1 D-modules

Weyl algebra (Heisenberg quantum mechanics)

$pq - qp = i\hbar$ (Heisenberg uncertainty principle)

K field (char. zero): say $\mathbb{R}$ or $\mathbb{C}$

$K[x]$ $\quad x = (x_1, \dots, x_n)$ ring of polynomials in n variables (commuting)

$\mathrm{End}_K(K[x])$ = linear operators $L : K[x] \to K[x]$

$K[x]$ ∞-dim vector space over K (no topology for now)

- $\hat{x}_i$ = multiplication operators

$$(\hat{x}_i f)(\underline{x}) = x_i \cdot f(\underline{x})$$

- derivations $\partial_i(f) = \dfrac{\partial f}{\partial x_i}$

- n-th Weyl algebra W_n

subalgebra of $\mathrm{End}_K(K[x])$ generated by $\hat{x}_i$ and ∂_i $\quad i = 1, \dots, n$

noncommutative

$$\partial_i \hat{x}_i f = x_i \partial_i f + f$$

$$\partial_i \hat{x}_i - \hat{x}_i \partial_i = \mathbb{1} \text{ (identity)}$$

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$$[\partial_i, \hat{x}_j] = \delta_{ij} \cdot 1$$
$$[\partial_i, \partial_j] = [\hat{x}_i, \hat{x}_j] = 0$$
} relations in the Weyl algebra

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\alpha = (\alpha_1, \dots, \alpha_n)$ $\quad |\alpha| = \alpha_1 + \dots + \alpha_n$

$\beta = (\beta_1, \dots, \beta_n)$ $\quad \partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$

$\mathcal{B} = \{ x^\alpha \partial^\beta \}$ basis of W_n as k-vector space

So can think of W_n as the ring of diff. operators w/ polynomial coeff's

$D \in W_n$ diff op

__Note__: "degree" in Weyl alg not same as usual deg of diff op (max derivation order) but accounts also for deg of polynomial coeff's

e.g. $\deg_{W_n}(2x_1\partial_2 + x_1 x_2^3 \partial_1 \partial_2) = 6$

while deg usual sense is 2

__Reason__: want to maintain "symmetry" between position & momentum variables x, ∂

$$\deg_{W_n}(D + D') \leq \max\{\deg_{W_n}(D), \deg_{W_n}(D')\}$$
$$\deg_{W_n}(DD') = \deg_{W_n}(D) + \deg_{W_n}(D')$$

$$\deg_{W_n}[D, D'] \leq \deg_{W_n}(D) + \deg_{W_n}(D') - 2$$

$\boxed{3}$ $[\partial^\beta, x^\alpha] = \partial_i [\partial^{\beta - e_i}, x^\alpha] + [\partial_i, x^\alpha] \partial^{\beta - e_i}$

by induction $\deg_{W_n} [\partial^{\beta-e_i}, x^\alpha] \leq |\alpha| + |\beta| - 3$

$(\ |\beta - e_i| = |\beta| - 1\ )$

and $\deg_{W_n} [\partial_i, x^\alpha] \leq |\alpha| + 1 - 2 = |\alpha| - 1$

so $\deg_{W_n} [\partial^\beta, x^\alpha] \leq |\alpha| + |\beta| - 2$ (using sum est.!)

then $\partial^\beta x^\alpha = [\partial^\beta, x^\alpha] + x^\alpha \partial^\beta$

$\deg_{W_n} (x^\alpha \partial^\beta) = |\alpha| + |\beta|$

$\Rightarrow \deg_{W_n} (\partial^\beta x^\alpha) \leq \max\{ |\alpha| + |\beta|, |\alpha| + |\beta| - 2 \}$
$= |\alpha| + |\beta|$

then for $D = x^\sigma \partial^\beta$ $\quad D' = x^\alpha \partial^\eta$

$$DD' = x^{\sigma+\alpha} \partial^{\beta+\eta} + \overbrace{x^\sigma \underbrace{P}_{} \partial^\eta}^{Q_1}$$

$P = [\partial^\beta, x^\alpha]$ $\qquad \deg(Q_1) \leq \deg(D) + \deg(D') - 2$
by induction hyp.

$\Rightarrow \deg(DD') \leq \deg(x^{\sigma+\alpha} \partial^{\beta+\eta}) = \deg(D) + \deg(D')$

similarly for $\deg(D'D) = x^{\sigma+\alpha} \partial^{\beta+\eta} + Q_2$
$\deg(Q) \leq \deg(D) + \deg(D') - 2$

$\Rightarrow [D, D'] = Q_1 - Q_2$ $\qquad \deg[D, D'] \leq \max\{\deg Q_1, \deg Q_2\}$
$= \deg D + \deg D' - 2$

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W_n is simple (no two-sided ideals)
non-zero ← $aI \subset I$, $Ib \subset I$ $\forall a, b \in W_n$

if $\exists D \neq 0$ in $I \subset W_n$ of <u>min degree</u>

$\exists$ some i s.t. $[x_i, D] \neq 0$ (D contains ∂_i)

or $\exists$ some i: $[\partial_i, D] \neq 0$ (D contains x_i)

but then because ideal $[x_i, D]$ or $[\partial_i, D]$ also in I
but both are of lower degree ↯

$\Rightarrow$ consequence every endomorphism of W_n is injective
(kernel would be a two sided ideal)

<u>question</u>: is every endom of W_n an automorphism?
(since know injectivity: is every end surjective?)
This is <u>Jacobian conjecture</u> (major open problem in mathematics)

More general "rings of diff operators" over a ring R ← specifies type of coefficients of the diff operators

e.g. $R = \mathcal{C}(\Omega)$ continuous functions
$R = \mathcal{C}^\infty(\Omega)$ smooth functions etc.
$R = \mathbb{C}((z))$ rational functions

Weyl algebra is ring of diff op's where $R = K[x_1, \dots, x_n]$ polynomial ring

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Modules over the Weyl algebra

R ring R-module M (rep of R)
$m \mapsto r.m$ (left / right mod if R n.c.)

e.g. $K[x] = k[x_1, .., x_n]$ is a module over W_n

$$K[x] = W_n / \sum_{i=1}^{n} W_n \partial_i$$

also $K[\partial] = K[\partial_1, .., \partial_n]$ (diff. op's w/ constant coefficients)
is mod over W_n

$$K[\partial] = W_n / \sum_{i=1}^{n} W_n \cdot x_i$$

$U \subset \mathbb{C}$ open set $\mathcal{H}(U) =$ holom. functs on U

$$\begin{array}{c}\mathcal{H}(U)\\ \cup \\ \mathbb{C}[z]\end{array}$$

also $W_1(\mathbb{C})$ module
z $\frac{d}{dz}$ generators acting by multipl. & derivation on $\mathcal{H}(U)$

a "torsion element" in this module (some $D \in W_1(\mathbb{C})$ w/ $Df = 0$)
is a holom. function satisfying
a diff equation w/ polynomial coefficients

6 D-module of a differential equation

$D \in W_n$ diff equatn (w/ polyn. coeff's)

$$D = \sum_\alpha g_\alpha \partial^\alpha \rightsquigarrow \text{equation } D(f) = \sum_\alpha g_\alpha \partial^\alpha(f) = 0$$

or systems of equations:

$$(*) \begin{cases} D_1(f) = 0 \\ \vdots \\ D_k(f) = 0 \end{cases} \quad D_1, \dots, D_k \in W_n$$

D-module associated to equation $(*)$

$$M = W_n \Big/ \sum_{i=1}^{k} W_n \cdot D_i$$

a polynomial solution of $(*)$: $f \in K[X]$ s.t. $D_i(f) = 0$ $i = 1, \dots, k$

Thm: vector space of polynomial solutions of $(*)$ is $\operatorname{Hom}(M, K[X])$ morphisms of W_n-modules

in fact: if $f \in K[X]$ polyn. sol'n of $(*)$

$\sigma_f : W_n \to K[X]$ mapping $1 \in W_n$ to f (& $D \in W_n \mapsto D(f)$)

then if $\tilde{D} \in J = \sum_{i=1}^{k} W_n D_i$ i.e. $\tilde{D} = \sum_{i=1}^{k} \delta_i D_i$

$\rightsquigarrow \tilde{D}(f) = 0$ (since all $D_i(f) = 0$)

$\Rightarrow \sigma_f : M \to K[X]$

[7] so polyn. sol'n $f \longrightarrow \sigma_f \in \mathrm{Hom}(\mathcal{M}, K[X])$

$f \longmapsto \sigma_f$ linear map (since D_i linear operators)

conversely:

$\tau \in \mathrm{Hom}(\mathcal{M}, K[x])$

take $1 \in \mathcal{M}$ (i.e. $1 \in W_n$ modulo $\sum_i W_n D_i$)

$\rightsquigarrow$ $\underset{\substack{\| \\ h}}{\tau(1)}$ some polynomial in $K[x]$

this h is a solution of $D_i(h) = 0$

(and $\tau = \sigma_h$)

$\tau(1 + \sum_i \delta_i D_i) = h$

$\tau(\sum_i \delta_i D_i) = 0$

but $\tau(1) = h \Rightarrow \tau(\tilde{D}) = \tilde{D}(h)$

so $\tau(\sum_i \delta_i D_i) = \sum_i \delta_i D_i(h) = 0$ for all δ_i

$\Rightarrow D_i(h) = 0$

Note:
$\mathrm{Hom}(\mathcal{M}, K[x])$ is a K-vector space (can be ∞-dim)
(not a module over W_n nor over $K[x]$)

Not only polyn. solutions e.g.

$\mathcal{C}^\infty(\Omega)$ solutions would be (same argument)

$\mathrm{Hom}(\mathcal{M}, \mathcal{C}^\infty(\Omega))$ same $\mathcal{M}$ = D-module of the equation

homomorphisms of W_n-modules

If look for solutions of $D(f) = 0$ of a certain type $f \in \mathcal{S}$ $\Rightarrow$ $\mathrm{Hom}(\mathcal{M}, \mathcal{S})$

<u>Note</u>: "microfunctions"
"microlocal analysis"

$\rightsquigarrow$ direct limits of modules

e.g. $\mathcal{H}(D_\varepsilon^*)$ hol. funct's on a punctured disk D_ε^* of radius ε

and direct limit as $\varepsilon \to 0$ as W_1-modules

objects of this lim "microfunctions"

can again interpret in this way

convergence of smooth funct's $f_\varepsilon \to \delta$ distributions

(Malgrange)

Maximally overdetermined systems in PDEs

there is a notion of <u>dimension</u> for modules M over W_n (generalizing notion of dim for vectorspaces)

(uses a filtration Γ of M

and <u>Hilbert polyn.</u> $\chi(t,\Gamma,M) = \sum_{i=0}^{t} \dim_K (\Gamma_i/\Gamma_{i-1}) = \dim_K(\Gamma_t)$

degree of as polyn. in t is $\dim(M)$)

$M = \bigoplus_{i \geq 0} M_i \qquad \sum_{i=0}^{s} \dim_K(M_i) = \chi(s)$ Hilbert polyn)

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Holonomic D-modules $\mathcal{M}$ s.t. $\dim(\mathcal{M}) = n$

$\mathcal{M}$: modules over W_n

examples $K[x_1, \dots, x_n]$, $K[\partial_1, \dots, \partial_n]$ hol D-modules for W_n (dim $2n$ as mod)

like Lagrangians (only positions / only momenta) in symplectic space (q, p) of positions & momenta w/ symplectic form $\omega = d\lambda = \sum_i dp_i \wedge dq_i$

Liouville 1-form $\lambda = \sum_i p_i \, dq_i$

Lagrangian: middle dim. & in ker of ω

(pairs q, p w/ canonical relations $[\hat{q}, \hat{p}] = 1$ give pairs of Lagr. that count pos/momenta)

this relation to sympl. geometry can be made more precise through a geom. construction

This is in fact the description of the symplectic structure of the cotangent bundle $T^*\Omega$

$q \in \Omega$ $\quad p \in T^*_x \Omega$

$\lambda = \sum_i p_i \, dq^i$ Liouville 1-form on $T^*\Omega$

what does it describe? $T^*\Omega \leadsto$ sections are 1-forms on Ω

$\boxed{10}$ $q \in \Omega$ $\quad \lambda_q \in T_q^*\Omega$ $\quad \lambda_q = df_q$ or combin. of such

if want 1-form on $T^*\Omega$ (not on Ω)

at each $(q,\omega) \in T^*\Omega$ $\quad \omega \in T_x^*\Omega$

assign $\lambda_{(q,\omega)}$ 1-form

but ω is a 1-form on Ω and have

projection $T^*\Omega \xrightarrow{\pi} \Omega$

$(q,\omega) \longmapsto x$

so $\pi^*\omega$ = 1-form on $T^*\Omega$

$\lambda_{(q,\omega)} = \pi^*\omega$ Liouville form

i.e. if $V = \sum_i a_i \partial_{q_i} + \sum_j b_j \partial_{p_j}$

$\pi_* V = \sum_i a_i \partial_{q_i}$ $\qquad \lambda_{(q,p)}(V) = p(\pi_* V)_q$

<u>Symplectic form</u>: 2-form on $T^*\Omega$ $\qquad \underbrace{(V,W)}_{\text{pairs w/ vector fields}} \mapsto \omega(V,W)$

$\omega = \sum_{i,j} \omega_{ij}\, dq_i \wedge dp_j$ $\qquad \omega: T\Omega \otimes T\Omega \to \mathbb{R}$

$d\omega = 0$ closed and $\omega^n = \omega \wedge \dots \wedge \omega \neq 0$ nowhere vanishing

scalar density (vol. form)

Classical Hamiltonian mechanics is closely related to the symplectic geometry of $T^*\Omega$

$T^*\Omega$ = phase space

$f : \Omega \to \mathbb{R}$ smooth function df 1-form

X_f vector field defined by $\omega(V, X_f) = df(V)$

for any $V \in T\Omega$ vector field

Poisson bracket $\{f, g\} = \omega(X_g, X_f)$

$H = H(q,p)$ Hamiltonian (smooth function on symplectic $T^*\Omega$)

Hamiltonian equations of classical mechanics

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i} \\[2mm] \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q_i} \end{cases} \iff \frac{df}{dt} = \{f, H\}$$

Poisson brackets evolution of classical observables

applied to $f = q_i$ or $f = p_i$

$\{f, H\} = 0$ invariants: conserved quantities

Lagrangians in $(T^*\Omega, \omega)$ symplectic

L Lagrangian if $\omega|_L \equiv 0$ and

$\dim L = n$ half-dimensional in $T^*\Omega$

$s_\alpha : \Omega \to T^*\Omega$ section of cotg bdle: $\pi \circ s_\alpha = id|_\Omega$

$s_\alpha(x) = \alpha_x$ 1-form $\alpha_x \in T^*_x\Omega$

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$\Omega_\alpha = s_\alpha(\Omega) \subset T^*\Omega$ image of section

Lagrangian if $\omega|_{\Omega_\alpha} \equiv 0$

Liouville 1-form $\omega = d\lambda$

$\omega|_{\Omega_\alpha (x,\alpha_x)} = s_\alpha^*(\omega)_x = s_\alpha^*(d\lambda)_{(x,\alpha_x)}$

$= s_\alpha^* d\pi^* \alpha_x = d(\pi \circ s_\alpha)^* \alpha_x = d\alpha_x$

$d\alpha = 0$ So Ω_α Lagrangian iff $d\alpha = 0$

so $f : \Omega \to \mathbb{R}$ $\{(x, df_{|x})\}$ = Lagrangian in $T^*\Omega$

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associated to $\mathcal{M} \rightsquigarrow Ch(\mathcal{M})$ characteristic variety of D-module $\mathcal{M}$

holonomic D-module $\longleftrightarrow$ char variety $Ch(\mathcal{M})$ Lagrangian (standard sympl. str. in $\mathbb{C}^{2n}$)

$Ch(M)$ characteristic variety of the D-module M

- for D single differential operator

$\sigma(D)$ = principal symbol of D

$$\sigma(D)(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha \quad \text{for} \quad D = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$$

$Ch(M) = \{ \xi \mid \sigma(D)\xi = 0 \}$ hypersurface

M D-module of equation $D(f) = 0$

(not holonomic unless $n=1$ as $\xi \in T^*\Omega$ dim $2n$ and $Ch(M)$ hypersurf dim $2n-1$ not dim n)

- for a system of PDEs

$$D_i f = 0$$

then $Ch(M)$ has codimension at most n
codim $Ch(M) \le n$

$$Ch(M) = \{ \xi \mid \sigma(D_i)\xi = 0 \}$$

if $Ch(M)$ has codim$=n$ (hence also dim$=n$) and Lagrangian submanifold of $T^*\Omega$

Holon. D-modules are
max overdetermined systems of PDE's

$$\begin{cases} D_1(f) = 0 \\ \vdots \\ D_n(f) = 0 \end{cases} \qquad D_i \in W_n$$

and all lin. indep.
(codim $= n$)

14 In general: overdetermined systems of PDE's

$$\begin{cases} \sum_{i=1}^{m} D_{i,1}(f_i) = 0 \\ \vdots \\ \sum_{i=1}^{m} D_{i,r}(f_i) = 0 \end{cases}$$

$\underline{f} = (f_1, \dots, f_m)$ unknown functions

$D_{i,j}$ = linear diff. operators $j = 1, \dots, r$ in n-variables $\underline{x} = (x_1, \dots, x_n)$

say first order for simplicity

overdetermined if

$m < r' \leq r$

number of lin ind. equations

maximally overdetermined:

$m = 1$ single f unknown

$$\begin{cases} D_1(f) = 0 \\ \vdots \\ D_r(f) = 0 \end{cases}$$

and $r' \leq r$ as large as possible

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case of first order D_i

if $r = 1$, $m = 1$ single eq.

$$D(f) = a_1 \frac{\partial f}{\partial x_1} + \dots + a_n \frac{\partial f}{\partial x_n} = 0$$

consider a system of ODE's

$$\frac{dx_1}{ds} = a_1 \quad \dots \quad \frac{dx_n}{ds} = a_n$$

15 suppose that $f_1, \dots, f_{n-1}$ are integrals of motion of $f_i(x_1, \dots, x_n)$

i.e. $$\frac{df_i}{ds} = \frac{\partial f_i}{\partial x_1}\frac{dx_1}{ds} + \dots + \frac{\partial f_i}{\partial x_n}\frac{dx_n}{ds} = 0$$

then general solution of $D(f) = 0$

$$f = \phi(f_1, \dots, f_{n-1})$$

as integral curves of vector field $a_1\frac{\partial}{\partial x_1} + \dots + a_n\frac{\partial}{\partial x_n}$

are curves cut out by eq'n

$$\begin{cases} f_1 = c_1 \\ \vdots \\ f_{n-1} = c_{n-1} \end{cases}$$

$m=1$

$r>1$: since at most n indep. equations $r \leq n$

& when $r=n$ only the trivial constant solutions as for n lin indep eq'ns can $\leadsto \frac{\partial f}{\partial x_i} = 0$ for each i

in other words: for first order lin. eqn's

$$\sigma(D_i)(\xi) = \sum_{k=1}^{n} a_{k,i}(x)\,\xi_k = 0$$ hyperplane in $T^*_x\Omega$

$Ch(M) = \{\xi \mid \sigma(D_i)\xi = 0$ intersection of n hyperplanes $\}$

$\{0\} \subset T^*_x\Omega \leadsto Ch(M) = \Omega$ as zero section of $T^*\Omega$

 for higher order more interesting loci

$\sigma(D_i)(\xi) = 0$ e.g. for order 2 quadratic forms

$$\sum_{j.k} q_{jk,i}(x)\,\xi_j\,\xi_k = 0 \qquad i = 1, \dots, r$$

intersection of r quadrics $Q_i(x) = \left\{ \sum_{jk} q_{jk,i}(x)\,\xi_j\,\xi_k = 0 \right\}$ in $T_x^*\Omega$ family over $x \in \Omega$

etc.