebraically by elements $a \in A$ and additional generators μ_s and μ_s^*, for $s \in S$, satisfying the relations $\mu_s^* \mu_s = 1$, $\mu_s \mu_s^* = \phi_s(1)$, where ϕ_s is the endomorphism of A corresponding to $s \in S$, and $\mu_{s_1 s_2} = \mu_{s_1} \mu_{s_2}$, $\mu_{s_2 s_1}^* = \mu_{s_1}^* \mu_{s_2}^*$, $\mu_s \, a = \phi_s(a) \, \mu_s$, and $a \, \mu_s^* = \mu_s^* \phi_s(a)$, for all $s, s_1, s_2 \in S$ and for all $a \in A$.

1.1.1. *Quantum statistical mechanical systems of endomotives.* At the analytic level, one considers the C^*-algebra $\mathcal{A} = C(X) \rtimes S$. One regards this as the algebra of observables of a quantum statistical mechanical system, with a time evolution given by a one parameter family of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$.

A covariant representation of the C^*-dynamical system $(\mathcal{A}, \sigma_t)$ is a pair (π, H) of a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of the algebra by bounded operators on a Hilbert space $\mathcal{H}$, and an operator H on $\mathcal{H}$ (usually unbounded) with the property that

$$\pi(\sigma_t(a)) = e^{-it\, H} \, \pi(a) \, e^{it\, H}. \tag{1.1}$$

One says that H is the Hamiltonian generating the the time evolution σ_t in the representation π.

The typical form of the time evolution considered in [8] arises from the modular automorphism group σ_t^φ associated to a state φ determined by a measure on X. Here we will give a construction of time evolutions on endomotives, based more generally on semigroup homomorphisms $g : S \to \mathbb{R}_+^*$ for which there exists an associated function h, with appropriate scaling properties, so that the pair (g, h) determine a time evolution and the corresponding Hamiltonian in an assigned representation (see Proposition 2.8 below).

1.1.2. *Partition function.* Given a C^*-dynamical system $(\mathcal{A}, \sigma_t)$ and a covariant representation (π, H), the partition function of the system is given by

$$Z(\beta) = \mathrm{Tr}(e^{-\beta\, H}). \tag{1.2}$$

In typical situations, there is a sufficiently large real β_0 such that for all real β with $\beta > \beta_0$ the operator $e^{-\beta H}$ is trace class. In physical terms one thinks of the variable β as an inverse temperature (up to the Boltzmann constant).

1.1.3. *Equilibrium states.* A state is a continuous linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, which satisfies positivity, $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$, and normalized by $\varphi(1) = 1$. States are the analog of probability measures on noncommutative spaces. An equilibrium state of a C^*-dynamical system $(\mathcal{A}, \sigma_t)$ is a state satisfying $\varphi(\sigma_t(a)) = \varphi(a)$ for all $a \in \mathcal{A}$ and all $t \in \mathbb{R}$.

An important class of equilibrium states on a system $(\mathcal{A}, \sigma_t)$ are the Gibbs states, which are of the form

$$\varphi(a) = \frac{\mathrm{Tr}(\pi(a)\, e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}, \tag{1.3}$$

for a covariant representation (π, H). These are well defined when $e^{-\beta H}$ is trace class. A more general class of equilibrium states generalizing Gibbs states is given by the KMS_β states (Kubo–Martin–Schwinger).

1.2. **Toric varieties and orbit-cone decomposition.** We recall some basic results from the theory of toric varieties, in particular the *orbit-cone correspondence*, which we will need to use extensively in the following. We refer the reader to the first three chapters of [12] for more details and complete proofs.

Let N be a lattice of rank d, and let M be its dual lattice. The lattices N and M are, respectively, the lattice of one-parameter subgroups and the character lattice of the toric variety. Let $N_\mathbb{R}$ and $M_\mathbb{R}$ be the real vector spaces obtained by tensoring N and M with $\mathbb{R}$, respectively.

Let Σ be a fan in $N_\mathbb{R}$, and X_Σ be the abstract toric variety associated to Σ.

Suppose that $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_m\}$, as a set of cones. Then, according to the orbit-cone correspondence, we can decompose X_Σ as a disjoint union of torus orbits

$$X_\Sigma = \bigsqcup_{k=1}^{m} O(\sigma_k), \tag{1.4}$$

with the orbits given by

$$O(\sigma_k) = \{\gamma : \sigma^\vee \cap M \to \mathbb{C} | \gamma(u) \neq 0 \Longleftrightarrow u \in \sigma^\perp \cap M\} \simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq T_{\sigma^\perp \cap M}\,, \tag{1.5}$$

where $\sigma^\perp$ denotes the orthonormal complement of σ in $M_\mathbb{R}$ and $\sigma^\vee$ the dual cone of σ.

Each torus orbit $O(\sigma_k)$ contains a distinguished point γ_k given by

$$\gamma_k \,:\, u \in \sigma_k^\vee \cap M \mapsto \begin{cases} 1 & \text{if } u \in \sigma_k^\perp \cap M \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

The orbit-cone correspondence will be the foundation of our construction of endomotives.

1.3. **Semigroup of an abstract toric variety.** We introduce here a semigroup of toric morphisms, associated to an abstract toric variety, which is defined directly in terms of the orbit-cone decomposition, and which will provide the semigroup for the endomotive construction.

We call a $\mathbb{Z}$-linear transformation ϕ of N compatible with the fan Σ if, for any cone σ_j in Σ, there exists a cone σ_k in Σ such that $\phi_\mathbb{R}(\sigma_j) \subseteq \sigma_k$, where $\phi_\mathbb{R} = \phi \otimes 1$ is the induced linear map on $N_\mathbb{R}$. For each such map ϕ and each cone σ_j in Σ, there exists a unique cone σ_k in Σ such that

$$\phi_\mathbb{R}(\mathrm{Relint}(\sigma_j)) \subseteq \mathrm{Relint}(\sigma_k). \tag{1.7}$$

Thus ϕ can be regarded as a self-map on $\{\sigma_1, \sigma_2, ..., \sigma_m\}$.

Let S_0 denote the set of all nonsingular linear transformations of N that are compatible with the fan Σ and bijective as a self-map on the cones in Σ. Then S_0 is a semigroup under composition (matrix multiplication if we identify the elements with $d \times d$-matrices).

A standard fact about toric varieties implies that each ϕ induces a toric morphism $\bar{\phi}$ of X_Σ in the following way: for each cone σ_j in Σ, let σ_k be the cone in Σ such that ϕ maps $\mathrm{Relint}(\sigma_j)$ into $\mathrm{Relint}(\sigma_k)$; then we have a map of corresponding torus orbits

$$\bar{\phi} : O(\sigma_k) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{C}^*) \to O(\sigma_j) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma_j^\perp \cap M, \mathbb{C}^*) \tag{1.8}$$

$$f_k \mapsto f_k \circ \phi^T, \tag{1.9}$$

where ϕ^T is the transposed matrix of ϕ. To simplify notation, we denote by S the semigroup consisting of the transposes of the matrices in S_0. Then S acts on X_Σ by precomposition.

2. Endomotives of abstract toric varieties

In this section we present our main construction of endomotives for abstract toric varieties. The algebra of observables and the time evolution are obtained from the orbit-cone decomposition and the semigroup introduced above, and from a class of functions with suitable scaling properties under composition with the toric morphisms of the semigroup.

2.1. **Multivariable Bost–Connes systems.**

We first recall briefly the case of a single torus, which corresponds to the "multivariable Bost–Connes systems" introduced in [19]. One considers in that case a variety that is an n-dimensional algebraic torus $\mathbb{T}^n = (\mathbb{G}_m)^n$. The semigroup S_n is given by $n \times n$-matrices with integer entries and positive determinant, $S_n = M_n(\mathbb{Z})^+$, with the action on $\mathbb{T}^n$ given by

$$t = (t_j)_{j=1}^n \overset{\alpha}{\mapsto} t^\alpha = (t_i^\alpha), \quad \text{where} \quad t_i^\alpha = \prod_j t_j^{\alpha_{ij}}, \tag{2.1}$$

where $\alpha = (\alpha_{ij}) \in M_n(\mathbb{Z})^+$. This generalizes to non-invertible transformations the usual action of *monomial Cremona transformations*, which corresponds to the case of $\alpha \in \mathrm{GL}_n(\mathbb{Z})$.

As in (1.6) above, one considers the distinguished point $\gamma = (1, 1, \ldots, 1)$ in $\mathbb{T}^n$. The endomotive in this case is constructed by considering the inverse images

$$X_\alpha = \{t \in \mathbb{T}^n \,|\, \alpha(t) = \gamma\}$$

for $\alpha \in M_n(\mathbb{Z})^+$. These form a projective system, under composition of the semigroup transformations, and one takes the corresponding projective limit

$$X = \varprojlim_\alpha X_\alpha, \tag{2.2}$$

or equivalently the direct limit

$$A = \varinjlim_\alpha A_\alpha,$$

where $X_\alpha = \mathrm{Spec}(A_\alpha)$ and $X = \mathrm{Spec}(A)$. It is shown in [19] that this limit is given by

$$A \simeq \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}.$$

The algebraic endomotive is then given by the $\mathbb{Q}$-algebra $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n} \rtimes M_n(\mathbb{Z})^+$ and the analytic endomotive by the C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z})^{\otimes n} \rtimes M_n(\mathbb{Z})^+$. A time evolution on this algebra is obtained by setting

$$\sigma_t(\mu_\alpha) = \det(\alpha)^{it} \mu_\alpha \quad \text{and} \quad \sigma_t(a) = a, \tag{2.3}$$

for all $\alpha \in M_n(\mathbb{Z})^+$ and for all $a \in C^*(\mathbb{Q}/\mathbb{Z})^{\otimes n}$.

In the construction of [19] the choice of the semigroup $S_n = M_n(\mathbb{Z})^+$ is large, hence one needs to either pass to a convolution algebra that eliminates the $\mathrm{SL}_n(\mathbb{Z})$-symmetry of the spectral levels of the Hamiltonian, or else computing the partition function with respect to a type II_1-trace, which gives

$$Z(\beta) = \sum_{\alpha \in M_n(\mathbb{Z})^+/\mathrm{SL}_n(\mathbb{Z})} \det(\alpha)^{-\beta} = \prod_{k=0}^{n-1} \zeta(\beta - k).$$

In the more general construction we present in the rest of this section, this problem of large $\mathrm{SL}_n(\mathbb{Z})$-symmetries is avoided, by working with the smaller semigroup S that we defined in Section 1.3 above. In the case of a single torus, our S in fact reduces to the much smaller $S = \{mI \,|\, m \in \mathbb{N}\}$ with I the $n \times n$-identity matrix, which computes the same projective limit (2.2) as the larger S_n. The construction of the time evolution we provide for arbitrary toric varieties will also be more general than (2.3), hence providing a broader range of possible zeta functions.

2.2. **Construction of the endomotive.** Let S be the semigroup of an abstract toric variety, constructed in §1.3 above. For each ϕ in S, let X_ϕ denote the preimage under ϕ of the set of distinguished points in all toric orbits,

$$X_\phi = \phi^{-1}(\sqcup_{k=1}^{m} \{\gamma_k\}). \tag{2.4}$$

For any ϕ_1, ϕ_2 and φ in S with $\phi_1 = \phi_2 \circ \varphi$, we have a transition map $\varphi : X_{\phi_1} \to X_{\phi_2}$, given by composition. Thus, all the sets X_ϕ's together with these maps form a projective system. We denote by X the projective limit. We have the following result.

Proposition 2.1. *As a topological space, the projective limit is*

$$X = \varprojlim_{\phi} X_\phi \simeq \sqcup_{k=1}^{m} \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \hat{\mathbb{Z}}) \tag{2.5}$$

and the algebra of continuos functions is correspondingly given by

$$C(X) \simeq \bigoplus_{k=1}^{m} C^*((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z}))\,. \tag{2.6}$$

Proof. Note that any matrix in S is invertible in $M_d(\mathbb{Q})$ and the inverse is also compatible with Σ and bijective as a map of cones, thus some integer multiple of this inverse lies in $M_d(\mathbb{Z})$ and hence in S, which shows that for any $\phi_1 \in S$ there exists $\phi_2 \in S$ and $n \in \mathbb{N}^+$such that $\phi_1\phi_2 = nI$ (here I denotes the d by d identity matrix). Therefore $\{nI | n \in \mathbb{N}^+\}$ is a cofinal subset of the indexing set, and it suffices to compute the projective limit over this subset.

Let $X_{n,k} = (nI)^{-1}(\gamma_k)$ be the preimage of the distinguished point γ_k under the action of nI. Then it is clear that

$$X_{nI} = \sqcup_{k=1}^{m} X_{n,k}.$$

Note that each transition map tI preserves the second index k of $X_{n,k}$, *i.e.* the image of $X_{tn,k}$ lies in $X_{n,k}$. Thus, the projective system $\{X_{nI} | n \in \mathbb{N}^+\}$ splits as the disjoint union of projective systems $\{X_{n,k} | n \in \mathbb{N}^+\}$ $(k = 1, ..., m)$. Hence we have

$$X \simeq \varprojlim_{n} X_n \simeq \sqcup_{k=1}^{m} \varprojlim_{n} X_{n,k}. \tag{2.7}$$

On the other hand, we have natural isomorphisms

$$X_{n,k} = (nI)^{-1}(\gamma_k) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}/n\mathbb{Z}), \tag{2.8}$$

which give

$$\varprojlim_{n} X_{n,k} \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \varprojlim_{n} (\mathbb{Z}/n\mathbb{Z})) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \hat{\mathbb{Z}}), \tag{2.9}$$

from which the first statement follows immediately.

As for the second one, clearly we have

$$C(X) \simeq C(\bigsqcup_{k=1}^{m} \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \hat{\mathbb{Z}})) \simeq \bigoplus_{k=1}^{m} C(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \hat{\mathbb{Z}})). \tag{2.10}$$

Moreover, by Pontryagin duality, we obtain canonical isomorphisms

$$C(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \hat{\mathbb{Z}})) \simeq C^*(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \hat{\mathbb{Z}})^{\wedge}), \tag{2.11}$$

where

$$\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \hat{\mathbb{Z}})^{\wedge} \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \hat{\mathbb{Z}}), \mathbb{Q}/\mathbb{Z}) \tag{2.12}$$

is the Pontryagin dual. Note that $\mathbb{Q}/\mathbb{Z}$ and $\hat{\mathbb{Z}}$ are Pontryagin duals to each other, hence we have a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \hat{\mathbb{Z}}), \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})), \mathbb{Q}/\mathbb{Z}). \tag{2.13}$$

Furthermore,

$$\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})) \simeq \mathrm{Hom}_{\mathbb{Z}}((\sigma_k^{\perp} \cap M) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}), \tag{2.14}$$

showing that we can identify the right-hand-side of (2.13) with the double dual of $(\sigma_k^{\perp} \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$, which is canonically isomorphic to $(\sigma_k^{\perp} \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$ itself.

Combining the identities above we obtain the second statement. □

For reasons related to the behavior of the resulting zeta functions, we also want to consider a variant of the construction of Proposition 2.1, where instead of working with the space X of (2.5), we work with a different space Y given by a *product*, instead of a *disjoint union*, of the limits $\varprojlim_n X_{n,k}$. Namely, we set

$$Y := \prod_{k=1}^{n} \varprojlim_{n} X_{n,k}. \tag{2.15}$$

The algebra of functions is correspondingly given by the *tensor product* instead of *direct sum*,

$$C(Y) = \bigotimes_{k=1}^{m} C(\varprojlim_{n} X_{n,k}) = \bigotimes_{k=1}^{m} C^*((\sigma_k^{\perp} \cap M) \otimes (\mathbb{Q}/\mathbb{Z})), \tag{2.16}$$

where the last equality follows as in Proposition 2.1.

2.3. **The crossed product algebra and Hilbert space representations.** Note that, since by construction M comes endowed with a basis, we may identify M as $\mathbb{Z}^d$, hence we can write $M \otimes (\mathbb{Q}/\mathbb{Z})$ as $(\mathbb{Q}/\mathbb{Z})^d$. In this way, each element of $(\sigma_k^{\perp} \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$ can be regarded as a d-tuple in $(\mathbb{Q}/\mathbb{Z})^d$, and there is an induced action of S on $M \otimes (\mathbb{Q}/\mathbb{Z}) \simeq (\mathbb{Q}/\mathbb{Z})^d$ given by matrix multiplication, which corresponds to the action given by the map $\phi \otimes 1$ on $M \otimes (\mathbb{Q}/\mathbb{Z})$.

It follows that this action via matrix multiplication again maps each $(\sigma_k^{\perp} \cap M) \otimes (\mathbb{Q}/\mathbb{Z}) \subseteq (\mathbb{Q}/\mathbb{Z})^d$ into the corresponding $(\sigma_{\phi(k)}^{\perp} \cap M) \otimes (\mathbb{Q}/\mathbb{Z}) \subseteq (\mathbb{Q}/\mathbb{Z})^d$. We will make use of this action and this identification later in Lemma 2.5 and 2.6.

The C^*-algebras of functions

$$C(X) \simeq \bigoplus_{k=1}^{m} C^*((\sigma_k^{\perp} \cap M) \otimes (\mathbb{Q}/\mathbb{Z})) \tag{2.17}$$

and

$$C(Y) \simeq \bigotimes_{k=1}^{m} C^*((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z})) \tag{2.18}$$

will serve as abelian parts of two variants of the endomotive construction. We represent them on Hilbert spaces as follows.

Lemma 2.2. *There is a natural representation of the algebra $C(X)$ of (2.17) as bounded linear operators on the Hilbert space*

$$\mathcal{H}_X = \bigoplus_{k=1}^{m} \ell^2(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})) \tag{2.19}$$

and of the algebra $C(Y)$ of (2.18) on

$$\mathcal{H}_Y = \bigotimes_{k=1}^{m} \ell^2(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})). \tag{2.20}$$

Proof. First we observe that

$$\mathrm{Hom}_{\mathbb{Z}}((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \hat{\mathbb{Z}}). \tag{2.21}$$

Moreover, $\mathbb{Z}$ naturally embeds into $\hat{\mathbb{Z}}$, hence we have a natural embedding of

$$\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \hat{\mathbb{Z}}), \tag{2.22}$$

which means that we can naturally identify each element of $\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})$ with a homomorphism

$$f_k : (\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}. \tag{2.23}$$

We can then represent in the following way the algebra $C^*((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z}))$ as bounded linear operators on the Hilbert space $\ell^2(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}))$. Consider the canonical basis

$$\{\epsilon_{f_k} | f_k \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})\},$$

with $\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{Z}}((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$. Then, for each generator $e(r_k)$ of $C^*((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z}))$, where $r_k \in (\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$, we set

$$e(r_k)\epsilon_{f_k} = \exp(2i\pi f_k(r_k))\epsilon_{f_k}. \tag{2.24}$$

From now on, for simplicity, we suppress the explicit notation $\exp(2i\pi\cdot)$ and we implicitly identify $f_k(r)$ with the corresponding root of unity.

Clearly, this preserves the multiplicative structure of the group algebra since each f_k is a homomorphism. Thus, we have a naturally induced representations of the direct sum $C(X) \simeq \oplus_{k=1}^m C^*((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z}))$ on $\mathcal{H}_X = \oplus_{k=1}^m \ell^2(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}))$, and of the tensor product $C(Y) \simeq \otimes_{k=1}^m C^*((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z}))$ on $\mathcal{H}_Y = \otimes_{k=1}^m \ell^2(\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}))$, respectively given by

$$e(\underline{r})(\oplus_{k=1}^m \epsilon_{f_k}) = \oplus_{k=1}^m f_k(r_k)\epsilon_{f_k}, \quad \text{and} \quad e(\underline{r})(\otimes_{k=1}^m \epsilon_{f_k}) = \otimes_{k=1}^m f_k(r_k)\epsilon_{f_k} \tag{2.25}$$

where $\underline{r} = (r_k)_{k=1,\dots,m}$, and $e(\underline{r}) := \oplus_{k=1}^m e(r_k)$ or $e(\underline{r}) := \otimes_{k=1}^m e(r_k)$, respectively. □

2.3.1. *Semigroup representation.* We represent the semigroup S as bounded on the same Hilbert spaces $\mathcal{H}_X$ and $\mathcal{H}_Y$.

Definition 2.3. *For $f_k \in \mathrm{Hom}(\sigma_k^\perp \cap M, \mathbb{Z})$, let $f'_1, ..., f'_m$ denote the compositions $f_1 \circ \phi, ..., f_m \circ \phi$, reordered in such a way that $f'_k \in \mathrm{Hom}(\sigma_k^\perp \cap M, \mathbb{Z})$.*

Notice that it is always possible to reorder the elements $f_1 \circ \phi, ..., f_m \circ \phi$ as described, since ϕ defines a permutation of the set $\{\sigma_k^\perp \cap M \mid k = 1, ..., m\}$. With a slight abuse of notation, in the following we will write $\epsilon_{f_k \circ \phi}$ instead of $\epsilon_{f'_k}$.

Lemma 2.4. *The actions given, respectively, by*

$$\mu_\phi(\oplus_{k=1}^m \epsilon_{f_k}) = \oplus_{k=1}^m \epsilon_{f_k \circ \phi} \quad \text{and} \quad \mu_\phi(\otimes_{k=1}^m \epsilon_{f_k}) = \otimes_{k=1}^m \epsilon_{f_k \circ \phi}, \tag{2.26}$$

define representations of the semigroup S on the Hilbert spaces $\mathcal{H}_X$ and $\mathcal{H}_Y$, where in both cases the adjoints are determined by

$$\mu_\phi^* \, \epsilon_{f_k} = \delta(f_k = g_k \circ \phi) \, \epsilon_{g_k}, \tag{2.27}$$

where

$$\delta(f_k = g_k \circ \phi) = \begin{cases} 1 & f_k = g_k \circ \phi \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

Proof. By construction, $\phi \mapsto \mu_\phi$ defines a semigroup homomorphism,

$$\mu_{\phi_1 \circ \phi_2} = \mu_{\phi_1} \mu_{\phi_2},$$

from S to isometries in the algebra of bounded operators on $\mathcal{H}_X$ and $\mathcal{H}_Y$, respectively.

The adjoint μ_ϕ^* is defined by the relation

$$\langle \mu_\phi^* \, \xi, \eta \rangle = \langle \xi, \mu_\phi \, \eta \rangle,$$

for all ξ, η in the Hilbert space. It suffices to check the identity on the elements of the canonical basis, hence we consider ϵ_{f_k} and ϵ_{h_k}, for some f_k and h_k in $\mathrm{Hom}(\sigma_k^\perp \cap M, \mathbb{Z})$. Then we have

$$\langle \xi, \mu_\phi \, \eta \rangle = \delta(h_k = f_k \circ \phi) = \langle \mu_\phi^* \, \xi, \eta \rangle,$$

for μ_ϕ^* as in (2.27). It is clear by construction that μ_ϕ^* is a left inverse of μ_ϕ and a right inverse on the image of μ_ϕ. ☐

2.3.2. *Crossed product relations.* We have the following lemma relating the operators $e(r_k)$ and μ_ϕ.

Lemma 2.5. *The action of S on $(\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$ satisfies*

$$e(\phi \cdot \underline{r}) = \mu_\phi^* \, e(\underline{r}) \, \mu_\phi \tag{2.29}$$

Proof. For any ϵ_{f_k} we have

$$\mu_\phi^* \, e(\underline{r}) \, \mu_\phi \, \epsilon_{f_k} = \mu_\phi^* e(\underline{r}) \, \epsilon_{f_k \circ \phi} = f_{\phi(k)} \circ \phi(r_k) \;\; \mu_\phi^* \, \epsilon_{f_k \circ \phi},$$

where by $\phi(k)$ we mean the permutation of the indices of the $\sigma_k^\perp \cap M$ induced by ϕ. By (2.27), we then write the above as

$$f_{\phi(k)} \circ \phi(r_k) \, \epsilon_{f_k} = e(\phi \cdot \underline{r}) \, \epsilon_{f_k},$$

hence the conclusion holds. ☐

Lemma 2.5 immediately implies the following result.

Lemma 2.6. *Let $\lambda = 1$ in the additive case of $(\mathcal{A}_X, \mathcal{H}_X)$ and $\lambda = m$ in the multiplicative case $(\mathcal{A}_Y, \mathcal{H}_Y)$. The operators $e(\underline{r})$ and μ_ϕ satisfy the relations*

$$\mu_\phi \; e(\underline{r}) \; \mu_\phi^* = \frac{1}{|\det(\phi)|^\lambda} \sum_{\phi\cdot\underline{s}=\underline{r}} e(\underline{s}). \tag{2.30}$$

Proof. First we show that the number of solutions $\underline{s}$ to the equation $\phi \cdot \underline{s} = \underline{r}$ is exactly $|\det(\phi)|$. By definition of the action of ϕ it is clear that $\phi : s_{k'} \mapsto r_k$ only if $s_{k'} \in (\sigma^\perp_{\phi^{-1}(k)} \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$.

On the other hand, as we observe in Lemma 2.13 below, $\sigma^\perp_{\phi^{-1}(k)}$ and $\sigma^\perp_k$ have the same dimension, and ϕ gives an isomorphism between them. Thus, for any $r \in (\mathbb{Q}/\mathbb{Z})^d \backslash (\sigma^\perp_{\phi^{-1}(k)} \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$, $(\phi \otimes 1)(r) \nsubseteq (\sigma^\perp_k \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$, hence, under matrix multiplication, $\phi r \neq r_k$. Thus, the set of $s_{k'}$ with $\phi \cdot s_{k'} = r_k$ is the set $\{\phi^{-1} r_k, \phi^{-1}(r_k + \underline{1}), ..., \phi^{-1}(r_k + \underline{(\det(\phi) - 1)})\} \mod \mathbb{Z}^d$, where ϕ^{-1} denotes the inverse matrix of ϕ in $GL_d(\mathbb{Q})$ and $\underline{1} = (1, 1, \ldots, 1) \in \mathbb{Q}^d$. This set can be written as $\phi^{-1} r_k \mod \mathbb{Z}^d + \mathrm{Ker}\phi$, where $\mathrm{Ker}\phi$ is a cyclic subgroup of order $\ell = |\det(\phi)|$ of $(\mathbb{Q}/\mathbb{Z})^d$.

Now we prove (2.30). For each $\underline{s}$ with $\phi \cdot \underline{s} = \underline{r}$, Lemma 2.5 gives $e(\underline{r}) = \mu_\phi^* e(\underline{s}) \mu_\phi$, hence

$$\mu_\phi e(\underline{r}) \mu_\phi^* = \mu_\phi \mu_\phi^* \; e(\underline{s}) \; \mu_\phi \mu_\phi^*, \tag{2.31}$$

so that the desired identity holds on the range of μ_ϕ on the Hilbert space, where the projector $\mu_\phi \mu_\phi^*$ is the identity operator.

Now let ϵ_{f_k} be chosen so that it is not in the range of μ_ϕ. Then, in particular, $f_{\phi^{-1}(k)} \neq f \circ \phi$ for any $f \in \mathrm{Hom}_\mathbb{Z}(\sigma^\perp_k \cap M, \mathbb{Z})$. The operator on the left-hand-side in (2.30) maps ϵ_{f_k} to 0, and it suffices to show so does the one on the right-hand-side.

For simplicity we write $\hat{r}_k$ for $\phi^{-1} r_k \mod \mathbb{Z}^d \in (\mathbb{Q}/\mathbb{Z})^d$. Then, as noted above, we can write the sum in the right-hand-side of (2.30) as a sum over $a_k \in \mathrm{Ker}\phi$, and the action of $e(r_k)$ on ϵ_{f_k} as multiplication by $f_{\phi^{-1}(k)}(\hat{r}_k + a_k)$. We also reinstate here the explicit notation $\exp(2\pi i \cdot)$ that we suppressed before, so that this action is, in fact, given by multiplication by the phase factor $\exp(2\pi i \, f_{\phi^{-1}(k)}(\hat{r}_k)) \exp(2\pi i \, f_{\phi^{-1}(k)}(a_k))$. Thus, we can write the right-hand-side of (2.30) as

$$\begin{aligned} \sum_{\phi\cdot s=r_k} e(s)\epsilon_{f_k} &= \sum_{\phi\cdot s=r_k} f_{\phi^{-1}(k)}(s) \; \epsilon_{f_k} \\ &= \sum_{a_k\in \mathrm{Ker}\phi} \exp(2\pi i \, f_{\phi^{-1}(k)}(\hat{r}_k)) \exp(2\pi i \, f_{\phi^{-1}(k)}(a_k)) \; \epsilon_{f_k}. \end{aligned} \tag{2.32}$$

Note then that the map $f_{\phi^{-1}(k)} : (\sigma^\perp_{\phi^{-1}(k)} \cap M) \otimes (\mathbb{Q}/\mathbb{Z}) \to (\mathbb{Q}/\mathbb{Z})$ cannot be identically zero on $\mathrm{Ker}\phi$, otherwise it would factor through $(\sigma^\perp_{\phi(k)} \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$, contradicting the fact that no $f \in Hom_\mathbb{Z}(\sigma^\perp_k \cap M, \mathbb{Z})$ in $Hom_\mathbb{Z}((\sigma^\perp_{\phi(k)} \cap M) \otimes (\mathbb{Q}/\mathbb{Z}), (\mathbb{Q}/\mathbb{Z}))$ satisfies $f_{\phi^{-1}(k)} = f \circ \phi$. In particular, $\xi := \exp(2\pi i f_{\phi^{-1}(k)}(a))$ is a nontrivial root of unity of order ℓ, where a is a generator of the finite cyclic group $\mathrm{Ker}\phi$ of order ℓ. Therefore we have

$$\sum_{a\in \mathrm{Ker}\phi} \exp(2\pi i \, f_{\phi^{-1}(k)}(a)) = \sum_{j=1}^{\ell} \xi^j = 0, \tag{2.33}$$

which in conjunction with (2.32) shows that

$$\sum_{\phi\cdot\underline{s}=\underline{r}} e(\underline{s}) = 0 \tag{2.34}$$

on the complement of the range of μ_ϕ, and this completes the proof. □

Now we are ready to define our endomotives for toric varieties.

Definition 2.7. *The* additive *endomotive associated with a toric variety X_Σ is given by the semi-group crossed product algebra*

$$\mathcal{A}_{X,\Sigma} = C(X) \rtimes_\rho S \simeq (\oplus_{k=1}^m C^*((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z})) \rtimes_\rho S \tag{2.35}$$

and the multiplicative *endomotive associated with X_Σ is given by the crossed product algebra*

$$\mathcal{A}_{Y,\Sigma} = C(Y) \rtimes_\rho S \simeq (\otimes_{k=1}^m C^*((\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z})) \rtimes_\rho S, \tag{2.36}$$

where in both cases the semigroup action is given by $\rho_\phi(e(\underline{r})) = \mu_\phi e(\underline{r})\mu_\phi^*$ under the representation specified in Lemma 2.2.

Note that, by Lemma 2.6, $\rho_\phi(e(\underline{r})) = \mu_\phi e(\underline{r})\mu_\phi^*$ is indeed an element of $\oplus_{k=1}^m C^*(\mathrm{Hom}_\mathbb{Z}(\sigma_k^\perp \cap M, \mathbb{Q}/\mathbb{Z}))$ (respectively, $\otimes_{k=1}^m C^*(\mathrm{Hom}_\mathbb{Z}(\sigma_k^\perp \cap M, \mathbb{Q}/\mathbb{Z}))$), hence ρ gives a well-defined action.

2.4. **Time evolution and Hamiltonian.** It is natural to consider on the algebras $\mathcal{A}_{X,\Sigma}$ and $\mathcal{A}_{Y,\Sigma}$ a time evolution and quantum statistical mechanical properties that generalize the corresponding ones of the Bost-Connes system. We discuss first the case of the multiplicative endomotive (2.36), and then the similar case for the additive (2.35).

The following result describes a general approach to construct a time evolution together with a generating Hamiltonian.

Proposition 2.8. *Let $g : S \to \mathbb{R}_+^*$ be a semigroup homomorphism, and for $k = 1, \ldots, m$, let*

$$h_k : \mathrm{Hom}_\mathbb{Z}(\sigma_k^\perp \cap M, \mathbb{Z}) \to \mathbb{R}_+^* \tag{2.37}$$

be positive-valued functions such that

$$h_k(f_k') = g(\phi)\, h_k(f_k), \tag{2.38}$$

for all $f_k \in \mathrm{Hom}_\mathbb{Z}(\sigma_k^\perp \cap M, \mathbb{Z})$ and for all $\phi \in S$, where the f_k' are as in Definition 2.3. Let $\lambda = 1$ in the additive case of $(\mathcal{A}_X, \mathcal{H}_X)$ and $\lambda = m$ in the multiplicative case of $(\mathcal{A}_Y, \mathcal{H}_Y)$. Then setting

$$\sigma_t(\mu_\phi) = g(\phi)^{i\lambda t}\, \mu_\phi, \quad \text{and} \quad \sigma_t(e(\underline{r})) = e(\underline{r}), \tag{2.39}$$

for all $\phi \in S$ and for all $r_k \in (\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z})$, with $\underline{r} = (r_k)$, determines a time evolution σ_t on the endomotive algebras $\mathcal{A}_X$ and $\mathcal{A}_Y$, generated by a Hamitonian H determined by the operator

$$H\ \epsilon_{f_k} = \log(h_k(f_k))\ \epsilon_{f_k}, \tag{2.40}$$

for all $f_k \in \mathrm{Hom}_\mathbb{Z}(\sigma_k^\perp \cap M, \mathbb{Z})$.

Proof. As in Lemma 2.2, with a slight abuse of notation, we write $h_k(f_k \circ \phi)$ instead of $h_k(f_k')$.

It is easy to check that (2.39) defines a one-parameter family of automorphisms, both in the case of $\mathcal{A}_X$ and of $\mathcal{A}_Y$. The operator (2.46) uniquely determines self-adjoint positive linear operators, which we still denote by H, on both $\mathcal{H}_X$ and $\mathcal{H}_Y$. We check that H is indeed the Hamiltonian. In both the additive and the multiplicative case we have

$$e^{itH} e(\underline{r}) e^{-itH} = e(\underline{r}) e^{itH} e^{-itH} = e(\underline{r}). \tag{2.41}$$

In the additive case of $\mathcal{A}_X$ and $\mathcal{H}_X$ we have

$$\begin{aligned} e^{itH} \mu_\phi e^{-itH} \bigoplus_{k=1}^m \epsilon_{f_k} &= \bigoplus_{k=1}^m h_k(f_k)^{-it}\, h_k(f_k \circ \phi)^{it}\ \epsilon_{f_k \circ \phi} \\ &= g(\phi)^{it} \bigoplus_{k=1}^m \epsilon_{f_k \circ \phi} \\ &= \sigma_t(\mu_\phi) \bigotimes_{k=1}^m \epsilon_{f_k}. \end{aligned} \tag{2.42}$$

Similarly, in the multiplicative case of $\mathcal{A}_Y$ and $\mathcal{H}_Y$, we have

$$\begin{aligned} e^{itH}\mu_\phi e^{-itH} \bigotimes_{k=1}^m \epsilon_{f_k} &= \bigotimes_{k=1}^m h_k(f_k)^{-it}\, h_k(f_k\circ\phi)^{it}\ \epsilon_{f_k\circ\phi} \\ &= g(\phi)^{imt} \bigotimes_{k=1}^m \epsilon_{f_k\circ\phi} \\ &= \sigma_t(\mu_\phi) \bigotimes_{k=1}^m \epsilon_{f_k}. \end{aligned} \tag{2.43}$$

This proves the statement. ☐

In the multiplicative case, we can consider the more general form of the time evolution given below, which agrees with the one above in the case where $h = h_1 \otimes \cdots \otimes h_m$.

Corollary 2.9. *Let $g : S \to \mathbb{R}_+^*$ be a semigroup homomorphism, and*

$$h : \prod_{k=1}^m \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}) \to \mathbb{R}_+^*$$

be a positive-valued function such that

$$h(f_1', ..., f_m') = g(\phi)\, h(f_1, ..., f_m), \forall f_k \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}), \forall \phi \in S \tag{2.44}$$

for all $f_k \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})$ and for all $\phi \in S$. Then setting

$$\sigma_t(\mu_\phi) = g(\phi)^{it}\, \mu_\phi,\ \forall \phi \in S, \quad \text{and} \quad \sigma_t(e(\underline{r})) = e(\underline{r}),\ \forall \underline{r} = (r_k) \in \prod_{k=1}^m (\sigma_k^\perp \cap M) \otimes (\mathbb{Q}/\mathbb{Z}), \tag{2.45}$$

defines a time evolution $\sigma : \mathbb{R} \to Aut(\mathcal{A}_Y)$ generated by the Hamitonian

$$H\ (\otimes_{k=1}^m \epsilon_{f_k}) = \log(h(f_1, ..., f_k)) \bigotimes_{k=1}^m \epsilon_{f_k}, \quad \forall f_k \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}). \tag{2.46}$$

Proof. As before, we write $h(f_1\circ\phi, ..., f_m\circ\phi)$ instead of $h(f_1', ..., f_m')$. The argument follows exactly as in Proposition 2.8, with (2.43) replaced by

$$\begin{aligned} e^{itH}\mu_{\otimes,\phi}e^{-itH}(\bigotimes_{k=1}^m \epsilon_{f_k}) &= h(f_1, ..., f_k)^{-it}\, h(f_1\circ\phi, ..., f_m\circ\phi)^{it}\, (\bigotimes_{k=1}^m \epsilon_{f_k\circ\phi}) \\ &= g(\phi)^{it}\, (\bigotimes_{k=1}^m \epsilon_{f_k\circ\phi}) \\ &= \sigma_t(\mu_\phi)(\bigotimes_{k=1}^m \epsilon_{f_k}). \end{aligned} \tag{2.47}$$

☐

2.4.1. *Symmetries.* By symmetries of a C^*-dynamical system $(\mathcal{A}, \sigma_t)$ we mean here a group $\mathcal{G} \subset \mathrm{Aut}(\mathcal{A})$ of automorphisms of the C^*-algebra of observables that is compatible with the time evolution, $\sigma_t \circ \gamma = \gamma \circ \sigma_t$, for all $t \in \mathbb{R}$ and for all $\gamma \in \mathcal{G}$.

Lemma 2.10. *The group $\mathcal{G} = \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \hat{\mathbb{Z}}^*)$ acts on $(\mathcal{A}_{X,\Sigma}, \sigma_t)$ and $(\mathcal{A}_{Y,\Sigma}, \sigma_t)$ as symmetries.*

Proof. For $\gamma \in \mathcal{G}$ we set $\gamma\, \mu_\phi = \mu_\phi$, for all $\phi \in S$, and on $C^*((\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z})$ as $\gamma(v \otimes t) = v \otimes \gamma_v(t)$, for $v \otimes t$ in $(\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z}$ and $\gamma \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \hat{\mathbb{Z}}^*)$, with $\gamma_v = \gamma(v) \in \hat{\mathbb{Z}}^*$. The action is compatible with the time evolution, since σ_t fixes the algebra $C^*((\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z})$ and only acts nontrivially on the semigroup elements μ_ϕ, which are fixed by $\mathcal{G}$. ☐

It was shown in [10] that often, in the setting of quantum statistical mechanical systems it is useful to consider not only symmetries given by automorphisms but also endomorphisms, but here we will restrict our attention to automorphisms, as that will suffice for our purposes.

2.5. **The partition function.** One obtains from the covariant representations of Proposition 2.8 the following zeta functions.

Lemma 2.11. *The partition functions of the systems described in Proposition 2.8 are of the form*

$$Z(\beta) = \sum_{k=1}^{m} \sum_{f \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})} h_k(f)^{-\beta}, \tag{2.48}$$

in the additive case of $(\mathcal{A}_X, \sigma_t)$ *represented on* $\mathcal{H}_X$*, and*

$$Z(\beta) = \prod_{k=1}^{m} \sum_{f \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})} h_k(f)^{-\beta} \tag{2.49}$$

in the multiplicative case $(\mathcal{A}_Y, \sigma_t)$ *on* $\mathcal{H}_Y$*, which in the case of Corollary 2.9 takes the form*

$$Z(\beta) = \sum_{\underline{f} = (f_k) \in \prod_k \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})} h(f_1, ..., f_m)^{-\beta}. \tag{2.50}$$

Proof. The partition function for the system $(\mathcal{A}_X, \sigma_t)$ with the covariant representation on $\mathcal{H}_X$ with Hamiltonian H as in Proposition 2.8 is given by

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \sum_{\oplus_{k=1}^m \epsilon_{f_k} \in \mathcal{H}_X} \langle \oplus_{k=1}^m \epsilon_{f_k}, e^{-\beta H} \ \oplus_{k=1}^m \epsilon_{f_k} \rangle = \sum_{f_k \in \sqcup_{k=1}^m \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})} h_k(f_k)^{-\beta}. \tag{2.51}$$

The partition function for the system $(\mathcal{A}_Y, \sigma_t)$, with the time evolution specified in Proposition 2.8, is given by

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \sum_{\bigotimes_{k=1}^m \epsilon_{f_k} \in \mathcal{H}_Y} \langle \bigotimes_{k=1}^m \epsilon_{f_k}, e^{-\beta H} \bigotimes_{k=1}^m \epsilon_{f_k} \rangle = \sum_{\underline{f} = (f_k) \in \prod_k \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})} (h_1(f_1) \cdots h_m(f_m))^{-\beta}, \tag{2.52}$$

which clearly extends to the case of Corollary 2.9 as

$$Z(\beta) = \sum_{\bigotimes_{k=1}^m \epsilon_{f_k} \in \mathcal{H}_Y} \langle \bigotimes_{k=1}^m \epsilon_{f_k}, e^{-\beta H} \bigotimes_{k=1}^m \epsilon_{f_k} \rangle = \sum_{\underline{f} = (f_k) \in \prod_k \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})} h(f_1, ..., f_m)^{-\beta}. \tag{2.53}$$

□

Remark 2.12. The main difference between the additive and the multiplicative cases $\mathcal{A}_X$ and $\mathcal{A}_Y$, for a quantum statistical mechanical perspective is that, while the additive case corresponds to the geometric decomposition of the toric variety into torus orbits, the multiplicative case corresponds, as a quantum mechanical system, to regarding the torus orbits as independent systems, hence the decomposition of the partition function as a product of the partition functions associated to the different orbits.

In practice one has many ways to define appropriate functions g and h and thus the corresponding partition functions. Here we describe one natural way. First we need some additional results on the semigroup S.

2.5.1. *Semigroup and symmetries of the fan.* Recall that S_0 is the semigroup of all nonsingular linear transformations of N that is compatible with the fan Σ and bijective as a map of the relative interiors of cones in Σ. Since the lattice N has a fixed basis, we may identify elements of S_0 with $d \times d$ matrices. We have the following lemma regarding the elements in S_0.

Lemma 2.13. *Let $\phi \in S_0$, then the following holds:*

1. *$n\phi \in S_0$ for all $n \in \mathbb{N}$;*
2. *$\phi : \sigma_k \to \sigma_{\phi(k)}$ is a bijective linear map of cones, for all k;*
3. *ϕ leaves every cone invariant if and only if $\phi = nI$, for some $n \in \mathbb{N}$;*
4. *$\phi^T : \sigma_k^\perp \to \sigma_{\phi^{-1}(k)}^\perp$ is an isomorphism of vector spaces, for all k.*

Proof. These statements follow immediately from the geometry of the fan. □

Lemma 2.14. *Let $G_0 \subset S_0$ be defined by*

$$G_0 = \{\phi \in S_0 \,|\, \phi \neq n\phi',\ \forall \phi' \in S_0,\ \forall n \in \mathbb{N},\ n > 1\}. \tag{2.54}$$

Then G_0 is a subgroup of the permutation group of the cones of Σ.

Proof. Note that, two distinct elements $\phi_1 \neq \phi_2 \in G_0$ cannot induce the same permutation of cones in Σ, otherwise we would have an element $\phi_2^{-1}\phi_1 \in \mathrm{GL}_d(\mathbb{Q})$ that keeps all the cones invariant, which means that there exists an element $t \in \mathbb{N}$, such that $t\phi_2^{-1}\phi_1 \in S_0$, and therefore $t\phi_2^{-1}\phi_1 = nI$, for some $n \in \mathbb{N}$, leading to a contradiction. Thus, we can identify G_0 with a subgroup of the permutation group of the cones in Σ. □

Remark 2.15. Although it is not true in general that G_0 is a subgroup of $\mathrm{GL}_d(\mathbb{Z})$, i.e. that every element in G_0 has determinant ± 1, it does hold in many cases, including the case of the fan corresponding to the projective space $\mathbb{P}^d$.

2.5.2. *Time evolution and G-orbits.* Now we restrict our attention to those toric varieties X_Σ for which the group G_0 of (2.54) is indeed a subgroup of $\mathrm{GL}_d(\mathbb{Z})$. In these cases we have $S_0 = \mathbb{N} \times G_0$. Correspondingly we have $S = \mathbb{N} \times G$, where G is the group consisting of transposes of elements in G_0, which is again a subgroup of $\mathrm{GL}_d(\mathbb{Z})$.

Let N_k is the sublattice of N consisting of vectors orthogonal to $\sigma_k^\perp \cap M$, or equivalently the intersection of N with the vector space spanned by σ_k.

Theorem 2.16. *Let Σ be a fan for which $S = \mathbb{N} \times G$, with $G \subset \mathrm{GL}_d(\mathbb{Z})$. Given the choice of a basis of $N/N_1 \simeq \mathbb{Z}^{d_1}$, let $N_1^* = \{\xi \in N/N_1 \,|\, \gcd(\xi) = 1\}$, where $\gcd(\xi)$ is the greatest common divisor of the coordinates of $\xi \in \mathbb{Z}^{d_1}$. The choice of a constant $c \in \mathbb{R}$ and of a function $h_1 : N_1^* \to \mathbb{R}_+^*$ that is constant on the orbits of the subgroup $G_1 \subset G$ that fixes σ_1 determine functions g and h satisfying the condition* (2.44)*, which therefore define a time evolution on the (multiplicative) endomotive of the toric variety.*

Proof. By duality we can naturally identify each $\mathrm{Hom}_\mathbb{Z}(\sigma_k^\perp \cap M, \mathbb{Z})$ with the quotient N/N_k. If f_k corresponds to ξ_k under this identification, then $f_k \circ \phi$ corresponds to $\phi^T \xi_k$. Note that, since ϕ defines a permutation of the cones, there is also an action of G on the set of cones by permutation.

We define g as follows. Fix a constant $c \in \mathbb{R}$. Each element in S can be uniquely written as $n\phi$ for some $n \in \mathbb{N}$ and some $\phi \in G$. Then we set

$$g(n\phi) = n^c. \tag{2.55}$$

As for h, we define it separately for each orbit of the action of G on the set of cones. Suppose $\{\sigma_1, ..., \sigma_p\}$ is such an orbit, and let G_1 be the subgroup of G that leaves σ_1 invariant. Assuming for the moment that we have defined a positive-valued map h_1 on N/N_1 that is compatible with the action of $\mathbb{N} \times G_0$. We define h_j on N/N_j by fixing a $\phi \in G$ that takes σ_1 to σ_j and setting

$$h_j(\phi^T \xi) = h_1(\xi), \quad \forall \xi \in N/N_1. \tag{2.56}$$

It is easy to check that this is well-defined since ϕ^T gives an isomorphism between N/N_1 and N/N_j and h_1 is compatible with the action of $\mathbb{N} \times G_0$. In this way, we define positive valued functions $h_1, ..., h_p$ that are together compatible with the action of S, and we have, for each j,

$$\sum_{\xi_j \in N/N_j} h_j(\xi_j)^{-s} = \sum_{\xi_1 \in N/N_1} h_1(\xi_1)^{-s}. \tag{2.57}$$

To define h_1, we choose a basis of N/N_1, and we identify it with vectors in $\mathbb{Z}^{d_1}$, where d_1 is the rank of the quotient lattice N/N_1. Note that this is indeed a lattice, since it is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^{\perp} \cap M, \mathbb{Z})$. Then consider the orbit of the action of G_1 on N/N_1. For each $\xi \in \mathbb{Z}^{d_1}$, let $\gcd(\xi)$ denote, as above, the greatest common divisor of the d_1 coordinates of ξ. Then it can be easily shown that $\gcd(\xi)$ is the same for all ξ in one orbit. Let $N_1^* = \{\xi \in N/N_1 \simeq \mathbb{Z}^{d_1} \mid \gcd(\xi) = 1\}$. Then it follows that G_1 also acts on N_1^*. Let $h_1 : N_1^* \to \mathbb{R}_+^*$ be any function that is constant on each orbit, and we extend h_1 to N/N_1 by setting

$$h_1(n\xi^*) = n^c h_1(\xi^*), \forall \xi^* \in N_1^*, \tag{2.58}$$

which gives the desired h_1. Finally we set

$$h(f_1, ..., f_m) = \prod_{k=1}^{m} h_k(\xi_k). \tag{2.59}$$

Then it is clear that the desired identity

$$h(f_1 \circ \phi, ..., f_m \circ \phi) = g(\phi)\, h(f_1, ..., f_m)$$

holds for these functions. □

In this case the partition function is then of the following form.

Corollary 2.17. *Let Σ be a fan for which $S = \mathbb{N} \times G$, with $G \subset \mathrm{GL}_d(\mathbb{Z})$, and let σ_t be a time-evolution on the endomotives of the toric variety, defined using the functions g and h constructed as in Theorem 2.16. The partition function is then of the form*

$$Z(\beta) = \zeta(\beta)^m \prod_{k=1}^{m} \Big(\sum_{\xi_k^* \in N_k^*} h_k(\xi_k^*)^{-\beta} \Big). \tag{2.60}$$

Proof. We have

$$Z(\beta) = \sum_{(\xi_1, ..., \xi_m) \in \prod_{k=1}^{m} N/N_k} \prod_{k=1}^{m} h_k(\xi_k)^{-\beta},$$

which we write equivalently as

$$Z(\beta) = \prod_{k=1}^{m} \Big(\sum_{\xi_k \in N/N_k} h_k(\xi_k)^{-\beta} \Big) = \zeta(\beta)^m \prod_{k=1}^{m} \Big(\sum_{\xi_k^* \in N_k^*} h_k(\xi_k^*)^{-\beta} \Big).$$

□

Remark 2.18. In the most general cases, if the fan Σ is quite large and lacks symmetry, then the semigroup S only consists of matrices of the form nI (alternatively, for any fan Σ we can always replace S in our above construction with the sub-semigroup $\{nI | n \in \mathbb{N}^+\}$ and everything else follows exactly the same). In this case G is the trivial group and thus we can define each h_k on N/N_k independently, and h_k can be obtained from an arbitrary positive-valued function on N_k^*.

A more concrete example of the construction of Theorem 2.16 is obtained in the following way.

Corollary 2.19. *Let $\|\cdot\|_k$ be a norm on each real vector space $(N/N_k)_{\mathbb{R}}$. Setting $h_k(\xi) = \|\xi\|_k^c$ satisfies the identity* (2.44)*, with g as in* (2.55)*, hence it defines a time evolution. The corresponding partition function on the (multiplicative) endomotive is given by*

$$Z(\beta) = \prod_{k=1}^{m} \Big(\sum_{\xi \in N/N_k} ||\xi||_k^{-c\beta} \Big). \tag{2.61}$$

The additive cases are analogous, with the partition functions given by sums instead of products over the set of torus orbits.

2.5.3. *Projective spaces.* We consider the example of projective spaces, where several of the general properties discussed above can be seen more explicitly.

Lemma 2.20. *In the case of $\mathbb{P}^d$, the orbit space of M under the action of G is the orbit space of the subspace of $\mathbb{Z}^{d+1}$ given by solutions of $x_1 + ... + x_{d+1} = 0$ under coordinate permutations.*

Proof. The fan Σ_{Δ_d} associated to $\mathbb{P}^d$, consists of the cones generated by all proper subsets of $\{e_0, e_1, ..., e_d\}$, where $e_1, ..., e_d$ form the standard basis of N and $e_0 = -e_1 - ... - e_d$. Then it is not hard to see that G consists of all matrices whose d rows are d distinct vectors from $\{e_0, e_1, ..., e_d\}$. Then the orbit of a point $(a_1, ..., a_d)$ in M consists of all vectors whose d coordinates are d different elements of the multiset $\{a_1, a_2, ..., a_d, -a_1 - ... - a_d\}$. Therefore, we may identify the orbit space of M under G with the set of all $(d+1)$-subsets of $\mathbb{Z}$ that sum up to 0, or equivalently the orbit space of the subspace of $\mathbb{Z}^{d+1}$ defined by $x_1 + ... + x_{d+1} = 0$ under the action of coordinate permutation. □

The corresponding partition functions implicitly encode the information about this symmetry. In the Section 4 below we will give a more concrete construction for projective spaces, using the arithmetic height functions.

3. Endomotives of toric varieties and $\mathbb{F}_1$-geometry

There are currently many different approaches aimed at developing a form of algebraic geometry over the "field with one element" $\mathbb{F}_1$. For an overview of various contribution and their interrelatedness, we refer to reader to the survey [16].

Toric varieties play a crucial role in $\mathbb{F}_1$. They are the only class of $\mathbb{Z}$-varieties that admit $\mathbb{F}_1$-structures according to *all* of the existing variants of $\mathbb{F}_1$-geometry. In some of the strongest formulations, they are essentially the only varieties that descend to $\mathbb{F}_1$ (see for instance [22] for a comparative analysis). While other approaches allow for a broader range of varieties over $\mathbb{F}_1$, toric varieties remain an important class on which different constructions can be compared.

We discuss here the relation between the endomotive construction and the $\mathbb{F}_1$-structure on the toric variety. Relations between endomotives and $\mathbb{F}_1$-geometry had already been considered in [9], for the case of the Bost–Connes endomotive, and in [19], for its multivariable generalizations. We will consider here three different connections to $\mathbb{F}_1$-geometry: the relation between the semigroup action and the Λ-ring structure, following Borger's approach to $\mathbb{F}_1$-geometry via Λ-rings, [5]; the relation to Soulé's notion of varieties over $\mathbb{F}_1$, as in [21]; a weaker form of the endomotive construction that extends from the case of toric variety to $\mathbb{F}_1$-varieties defined by torified spaces, in the sense of [15].

3.1. **Endomotives and Λ-ring structures.** As in [5], an integral Λ-ring structure on a commutative ring R, whose underlying abelian group is torsion free, is given by an action of the semigroup $\mathbb{N}$ by endomorphisms of R, so that, for each prime p, the action ϕ_p of p on R is a Frobenius lift,

$$\phi_p(r) - r^p \in pR, \quad \forall r \in R. \tag{3.1}$$

As in [19], a $\mathbb{Q}$-algebra A has a Λ-ring structure if it has an action of $\mathbb{N}$ by endomorphisms, and $A = R \otimes \mathbb{Q}$, with R a commutative ring with a Λ-ring structure as above, inducing the same $\mathbb{N}$-action on A.

In order to compare the construction of endomotives of toric varieties given above with Λ-ring structures, we need to work with algebraic endomotives instead of the analytic ones discussed above.

Definition 3.1. *Given an abstract toric variety X_Σ defined over $\mathbb{Q}$, the algebraic abelian subalgebra of the endomotives of X_Σ are, respectively, the $\mathbb{Q}$-algebras*

$$\oplus_{k=1}^m \mathbb{Q}[(\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z}], \quad \text{and} \quad \otimes_{k=1}^m \mathbb{Q}[(\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z}]. \tag{3.2}$$

The semigroup S acts on both by endomorphisms, and one obtains the algebraic additive and multiplicative endomotives of X_Σ as the algebraic semigroup crossed products

$$\mathcal{A}_{X,\Sigma,\mathbb{Q}} = (\oplus_{k=1}^m \mathbb{Q}[(\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z}]) \rtimes S, \quad \text{and} \quad \mathcal{A}_{Y,\Sigma,\mathbb{Q}} = (\otimes_{k=1}^m \mathbb{Q}[(\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z}]) \rtimes S. \tag{3.3}$$

We also consider the subalgebras $\mathcal{A}^0_{X,\Sigma\mathbb{Q}}$ and $\mathcal{A}^0_{Y,\Sigma\mathbb{Q}}$ obtained, as above, as algebraic crossed products by the subsemigroup $S_0 \subset S$, with $S_0 = \{nI \,|\, n \in \mathbb{N}\}$.

Proposition 3.2. *Let X_Σ be an abstract toric variety defined over $\mathbb{Q}$. The abelian algebras* (3.2) *are direct limits of Λ-rings, with the Λ-ring structure given by the action of $S_0 = \{nI \,|\, n \in \mathbb{N}\}$. In the additive case, there are embeddings*

$$\sqcup_{k=1}^m X_{n,k} \simeq \sqcup_{k=1}^m \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}/n\mathbb{Z}) \hookrightarrow \sqcup_{k=1}^m O(\sigma_k) \subseteq X_\Sigma, \tag{3.4}$$

determined by embeddings of $\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}/n\mathbb{Z})$ into the torus orbit $O(\sigma_k)$ of X_Σ, which induce corresponding maps of Λ-rings, with respect to the Λ-ring structure on the toric variety.

Proof. The abelian algebra $\oplus_{k=1}^m \mathbb{Q}[(\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z}]$ is a direct limits of $A_n = \oplus_{k=1}^m \mathbb{Q}[(\sigma_k^\perp \cap M) \otimes \mathbb{Z}/n\mathbb{Z}] = R_n \otimes \mathbb{Q}$ with $R_n = \oplus_{k=1}^m \mathbb{Z}[(\sigma_k^\perp \cap M) \otimes \mathbb{Z}/n\mathbb{Z}]$. The action of $\mathbb{N}$ is the one given by $e(r_k) \mapsto e(\phi_p(r_k))$, which on the limit $A = \varinjlim_n A_n$ corresponds to $\mu_\phi^* e(r_k) \mu_\phi$.

The embeddings (3.4) are determined by the identification $O(\sigma_k) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{G}_m)$, as in (1.8). In particular, the subsemigroup S_0 acts on the part of the limit set $X_k = \varprojlim_n X_{n,k}$ in the torus $T_{\sigma_k^\perp \cap M} = O(\sigma_k)$ as a restriction to $X_k \subset T_{\sigma_k^\perp \cap M}$ of the action of $\mathbb{N}$ on the torus $T_{\sigma_k^\perp \cap M}$ given on the coordinates by $\sigma_p : t_j \mapsto t_j^p$. The compatibility of this action with the Frobenius action, showing that it defines a Λ-ring structure, follows in the same way as the analogous result for the original Bost–Connes case, given in [9] and [19]. The embeddings (3.4) determine maps of Λ-rings, since the Λ-ring structure on a toric variety X_Σ is compatible with the decomposition into torus orbits, namely, it induces the compatible Λ-ring structures described above on all the torus orbits $O(\sigma_k)$, see §2.4 of [5]. ◻

3.2. **Endomotives and Soulé's varieties over $\mathbb{F}_1$.** A relation between the endomotive of the Bost–Connes system and Soulé's notion of varieties over $\mathbb{F}_1$ was described in [9], based on the construction of a model over $\mathbb{Z}$ of the endomotive. We show here that, in a similar way, we can obtain models over $\mathbb{Z}$ for the endomotives of toric varieties.

3.2.1. *Integer models of the endomotives.* As in the Bost–Connes case analyzed in [9], the crossed product algebras (3.3) of our algebraic endomotives of toric varieties admits a model over $\mathbb{Z}$, which is obtained in the following way.

Consider the algebras $\mathcal{A}_{X,\Sigma,\mathbb{Z}}$ and $\mathcal{A}_{Y,\Sigma,\mathbb{Z}}$ generated by

$$\mathcal{C}_{X,\Sigma,\mathbb{Z}} := \oplus_{k=1}^{m} \mathbb{Z}[(\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z}]$$

and

$$\mathcal{C}_{Y,\Sigma,\mathbb{Z}} := \otimes_{k=1}^{m} \mathbb{Z}[(\sigma_k^\perp \cap M) \otimes \mathbb{Q}/\mathbb{Z}],$$

respectively, and by elements μ_ϕ^* and $\tilde{\mu}_\phi$, for all $\phi \in S$, satisfying the relations $\tilde{\mu}_{\phi_1}\tilde{\mu}_{\phi_2} = \tilde{\mu}_{\phi_1\phi_2}$, $\mu_{\phi_1}^*\mu_{\phi_2}^* = \mu_{\phi_1\phi_2}^*$, for all $\phi_1, \phi_2 \in S$, and $\mu_\phi^*\tilde{\mu}_\phi = |\det(\phi)|^\lambda$, where $\lambda = 1$ in the additive case and $\lambda = m$ in the multiplicative case, and

$$\mu_\phi^*\, a = \sigma_\phi(a)\, \mu_\phi^* \quad \text{ and } \quad a\, \tilde{\mu}_\phi = \tilde{\mu}_\phi\, \sigma_\phi(a),$$

for all $\phi \in S$ and for all $a \in \mathcal{C}_{X,\Sigma,\mathbb{Z}}$ or $\mathcal{C}_{Y,\Sigma,\mathbb{Z}}$, where $\sigma_\phi(e(\underline{r})) = e(\phi \cdot \underline{r})$.

3.2.2. *Soulé's gadgets and varieties.* Soulé's approach to $\mathbb{F}_1$-geometry is based on the concept of an $\mathbb{F}_1$-gadget and of an $\mathbb{F}_1$-variety. A gadget consists of data $(X, \mathcal{A}_X, e_{x,\sigma})$ with $X : \mathcal{R} \to$ Sets a covariant functor from the category $\mathcal{R}$ of finitely generated flat rings, $\mathcal{A}_X$ a complex algebra, and evaluation maps given by algebra homomorphisms $e_{x,\sigma} : \mathcal{A}_X \to \mathbb{C}$, for all $x \in X(R)$ and $\sigma : R \to \mathbb{C}$, satisfying $e_{f(y),\sigma} = e_{y,\sigma\circ f}$, for any ring homomorphism $f : R' \to R$. An affine variety $V_\mathbb{Z}$ over $\mathbb{Z}$ determines a gadget with $X_V(R) = \mathrm{Hom}(\mathcal{O}(V), R)$ and $\mathcal{A}_X = \mathcal{O}(V) \otimes \mathbb{C}$. A gadget is an affine $\mathbb{F}_1$-variety is $X(R)$ is finite and there is an affine variety $W_\mathbb{Z}$ with a morphism of gadgets $X \to X_W$ such that morphisms of gadgets $X \to X_V$ are induced by morphisms of varieties $W_\mathbb{Z} \to V_\mathbb{Z}$. Heuristically, one should think of the case where $R = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$, for which $X(R)$ gives the cyclotomic points.

3.2.3. *Endomotives as $\mathbb{F}_1$-varieties.* In [9], the Bost–Connes endomotive is described in terms of a family of $\mathbb{F}_1$-varieties $\mu^{(k)}$, in the sense of Soulé, determined by the functor $\underline{\mu}^{(k)} : \mathcal{R} \to$ Sets given by $\underline{\mu}^{(k)}(R) = \{r \in R \,|\, r^k = 1\}$, represented by $\underline{\mu}^{(k)}(R) = \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}], R)$. The algebra $\mathcal{A}_{\mu^{(k)}}$ of the gadget is given by $\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}] \otimes_\mathbb{Z} \mathbb{C} = \mathbb{C}[\mathbb{Z}/k\mathbb{Z}]$. The projective limit $\mu^{(\infty)} = \varprojlim_k \mu^{(k)}$ is given by the functor that assigns $\underline{\mu}^{(\infty)}(R) = \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], R)$. The maps in this projective limit are exactly the ones used in the construction of the Bost–Connes endomotive, coming from the action of the semigroup $\mathbb{N}$. This tower of zero-dimensional affine $\mathbb{F}_1$-varieties $\mu^{(k)}$ describes the inductive system of extensions $\mathbb{F}_{1^k}$, as defined by Kapranov–Smirnov, with $\mathbb{F}_{1^\infty} = \varinjlim_k \mathbb{F}_{1^k}$, where the "extension of coefficients to $\mathbb{Z}$" is formally given by

$$\mathbb{F}_{1^k} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[t, t^{-1}]/(t^k - 1),$$

with $\mathbb{Z}[t, t^{-1}]/(t^k - 1) \simeq \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$.

We have an analogous construction in the case of endomotives of toric varieties.

Proposition 3.3. *Let X_Σ be an abstract toric variety defined over $\mathbb{Z}$. To the abelian parts $\mathcal{C}_{X,\Sigma,\mathbb{Z}}$ and $\mathcal{C}_{Y,\Sigma,\mathbb{Z}}$ of the endomotives $\mathcal{A}_{X,\Sigma,\mathbb{Z}}$ and $\mathcal{A}_{Y,\Sigma,\mathbb{Z}}$ one can assign projective systems of affine $\mathbb{F}_1$-varieties in the sense of Soulé, where the maps in the projective systems are induced by the action of the semigroup S of the endomotives.*

Proof. Consider the functors $\underline{\mu}_{X,\Sigma}^{(n)} : \mathcal{R} \to$ Sets given by

$$\underline{\mu}_{X,\Sigma}^{(n)}(R) = \mathrm{Hom}_\mathbb{Z}(\oplus_{k=1}^{m} \mathbb{Z}[(\sigma_k^\perp \cap M) \otimes \mathbb{Z}/n\mathbb{Z}], R)$$

$$\underline{\mu}_{Y,\Sigma}^{(n)}(R) = \mathrm{Hom}_\mathbb{Z}(\otimes_{k=1}^{m} \mathbb{Z}[(\sigma_k^\perp \cap M) \otimes \mathbb{Z}/n\mathbb{Z}], R).$$

Using $X_{n,k} \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z}/n\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}((\sigma_k^\perp \cap M) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, we see that the varieties $\mu^{(n)}_{X,\Sigma}$ associated to the above functors form a projective system where the projective limits are, respectively, $\mathcal{C}_{X,\Sigma,\mathbb{Z}}$ and $\mathcal{C}_{Y,\Sigma,\mathbb{Z}}$, with the maps of the projective system coming from the elements $\phi \in S$. The analytic datum of the $\mathbb{F}_1$-gadgets is given by the $\mathbb{C}$-algebras $\mathcal{C}_{X,\Sigma,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathcal{C}_{Y,\Sigma,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. The construction is otherwise completely analogous to the case discussed in [9]. □

Unlike the Λ-ring structure of the toric variety discussed above, which replies of semigroup actions providing consistent liftings of the Frobenius action, the construction described here uses cyclotomic points on the toric variety and it fits with Soulé's and Manin's general philosophy, [17], [21] of cyclotomy as descent data from $\mathbb{Z}$ to $\mathbb{F}_1$.

3.3. **Endomotives and torified spaces.** We discuss here another approach to $\mathbb{F}_1$-geometry, based on "torifications", which was developed by López-Peña and Lorscheid in [15]. We recall their main definition of a torified space.

Definition 3.4. *Let X be a variety over $\mathbb{Z}$. A* torification *of X is a disjoint union $T = \sqcup_{j\in I} T_j$ of tori $T_j = \mathbb{G}_m^{d_j}$, together with a morphism $e_X : T \to X$, such that $e_X|_{T_j}$ is an immersion for all j and e_X a bijection of the set of $\mathbb{K}$-points, $T(\mathbb{K}) \simeq X(\mathbb{K})$, over any field $\mathbb{K}$.*

A toric variety is a torified space, through its decomposition into torus orbits $X_\Sigma = \sqcup_{k=1}^m O(\sigma_k)$. However, the notion of torification is much more general and it includes, for example, spaces with cell decompositions. More restrictive conditions on the torification (affine, regular) can be imposed that restrict the class of (affinely, regularly) torified spaces, see [16] for more details. For our purposes, we do not impose any of these stronger conditions, and we consider torifications as in Definition 3.4 above. We show that a simple variant of the construction of the multivariable Bost–Connes endomotives of [19] provides endomotives associated to arbitrary torified spaces, which generalize (in a weaker form) the construction we described for toric varieties.

Proposition 3.5. *Let X be a variety over $\mathbb{Z}$, which admits a torification, as in Definition 3.4, and let $T = \sqcup_{j\in I} T_j$ with $T_j = \mathbb{G}_m^{d_j}$ be a choice of a torification on X. For each torus T_j of the torification, consider the projective system*

$$X_n(T_j) = \{t \in \mathbb{T}^{d_j} \mid s_n(t) = \gamma\},$$

with $\gamma = (1, 1 \ldots, 1) \in \mathbb{T}^{d_j}$ and with $s_n : t = (t_i)_{i=1,\ldots,d_j} \mapsto s_n(t) = (t_i^n)_{i=1,\ldots,d_j}$. The semigroup $\mathbb{N}$ acts by endomorphisms on the projective limit $X(T_j) = \varprojlim_n X_n(T_j)$ and on the algebra $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes d_j}$ with $\mathrm{Spec}(X(T_j)) = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes d_j}$. This determines an additive algebraic endomotive $\mathcal{A}_{X(T),\mathbb{Q}} := (\oplus_{j\in I} \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes d_j}) \rtimes \mathbb{N}$ and a multiplicative $\mathcal{A}_{Y(T),\mathbb{Q}} := (\otimes_{j\in I} \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes d_j}) \rtimes \mathbb{N}$. The corresponding analytic endomotives are given by $\mathcal{A}_{X(T)} := (\oplus_{j\in I} C^*(\mathbb{Q}/\mathbb{Z})^{\otimes d_j}) \rtimes \mathbb{N}$ and $\mathcal{A}_{Y(T)} := (\otimes_{j\in I} C^*(\mathbb{Q}/\mathbb{Z})^{\otimes d_j}) \rtimes \mathbb{N}$, respectively. There are representation by bounded operators of $\mathcal{A}_{X(T)}$ and $\mathcal{A}_{Y(T)}$ on the Hilbert spaces $\mathcal{H}_{X(T)} = \oplus_{j\in I} \ell^2(\mathbb{N}^{d_j})$ and $\mathcal{H}_{Y(T)} = \otimes_{j\in I} \ell^2(\mathbb{N}^{d_j})$, respectively. Suppose given a semigroup homomorphism $g : \mathbb{N} \to \mathbb{R}_+^*$ and $\mathbb{R}_+^*$-valued functions h_j on $\mathbb{N}^{d_j}$ satisfying $h_j(s_n(\underline{n}_j)) = g(n)\, h_j(\underline{n}_j)$, for all $\underline{n}_j \in \mathbb{N}^{d_j}$ and $n \in \mathbb{N}$ with $s_n(\underline{n}_j) = n\underline{n}_j$ the coordinatewise multiplication. Then setting $\sigma_t(\mu_n) = g(n)^{it}$ and $\sigma_t(e_j(\underline{r})) = e_j(\underline{r})$, for all $n \in \mathbb{N}$, for all $e_j(\underline{r})$ in $C^*(\mathbb{Q}/\mathbb{Z})^{\otimes d_j})$, for all $j \in i$ and all $t \in \mathbb{R}$, defines a time evolution on $\mathcal{A}_{X(T)}$ and $\mathcal{A}_{Y(T)}$, respectively, whose Hamiltonian is determined by the operator

$$H\epsilon_{\underline{n}_j} = \log(h_j(\underline{n}_j))\ \epsilon_{\underline{n}_j},$$

where $\{\epsilon_{\underline{n}_j}\}$ denotes the canonical orthonormal basis of $\ell^2(\mathbb{N}^{d_j})$.

Proof. Everything follows the same construction as in [19], using only the subsemigroup $\{nI_{d_j}\}$ of $M_{d_j}(\mathbb{Z})^+$ for the endomotive construction, so we will not reproduce the details here. The construction of the time evolution and Hamiltonian is modeled on the analogous construction we gave in Section 2 for toric varieties and the argument follows in the same way. □

While in the endomotive construction for toric varieties the semigroup S depends not only on the decomposition of X_Σ into tori, but also on how these tori fit together as orbits of the same torus action (through the G subgroup of S), the construction for torified spaces is necessarily weaker and only contains the information on the decomposition into tori given by a choice of torification. One can think of the action of the semigroup $\mathbb{N}$ in the endomotive of a torified space that is not a toric variety as a weaker replacement for a Λ-ring structure, associated to a choice of torification.

4. Height functions and endomotives

Now we generalize the construction described in Section 2 to build endomotives of projective toric varieties, and study their properties using the arithmetic height function. The arithmetic height function is a key notion in diophantine geometry and it encodes much information concerning the arithmetic of the varieties, so it seems particularly interesting to include this kind of data as part of the quantum statistical mechanics of the endomotives of toric varieties.

We will focus on the concrete examples of projective spaces and affine spaces, though much of what we describe can be generalized to other toric varieties and their height functions.

4.1. **Height functions and toric varieties.**

Height functions play an important role in addressing questions on the distribution of rational points on algebraic varieties, see [1].

For a variety X defined over a number field $\mathbb{K}$, endowed with a choice of a line bundle $\mathcal{L}$ with an adelic metric and a section s in a neighborhood U of a point $x \in X(\mathbb{K})$, one defines a height function as

$$H_{\mathbb{K},\mathcal{L},s}(x) = \prod_{v\in \mathrm{Val}(\mathbb{K})} \|s(x)\|_v^{-1}.$$

For an overview of the properties of this type of functions, with respect to dependence on the data $\mathbb{K}$, $\mathcal{L}$, s, we refer the reader to the survey [7]. We simply write $H_{\mathcal{L}}(x)$ in the following.

4.1.1. *Height zeta functions.*

The height zeta function is the associated generating function,

$$Z_{X,\mathbb{K}}(\mathcal{L},\beta) = \sum_{x\in X(\mathbb{K})} H_{\mathcal{L}}(x)^{-\beta}.$$

For $\mathcal{L}$ an ample line bundle, Northcott's theorem implies that $N_X(\mathcal{L},B) = \#\{x \in X(\mathbb{K}) \,|\, H_{\mathcal{L}}(x) \leq B\}$ grows at most polynomially on B. If $N_X(\mathcal{L},B) < B^a$, then $Z_{X,\mathbb{K}}(\mathcal{L},\beta)$ converges for $\Re(\beta) > a$, see [7] for more details. For a general overview of properties of the height zeta functions and applications to the study of algebraic points on varieties, we also refer the reader to the survey [18]. For more background on the arithmetic height function, see also [4] and [20].

Height functions on toric varieties and the behavior of the height zeta function were studied in [2], [3], where it is shown that, for $\mathcal{L}$ with Chern class in the interior of the cone of effective divisors, the height zeta function on a smooth projective toric variety gives an asymptotic formula for the number of rational points of bounded height of the form $N_X(\mathcal{L},B) \sim c(X,\mathcal{L},\mathbb{K})\, B^{a(\mathcal{L})}(\log B)^{b(\mathcal{L})-1}$ where the exponents $a(\mathcal{L})$ and $b(\mathcal{L})$ and the constant $c(X,\mathcal{L},\mathbb{K})$ are determined by the geometry of X according to a conjecture of Manin's.

4.1.2. *Heights on projective spaces.* On projective spaces $\mathbb{P}^d$, one can see that the information coming from the height function is carried by the archimedean valuation, with the p-adic factor equal to one, see §3.1 of [20]. Thus, one sets

$$H_{\mathbb{Q}}(x) = \max\{|x_0|_\infty, \ldots, |x_d|_\infty\},$$

for $x \in \mathbb{P}^d(\mathbb{Q})$ and, which extends to points $x \in \mathbb{P}^d(\mathbb{K})$ over number fields $\mathbb{K}$ by

$$H_{\mathbb{K}}(x) = H_{\mathbb{Q}}(x)^{[\mathbb{K}:\mathbb{Q}]}.$$

The *absolute height* of a point $x \in \mathbb{P}^d(\bar{\mathbb{Q}})$, denoted by $H(x)$, is defined as $H(x) = H_{\mathbb{K}}(x)^{1/[\mathbb{K}:\mathbb{Q}]}$, where $\mathbb{K}$ is a number field such that $x \in \mathbb{P}^d(\bar{\mathbb{Q}})$, so that the result is independent of $\mathbb{K}$. The *absolute logarithmic height* on $\mathbb{P}^d(\bar{\mathbb{Q}})$ is the function

$$h(x) = \log(H(x)). \tag{4.1}$$

Remark 4.1. Clearly, an analogous height zeta function for the logarithmic height would not be convergent, but, as we will see below, one can restrict to suitable choices of subsets of the set of algebraic points that cut down the multiplicities to logarithmic size, for which one is then able to define a zeta function with the desired properties based on the logarithmic height. This choice seems unnatural from the usual point of view of height functions in diophantine geometry, but we will see that it is instead quite natural from the point of view of endomotives.

4.2. **Endomotives of projective spaces.** First we focus on the case of the projective space $\mathbb{P}^d$. This is a toric variety, with the lattice of one parameter subgroups given by $N = \mathbb{Z}^{d+1}/(1, ..., 1)$.

Fix a homogeneous coordinate system on the projective space $\mathbb{P}^d$, then, as we have seen in Lemma 2.20, the elements of the group G are precisely the permutations of coordinates.

4.2.1. *Algebraic points and the endomotive.* Let $X_0 \in \mathbb{P}^d(\bar{\mathbb{Q}})$ be a finite subset of the $\bar{\mathbb{Q}}$-algebraic points in the projective space that is invariant under the action of G. We replace the set of distinguished points in the torus orbits, which we used in our previous construction, with the set X_0, and we consider the preimages of X_0 under the action of the semigroup S, which again form a projective system.

The argument of Proposition 2.1 applies exactly in the same way here. It shows that the projective limit X is the disjoint union of the limits corresponding to each individual point in X_0. More precisely, as topological spaces, we have

$$X = \varprojlim_{\phi} \phi^{-1}(X_0) = \bigsqcup_{x \in X_0} \varprojlim_{\phi} \phi^{-1}(x) = \bigsqcup_{x \in X_0} \varprojlim_{n} (nI)^{-1}(x), \tag{4.2}$$

and the corresponding function algebras satisfy

$$C(X) = \bigoplus_{x \in X_0} C(\varprojlim_{n} (nI)^{-1}(x)). \tag{4.3}$$

As before, we also consider a multiplicative version, where instead of the disjoint union X we consider the product

$$Y = \prod_{x \in X_0} \varprojlim_{n} (nI)^{-1}(x), \tag{4.4}$$

and the corresponding algebra

$$C(Y) = \bigotimes_{x \in X_0} C(\varprojlim_{n} (nI)^{-1}(x)). \tag{4.5}$$

Now we analyze more closely the algebra $C(\varprojlim_n (nI)^{-1}(x))$. We obtain a more explicit description as follows.

Proposition 4.2. *The algebra $C(\varprojlim_n (nI)^{-1}(x))$ can be identified with*

$$C(\varprojlim_n (nI)^{-1}(x)) \simeq C^*(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell+1}/(1,...,1), \mathbb{Q}/\mathbb{Z})), \tag{4.6}$$

where $\ell+1$ is the number of nonzero coordinates of x.

Proof. Let $x = (x_0, ..., x_d)$. Without loss of generality, we suppose that $x_0, x_1, \ldots, x_\ell$ are all the nonzero coordinates. Then

$$(nI)^{-1}(x) = \{(\sqrt[n]{x_0}, ..., \sqrt[n]{x_\ell}, 0, .., 0) \in \mathbb{P}^d(\bar{\mathbb{Q}})\},$$

where $\sqrt[n]{x_j}$ denotes an n-th root of x_j. We show that the projective system consisting of the $(nI)^{-1}(x)$ is isomorphic to the natural projective system consisting of the $(\mathbb{Z}/n\mathbb{Z})^{\ell+1}/(1,...,1)$, as projective systems of discrete topological spaces.

For each $z \in \bar{\mathbb{Q}}^*$, we can uniquely write $z = re^{i\theta}$ where $r > 0$ and $0 \leq \theta < 2\pi$. Then we write $z^{1/n} = r^{1/n} e^{\frac{i\theta}{n}}$ for $n \in \mathbb{N}$, where $r^{1/n}$ is the unique n-th root of r in $\mathbb{R}^*_+$. Clearly, we have $(z^{1/n_1 n_2})^{n_2} = z^{1/n_1}, \forall n_1, n_2 \in \mathbb{N}$. With this notation, each $\sqrt[n]{x_j}$ can be uniquely written as $x_j^{1/n} \xi_n^k$, for some $k \in \{0, 1, ..., n-1\}$, where $\xi_n = e^{\frac{2\pi i}{n}}$. Therefore, the set of $(d+1)$-tuples $(\sqrt[n]{x_0}, ..., \sqrt[n]{x_\ell}, 0, .., 0)$ can be identified with $(\mathbb{Z}/n\mathbb{Z})^{\ell+1}$. In terms of homogeneous coordinates, we have $(nI)^{-1}(x) \simeq (\mathbb{Z}/n\mathbb{Z})^{\ell+1}/(1,...,1)$ as discrete spaces. Clearly we have the following commutative diagram:

$$\begin{array}{ccc} (n_1 n_2 I)^{-1}(x) & \xrightarrow{n_2 I} & (n_1 I)^{-1}(x) \\ \downarrow & & \downarrow \\ (\mathbb{Z}/n_1 n_2 \mathbb{Z})^{\ell+1}/(1,...,1) & \xrightarrow{n_2} & (\mathbb{Z}/n_1\mathbb{Z})^{\ell+1}/(1,...,1) \end{array} \tag{4.7}$$

where the upper map $n_2 I$ raises the homogeneous coordinates to n_2-th powers and the lower map is multiplication by n_2. Therefore,

$$\varprojlim_n (nI)^{-1}(x) \simeq \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^{\ell+1}/(1,...,1) \simeq (\mathbb{Z}^{\ell+1}/(1,...,1)) \otimes \hat{\mathbb{Z}} \tag{4.8}$$

as topological spaces, which implies isomorphisms of the corresponding commutative C^*-algebras of functions

$$C(\varprojlim_n (nI)^{-1}(x)) \simeq C((\mathbb{Z}^{\ell+1}/(1,...,1)) \otimes \hat{\mathbb{Z}}). \tag{4.9}$$

By Pontryagin duality, we also have

$$\begin{aligned} C((\mathbb{Z}^{\ell+1}/(1,...,1)) \otimes \hat{\mathbb{Z}}) &\simeq C^*(\mathrm{Hom}_{\mathbb{Z}}((\mathbb{Z}^{\ell+1}/(1,...,1)) \otimes \hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})) \\ &\simeq C^*(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell+1}/(1,...,1), \mathbb{Q}/\mathbb{Z})). \end{aligned} \tag{4.10}$$

Note that here by $\mathbb{Z}^{\ell+1}/(1,...,1)$ we actually mean the subgroup $(\mathbb{Z}^{\ell+1} \oplus 0^{d-\ell})/(1,...,1)$ of $N = \mathbb{Z}^{d+1}/(1,...,1)$, where the coordinates that vanish in $\mathbb{Z}^{d+1}$ correspond to the coordinates of x that vanish. This embedding into N provides a way to identify the algebras for points x in different torus orbits of the projective space. □

4.2.2. *Hilbert space representations.* As in Lemma 2.2, we construct crossed product algebras of the endomotive and represent them on Hilbert spaces.

Lemma 4.3. *Fix a nontorsion element $\alpha \in \bar{\mathbb{Q}}^*$. For $\ell = \ell(x)$ as above, let*

$$\mathcal{B}_{\alpha,x} := \{(\alpha^{k_0} x_0^{k_0'}, ..., \alpha^{k_\ell} x_\ell^{k_\ell'}) \in \mathbb{P}^d(\bar{\mathbb{Q}}) \,|\, k_0, k_0', \dots, k_\ell, k_\ell' \in \mathbb{Z}\}. \tag{4.11}$$

Then there is a natural representation of the C^*-algebra $C^*(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell+1}/(1,...,1), \mathbb{Q}/\mathbb{Z}))$ associated to x on the Hilbert space $\ell^2(\mathcal{B}_{\alpha,x})$, given by

$$e(f)\epsilon_{\alpha^k x^{k'}} = \exp(2\pi i f(k))\, \epsilon_{\alpha^k x^{k'}}, \tag{4.12}$$

for all $f \in Hom_{\mathbb{Z}}(\mathbb{Z}^{\ell+1}/(1,...,1), \mathbb{Q}/\mathbb{Z})$, and for all $k \in \mathbb{Z}^{\ell+1}/(1,...,1), k' \in \mathbb{Z}$, where we abbreviate $(\alpha^{k_0} x_0^{k'}, ..., \alpha^{k_\ell} x_\ell^{k'})$ by $\alpha^k x^{k'}$.

The proof is straghtforward and thus omitted.

Corollary 4.4. *It follows that we have naturally induced representations of the algebras*

$$C(X) = \bigoplus_{x \in X_0} C(\varprojlim_n (nI)^{-1}(x))$$

and

$$C(Y) = \bigotimes_{x \in X_0} C(\varprojlim_n (nI)^{-1}(x)),$$

respectively, on the Hilbert spaces

$$\bigoplus_{x \in X_0} \ell^2(\mathcal{B}_{\alpha,x}) = \ell^2(\mathcal{B}_{X,\alpha}), \qquad \text{where} \quad \mathcal{B}_{X,\alpha} = \bigsqcup_{x \in X_0} \mathcal{B}_{\alpha,x}, \tag{4.13}$$

and

$$\bigotimes_{x \in X_0} \ell^2(\mathcal{B}_{\alpha,x}) = \ell^2(\mathcal{B}_{Y,\alpha}), \qquad \text{where} \quad \mathcal{B}_{Y,\alpha} = \prod_{x \in X_0} \mathcal{B}_{\alpha,x}. \tag{4.14}$$

Again, this follows immediately from the previous statement. We then represent the semigroup on the same Hilbert spaces in the following way.

Lemma 4.5. *The semigroup S has representations on the Hilbert spaces* (4.13) *and* (4.14)*, determined by setting*

$$\mu_\phi \epsilon_{\alpha^k x^{k'}} = \epsilon_{\phi\cdot(\alpha^k x^{k'})}, \quad \forall \phi \in S, \ \forall \alpha^k x^{k'} \in \mathcal{B}_{\alpha,x}, \ \forall x \in X_0. \tag{4.15}$$

Proof. Here by $\phi \cdot (\alpha^k x^{k'})$ we mean the action of S on $\mathbb{P}^d(\bar{\mathbb{Q}})$. Note that we can write $\phi = n\phi_0$ for some $n \in \mathbb{N}$ and $\phi_0 \in G$, then

$$\phi \cdot (\alpha^k x^{k'}) = \phi_0 \cdot (\alpha^{nk} x^{nk'}) = \alpha^{nk} (\phi_0 \cdot x)^{nk'} \in \mathcal{B}_{\alpha, \phi_0 \cdot x} \tag{4.16}$$

where by $\phi_0 \cdot x$ we mean the action of G on X_0. It is then clear that (4.15) has the right properties and defines a representation of the semigroup S by isometries of the Hilbert space. □

Moreover, just like in the previous section, the adjoint of the isometry μ_ϕ is given by

$$\mu_\phi^* \epsilon_{\alpha^k x^{k'}} = \begin{cases} \epsilon_{\alpha^r x^{r'}} & \alpha^k x^{k'} = \phi \cdot (\alpha^r x^{r'}) \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

As in Lemma 2.5 and Lemma 2.6, the following lemma relates the operators $e(f)$ and μ_ϕ.

Let $\phi \cdot f_x$ denote the action of S on $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)+1}/(1,...,1), \mathbb{Q}/\mathbb{Z})$ given by precomposition by ϕ, considered as a linear map on the lattice $N = \mathbb{Z}^{d+1}/(1,...,1)$.

Lemma 4.6. *We have the following identities*

$$e(\phi \cdot f_x) = \mu_\phi^* e(f_x) \mu_\phi \tag{4.18}$$

and

$$\mu_\phi e(f_x) \mu_\phi^* = \frac{1}{n} \sum_{\phi \cdot f = f_x} e(f) \tag{4.19}$$

for all $\phi \in S$, *and all* $f_x \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)+1}/(1,...,1), \mathbb{Q}/\mathbb{Z})$, *for all* $x \in X_0$.

Proof. The proof is completely analogous to those of Lemma 2.2 and Lemma 2.5 and are therefore omitted. The action $\phi \cdot f_x$ on $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)+1}/(1,...,1), \mathbb{Q}/\mathbb{Z})$ determines corresponding actions of S on

$$\bigsqcup_{x \in X_0} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)+1}/(1,...,1), \mathbb{Q}/\mathbb{Z})$$

and

$$\prod_{x \in X_0} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)+1}/(1,...,1), \mathbb{Q}/\mathbb{Z}),$$

respectively, satisfying the relations (4.18), (4.19), where $\#\{f \,|\, \phi \cdot f = f_x\} = n$. ☐

We have crossed product algebras of the endomotive in the following form.

Definition 4.7. *The additive endomotive associated to* $\mathbb{P}^d$ *with the choice of a finite set* X_0 *of* $\bar{\mathbb{Q}}$*-algebraic points is the crossed product algebra*

$$\begin{aligned} \mathcal{A}_{X,X_0} := C(X) \rtimes_\rho S = &\ (\oplus_{x \in X_0} C(\varprojlim_n (nI)^{-1}(x))) \rtimes_\rho S \\ = &\ (\oplus_{x \in X_0} C^*(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)+1}/(1,...,1), \mathbb{Q}/\mathbb{Z}))) \rtimes_\rho S \end{aligned} \tag{4.20}$$

and the multiplicative endomotive

$$\begin{aligned} \mathcal{A}_{Y,X_0} := C(Y) \rtimes_\rho S = &\ (\otimes_{x \in X_0} C(\varprojlim_n (nI)^{-1}(x))) \rtimes_\rho S \\ = &\ (\otimes_{x \in X_0} C^*(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)+1}/(1,...,1), \mathbb{Q}/\mathbb{Z}))) \rtimes_\rho S. \end{aligned} \tag{4.21}$$

4.2.3. *Time evolution, Hamiltonian, and height function.* We construct time evolutions on the endomotives (4.20) and (4.21) using the same general technique that we described in the previous section. We need a preliminary lemma on the behavior of the logarithm height function under the action of the semigroup S.

Lemma 4.8. *The logarithm height function* h *on* $\mathbb{P}^d(\bar{\mathbb{Q}})$ *satisfies*

$$h(\phi x) = n h(x), \quad \forall \phi = n\phi_0 \in S, \ \phi_0 \in G. \tag{4.22}$$

Proof. This is clear since $\phi = n\phi_0$ acts on $\mathbb{P}^d(\bar{\mathbb{Q}})$ by permuting the homogeneous coordinates according to ϕ_0 and then raising them to n-th power. ☐

We then obtain the following construction of a time evolution on the endomotives.

Lemma 4.9. *Let* h *be the logarithm height functions on* $\mathbb{P}^d$. *For all* $t \in \mathbb{R}$, *setting*

$$\sigma_t(\mu_\phi) = n^{it} \mu_\phi, \quad \forall\, \phi = n\phi_0 \in S \ \text{ with } \ \phi_0 \in G, \quad \text{ and } \quad \sigma_t(e(f_x)) = e(f_x) \tag{4.23}$$

for all

$$f_x \in \bigsqcup_{x \in X_0} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)+1}/(1,...,1), \mathbb{Q}/\mathbb{Z}),$$

determines time evolutions on the algebras $\mathcal{A}_{X,X_0}$ *and* $\mathcal{A}_{Y,X_0}$ *of* (4.20) *and* (4.21). *The Hamiltonians implementing these time evolutions in the representations of* $\mathcal{A}_{X,X_0}$ *and* $\mathcal{A}_{Y,X_0}$ *on the* *respective Hilbert spaces* $\ell^2(\mathcal{B}_{X,\alpha})$ *and* $\ell^2(\mathcal{B}_{Y,\alpha})$ *are determined in both cases by the assignment, for* $\alpha^k x^{k'} \in \mathcal{B}_{\alpha,x}$*,*

$$H\epsilon_{\alpha^k x^{k'}} = \log(h(\alpha^k x^{k'}))\ \epsilon_{\alpha^k x^{k'}}. \tag{4.24}$$

Proof. By Lemma 4.8 we see that the logarithmic height function satisfies the condition required for Proposition 2.8, hence the operator H defines as in (4.24) is indeed the Hamiltonian of the time evolution, when extended to an operator on the direct sum, respectively the tensor product, of the spaces $\ell^2(\mathcal{B}_{\alpha,x})$, for $x \in X_0$. □

4.2.4. *Partition function and a logarithmic height zeta function.* We obtain the height zeta function of (multi)-sets $\mathcal{B}_{X,\alpha}$ and $\mathcal{B}_{Y,\alpha}$ of algebraic points in $\mathbb{P}^d(\bar{\mathbb{Q}})$ as the partition function of the quantum statistical mechanical system.

Lemma 4.10. *The partition function of the quantum statistical mechanical system* $(\mathcal{A}_{X,X_0}, \sigma_t)$*, represented on the Hilbert space* $\ell^2(\mathcal{B}_{X,\alpha})$ *is a zeta function of the form*

$$Z_{X,X_0,\alpha}(\beta) = \sum_{x \in \mathcal{B}_{X,\alpha}} h(x)^{-\beta}. \tag{4.25}$$

The multiplicative case is analogous.

Proof. This follows directly from Lemma 4.9, and the Hamiltonian (4.24) that the partition function of the system is given by

$$Z_{X,X_0,\alpha}(\beta) = \mathrm{Tr}(e^{-sH}) = \sum_{x \in \mathcal{B}_{X,\alpha}} h(x)^{-\beta}, \tag{4.26}$$

which is a logarithmic height zeta function of the (multi)-set $\mathcal{B}_{X,\alpha}$ of algebraic points in $\mathbb{P}^d(\bar{\mathbb{Q}})$. □

4.2.5. *Convergence.* One knows that the exponential height zeta function converges for all $\Re(\beta) > a$, for some sufficiently large $a > 0$. For a zeta function based on the logarithmic height to have similar convergence properties, one needs the size of the points with a given height bound inside the sampling set $\mathcal{B}_{X,\alpha}$ to be growing with only at most logarithmic speed rather than polynomially.

Lemma 4.8, with the scaling property (4.22) of the height function, together with the construction of the sets $\mathcal{B}_{X,\alpha}$ and $\mathcal{B}_{Y,\alpha}$, and the action of the semigroup on it as in (4.15), show that this is indeed the case.

4.2.6. *Height and degree bounds.* Lemma 3.3 implies that for any $a, b \in \mathbb{R}^+$ we can use

$$X_0 = \{x \in \mathbb{P}^d(\bar{\mathbb{Q}}) | h(x) \leq a, [\mathbb{Q}(x) : \mathbb{Q}] \leq b\}$$

as the starting point of the construction since G acts on it. In this case the endomotive system carries some of the arithmetic information on the $\bar{\mathbb{Q}}$-algebraic points in the projective space with bounded height and degree.

4.3. **Endomotives of affine spaces.** We modify the construction given above for projective spaces, to obtain endomotives associated to the affine spaces $\mathbb{A}^d$, with similar properties.

Note that as a toric variety $\mathbb{A}^d$ is given by the fan spanned by the canonical basis vectors $e_1, ..., e_d$ in $N = \mathbb{Z}^d$, and it is easy to see that the group G is given by permutations of cartesian coordinates in the affine space.

As in Section 4.2, we start with a G-invariant finite subset X_0 of $\mathbb{A}^d(\bar{\mathbb{Q}})$. Here we need the additional assumption that for each $x \in X_0$, each coordinate of x is a non-torsion element of $\bar{\mathbb{Q}}^*$. Then, we use the same preimage projective limit construction. The main difference is that now we are not working with homogeneous coordinates anymore, so that each point has a unique d-tuple coordinate corresponding to it, which simplifies the construction. The analog of Lemma 4.2 now takes the following form.

Lemma 4.11. *In the case of the affine space $\mathbb{A}^d$, the algebra $C(\varprojlim_n (nI)^{-1}(x))$ is isomorphic to $C^*(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\ell, \mathbb{Q}/\mathbb{Z}))$, where $\ell = \ell(x)$ is the number of nonzero coordinates of $x = (x_1, ..., x_d)$.*

Proof. The only difference, with respect to the analogous statement for $\mathbb{P}^d$, is that now we do not need to quotient the lattices by $(1, \dots, 1)$. This, the commutative diagram (4.7) now becomes

$$\begin{array}{ccc} (n_1 n_2 I)^{-1}(x) & \xrightarrow{n_2 I} & (n_1 I)^{-1}(x) \\ \downarrow & & \downarrow \\ (\mathbb{Z}/n_1 n_2 \mathbb{Z})^\ell & \xrightarrow{n_2} & (\mathbb{Z}/n_1\mathbb{Z})^\ell. \end{array} \tag{4.27}$$

The rest of the proof follows exactly as in Lemma 4.2. ∎

4.3.1. *The endomotive algebra.* The additive endomotive associated to $\mathbb{A}^d$ with the choice of a finite set X_0 of $\bar{\mathbb{Q}}$-algebraic points is the crossed product algebra

$$(\oplus_{x \in X_0} C^*(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)}, \mathbb{Q}/\mathbb{Z}))) \rtimes_\rho S \tag{4.28}$$

and the multiplicative endomotive

$$(\otimes_{x \in X_0} C^*(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)}, \mathbb{Q}/\mathbb{Z}))) \rtimes_\rho S. \tag{4.29}$$

4.3.2. *The representation.* The main difference, with respect to the analogous construction for projective spaces, appears in the Hilbert space we represent the algebra on.

Lemma 4.12. *Consider the set*

$$\mathcal{B}_x = \{(x_1^{k_1}, ..., x_\ell^{k_\ell}) \in \mathbb{A}^d(\bar{\mathbb{Q}}) \,|\, k_1, ..., k_\ell \in \mathbb{Z}\}. \tag{4.30}$$

Then there is a natural representation of the C^*-algebra $C^*(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\ell, \mathbb{Q}/\mathbb{Z}))$, with $\ell = \ell(x)$, for a point $x \in X_0$, on the Hilbert space $\ell^2(\mathcal{B}_x)$, given by

$$e(f)\epsilon_{x^k} = \exp(2\pi i f(k))\, \epsilon_{x^k}, \quad \forall f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\ell(x)}, \mathbb{Q}/\mathbb{Z}), \text{ and } \forall k \in \mathbb{Z}^{\ell(x)}, \tag{4.31}$$

where we abbreviate $(x_1^{k_1}, ..., x_\ell^{k_\ell})$ with the notation x^k.

The proof is again straghtforward and thus ommited. Note that since each x_j is non-torsion, $\mathcal{B}_x$ naturally identifies with $\mathbb{Z}^\ell$. Also note that, in this affine case, we do not need to introduce a new parameter α to define the representation.

The representation of S on the Hilbert spaces $\ell^2(\mathcal{B}_X)$ and $\ell^2(\mathcal{B}_Y)$ where

$$\mathcal{B}_X = \coprod_{x \in X_0} \mathcal{B}_x \quad \text{and} \quad \mathcal{B}_Y = \prod_{x \in X_0} \mathcal{B}_x$$

is the same as before, given by the action of S on $\mathcal{B}_X$ and $\mathcal{B}_Y$, and with the adjoints given by partial inverses.

Lemma 4.6 relating these operators also holds. This allows us to define the endomotive as the semigroup crossed product in the usual way, with the same time evolution and Hamitonian as in Section 4.2 (note that Lemma 4.8 also holds in the affine case).

4.3.3. *A logarithmic height zeta function.* The partition function of this system is then given by a logarithmic height zeta function on $\mathcal{B}_X$ or $\mathcal{B}_Y$ of the form

$$Z(\beta) = \sum_{x \in \mathcal{B}_X} h(x)^{-\beta}. \tag{4.32}$$

5. Gibbs states

We consider here again the general construction of endomotives of abstract toric varieties described in Section 2. Thus, we consider C^*-dynamical systems $(\mathcal{A}_{X,\Sigma}, \sigma_t)$ and $(\mathcal{A}_{Y,\Sigma}, \sigma_t)$ as above, with the time evolution and covariant representations constructed as in Proposition 2.8.

Lemma 5.1. *Let $g : S \to \mathbb{R}_+^*$ be a semigroup homomorphism as in Proposition 2.8, with functions h_k as in* (2.38)*. Suppose that there is a $\beta_g > 0$ such that for all $\Re(\beta) > \beta_g$ the zeta function*

$$Z_g(\beta) = \sum_{\phi \in S} g(\phi)^{-\beta} < \infty. \tag{5.1}$$

Then there is a choice of h_k as in (2.38) *such that the zeta function*

$$Z_k(\beta) = \sum_{f_k \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})} h_k(f_k)^{-\beta} < \infty \tag{5.2}$$

for all $\Re(\beta) > \beta_c$.

Proof. Let $\mathcal{F}_{k,S} \subset \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})$ be a fundamental domain for the action of the semigroup S. A function h_k as in Proposition 2.8 is determined by assigning h_k on $\mathcal{F}_{k,S}$ and by extending it using (2.38). In particular, one can choose the function h_k so that, for $\Re(\beta) > \beta_g$ the series

$$Z_{\mathcal{F}_{k,S}}(\beta) = \sum_{f \in \mathcal{F}_{k,S}} h_k(f)^{-\beta} < \infty. \tag{5.3}$$

By (2.38), the partition function (5.2) factors as a product

$$Z_k(\beta) = Z_g(\beta)\ Z_{\mathcal{F}_{k,S}}(\beta),$$

hence the statement follows. ∎

The previous statement can be reformulated as in Corollary 2.17, in the case where $S = \mathbb{N} \times G$ with $G \subset \mathrm{Gl}_d(\mathbb{Z})$.

5.1. **Polylogarithm-type functions on toric varieties.**

We now consider Gibbs states of the form

$$\varphi(a) = \frac{\mathrm{Tr}(\pi(a)\, e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}, \tag{5.4}$$

in a given covariant representation (π, H) of a C^*-dynamical system $(\mathcal{A}, \sigma_t)$. These are well defined and satisfy the KMS_β condition whenever $\mathrm{Tr}(e^{-\beta H}) < \infty$.

In the case of the original Bost–Connes system, the Gibbs states (5.4) are well defined for $\beta > 1$ and they are values at roots of unity ζ_r, $r \in \mathbb{Q}/\mathbb{Z}$,

$$\varphi(e(r)) = \zeta(\beta)^{-1} \sum_{n \geq 1} \frac{\zeta_r^n}{n^\beta} = \zeta(\beta)^{-1} \mathrm{Li}_\beta(\zeta_r)$$

of the polylogarithm function

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s},$$

normalized by the Riemann zeta function. These states depend on the choice of an element $\rho \in \hat{\mathbb{Z}}^*$ viewed as a choice of an embedding of the abstract roots of unity $\mathbb{Q}/\mathbb{Z}$ in $\mathbb{C}$, with $\zeta_r = \rho(r)$.

Gibbs states of endomotives of toric varieties provide an analog of polylogarithm functions on toric varieties, of the form

$$\varphi(e(\underline{r})) = \frac{1}{Z(s)} \sum_{k=1}^{m} \sum_{f_k \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})} \exp(2\pi i f_k(r_k))\ h_k(f_k)^{-\beta}, \tag{5.5}$$

normalized by the zeta function

$$Z(s) = \sum_{k=1}^{m} \sum_{f_k \in \mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \mathbb{Z})} h_k(f)^{-\beta}.$$

These are KMS_β states for the system $(\mathcal{A}_{X,\Sigma}, \sigma_t)$. The case of $\mathcal{A}_{Y,\Sigma}$ is similar. As in the case of the Bost–Connes system, we can change the KMS-state (5.5) by precomposing with an automorphism of the dynamical system in $\mathrm{Hom}_{\mathbb{Z}}(\sigma_k^\perp \cap M, \hat{\mathbb{Z}}^*)$, as in Lemma 2.10.

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