Convexity, Simplexes, Probabilities

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Convexity is the mathematical property that describes the operation of mixing

- notation: $\mathbb{E}^n$ Euclidean space in n-dimensions, $\mathbb{A}^n$ affine space (usually over $\mathbb{C}$ or $\mathbb{R}$), $\mathbb{P}^n = (\mathbb{A}^{n+1} \smallsetminus \{0\})/\mathbb{G}_m$ projective space (quotient by multiplicative group $\mathbb{C}^*$ or $\mathbb{R}^*$)
- over $\mathbb{R}$: line segment in $\mathbb{E}^n$

$$\ell_{\underline{x_1},\underline{x_2}} = \{\underline{x} = \lambda_1 \underline{x_1} + \lambda_2 \underline{x_2} \,|\, \lambda_i \geq 0, \;\; \lambda_1 + \lambda_2 = 1\}$$

convex linear combinations of extremal points $\underline{x_1}, \underline{x_2}$

- barycentric coordinates: simplex

$$\Delta_{\underline{x_1},\ldots,\underline{x_N}} = \{\underline{x} = \lambda_1 \underline{x_1} + \lambda_2 \underline{x_2} + \cdots + \lambda_N \underline{x_N} \,|\, \lambda_i \geq 0, \;\; \sum_i \lambda_i = 1\}$$

- convex set: each point in the set has unobstructed view of all other points in the set (line segment connecting them is entirely contained in the set)

convex hull

- set $S \subset \mathbb{A}^n$ convex hull $H(S)$ smallest convex set in $\mathbb{A}^n$ containing S
- if S is a finite set of points $H(S)$ is a polytope
- if S consists of $N+1$ points not all on any $N-1$ affine subspace, $H(S)$ is an N-simplex Δ_S (or $\Delta_N(S)$)
- C convex set in some ambient $\mathbb{A}^N$: dimension $\dim C = n$ if largest n for which C contains a simplex Δ_n
- slices $C \cap \mathbb{A}^k$ of convex sets by lower dimensional affine subspaces $\mathbb{A}^k \subset \mathbb{A}^N$ are also convex sets

Cones

- $C \subset \mathbb{A}^N$ convex set, cone $\hat{C} \subset \mathbb{A}^{N+1}$: take point $\underline{x} \in \mathbb{A}^{N+1}$ not in C and union $\hat{C}$ of all half-line from $\underline{x}$ passing through points of C (unbounded cone)
- cones have a partial ordering $\underline{x} \leq \underline{y}$ in $\hat{C}$ iff $\underline{y} - \underline{x} \in \hat{C}$
- cone $\hat{C} \subset \mathbb{E}^N$, dual cone: $\hat{C}^\vee =$ linear functionals $f \in \mathbb{E}^{n\vee} = \mathrm{Hom}(\mathbb{E}^n, \mathbb{R})$ such that $f(\underline{x}) \geq 0$ for all $\underline{x} \in \hat{C}$
- with Euclidean metric identify $\mathbb{E}^n \simeq \mathbb{E}^{n\vee}$, then cone is self-dual if $\hat{C} = \hat{C}^\vee$ in this identification

convex bodies

- topologically an n-dimensional convex set is always homemorphic to an n-ball (move along the rays from an interior point, center of the ball)
- convex body C a compact convex set (while convex cones $\hat{C}$ always non-compact)
- convex bodies and convex cones over a convex body always have extremal points that are not convex combinations of other points (pure points, pure states)
- but not true for other convex cones (eg half-space)
- k-dimensional face $F \subset C$ of an n-dimensional convex set: if a point $\underline{x} \in C$ is in F and $\underline{x} = \lambda \underline{x}_1 + (1-\lambda)\underline{x}_2$, for some $0 \leq \lambda \leq 1$, then $\underline{x}_1, \underline{x}_2$ also in F
- partial ordering by inclusion of the set of faces of a convex set (lattice of inclusions)

Facts about convex sets

- Minkowski: every convex body Y is convex hull $Y = C(X)$ of its set X of pure points
- Carathéodory: given $X \subset \mathbb{E}^n$ (so $\dim X \leq n$), any $\underline{x} \in H(X)$ is a convex combination of at most $n+1$ points of X

$$\underline{x} = \sum_i \lambda_i \underline{x}_i \quad \sum_i \lambda_i = 1, \quad \lambda_i \geq 0$$

Note that here $\lambda_i, \underline{x}_i$ depend on $\underline{x}$

- case of simplex: all points written as a mixture in a unique way (this property characterizes simplices)
- number of k-faces in an n-simplex

$$\#\{k\text{−faces}\} = \binom{n+1}{k+1} = \binom{n+1}{n-k} = \#\{(n-k-1)\text{−faces }\}$$

lattice of faces is self-dual (in fact it is also a Boolean lattice)

- supporting hyperplanes of convex sets: $C \subset \mathbb{E}^n$ real hyperplane H that intersects C and such that all of C lies in the same half-space cut out by H
- regular point of C, point $\underline{x}$ on ∂C such that there is only one support hyperplane through $\underline{x}$
- regular support hyperplane meets C at only one point
- regular convex set C all points and all support hyperplanes are regular
- a ball is regular a simplex is not
- Hahn–Banach: $C \subset \mathbb{E}^n$ convex body and $\underline{x}_0 \in \mathbb{E}^n \smallsetminus C$ then $\exists f : \mathbb{E}^n \to \mathbb{R}$ linear functional such that $f(\underline{x}) > 0$ for all $\underline{x} \in C$ and $f(\underline{x}_0) < 0$ (think in terms of support hyperplanes)
- convex functions $f : C \to \mathbb{R}$ from C convex body such that

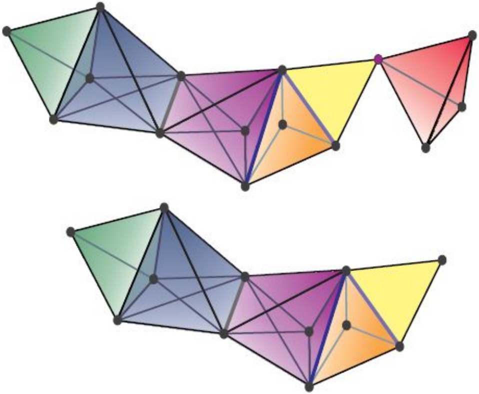
$$f(\lambda \underline{x} + (1-\lambda)\underline{y}) \leq \lambda f(\underline{x}) + (1-\lambda) f(\underline{y})$$

$\forall \underline{x}, \underline{y} \in C$ and $\forall \lambda \in [0,1]$

Another way to think of simplices: Simplicial Sets

Simiplicial Sets: intuition

Sets locally described by simplexes appropriately glued together along faces (subtleties: some simplexes can be degenerate etc.)



Idea: these sets locally have "mixing property" of simplexes (barycentric coordinates) but are not necessarily globally convex

Categories

- Category: $\mathcal{C}$ with objects $Obj(\mathcal{C})$ and morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for $X,Y \in Obj(\mathcal{C})$ (small categories: $Obj(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are sets); composition

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z), \quad (\phi,\psi) \mapsto \psi \circ \phi$$

  is associative; there is an identity morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ for every object X

- Functors $F : \mathcal{C} \to \mathcal{C}'$ between two categories, maps objects to objects $F : Obj(\mathcal{C}) \to Obj(\mathcal{C}')$ and morphisms to morphisms $F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))$ respecting composition and identities, $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ and $F(1_X) = 1_{F(X)}$

- Natural transformations of functors: $F, G : \mathcal{C} \to \mathcal{C}'$ functors, natural transformation $\eta : F \to G$
  - for all objects $X \in \mathrm{Obj}(\mathcal{C})$ a morphism $\eta_X : F(X) \to G(X)$, $\eta_X \in \mathrm{Hom}_{\mathcal{C}'}(F(X), G(X))$
  - for every morphism $\phi : X \to Y$, $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, compatibility $\eta_Y \circ F(\phi) = G(\phi) \circ \eta_X$

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(\phi)}\downarrow & & \downarrow{\scriptstyle G(\phi)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

Another way to think of simplices: Simplex Category (or Δ category)

- Simplex Category: objects in $Obj(\Delta)$ are totally ordered sets $[n] = \{0, 1, \dots, n\}$; morphisms in $\mathrm{Hom}_\Delta([n], [m])$ are nondecreasing maps $f : [n] \to [m]$

$$f(i) \leq f(j) \quad \text{for} \quad i \leq j$$

- this category "looks like a simplex": this can be seen by showing that the morphisms are generated by a set of faces and degeneracies morphisms (all morphisms are finite compositions of those)

Simplex Category: Faces and Degeneracies

- D_i increasing injection $[n] \to [n+1]$ not taking value i; S_i nondecreasing surjection taking value i twice
- morphisms are generated by maps $D_i : [n] \to [n+1]$ and $S_i : [n+1] \to [n]$

$$D_i[0,\dots,n] = [0,\dots,\hat{i},\dots,n], \quad S_i[0,\dots,n] = [0,\dots,i,i,\dots,n]$$

- in Δ^{op} the D_i become face maps $d_i : [n+1] \to [n]$ and S_i the degeneracy maps $s_i : [n] \to [n+1]$

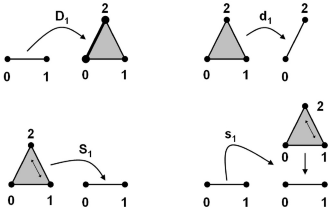


image of degeneracy s_1 degenerate 2-simplex image of collapse map S_1

- generators in Δ faces $D_i^n : [n] \to [n+1]$ and degeneracies $S_i^n : [n+1] \to [n]$
- relations

$$D_i^{n+1} \circ D_j^n = D_{j+1}^{n+1} \circ D_i^n \quad i \leq j$$

$$S_j^n \circ S_i^{n+1} = S_i^n \circ S_{j+1}^{n+1} \quad i \leq j$$

$$S_j^n \circ D_i^{n+1} = \begin{cases} D_i^n \circ S_{j-1}^{n-1} & i < j \\ \mathrm{Id}_n & i = j \text{ or } i = j+1 \\ D_{i-1}^n \circ S_j^{n-1} & j+1 < i \end{cases}$$

- any nondecreasing maps $f : [n] \to [m]$ is a finite composition of D_i^k's and S_j^ℓ's

Simplicial Sets: rigorous definition

- Simplicial set: functor $X : \Delta^{op} \to \mathcal{S}$ to the category of sets (contravariant functor from Δ)
- this means collection of sets $X_n = X([n])$ images of objcts of Δ-category endowed with face and degeneracy maps: n-skeleta of the simplicial set

Realization

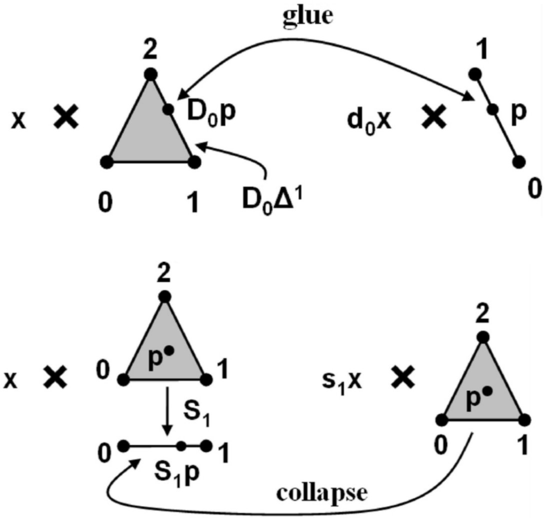
- Realization of a simplex: $|\Delta^n|$ the geometric simplex realization of combinatorial $\Delta^n = [n]$
- Standard simplex $|\Delta^n|$ is simplex $\Delta_n = \Delta_{e_0,\ldots,e_n}$ of standard o.n. basis in $\mathbb{E}^{n+1}$

$$|\Delta^n| = \Delta_n = \{\underline{x} \in \mathbb{E}^{n+1} \,|\, x_i \geq 0 \text{ and } \sum_{i=0}^{n+1} x_i = 1\}$$

- Realization of X

$$|X| := \sqcup_n (X_n \times |\Delta^n|) \,/ \sim$$

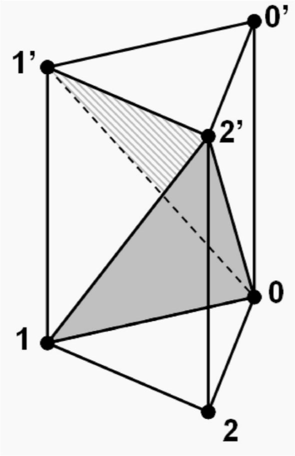
  modulo equivalence relation $(x, S_i(t)) \sim (s_i(x), t)$ and $(x, D_i(t)) \sim (d_i(x), t)$
- interpret as recipe for gluing the geometric simplexes $|\Delta^n|$ together according to the combinatorial scheme prescribed by the X_n so that faces and degeneracies match



glue
2
x
D$_0$p
1
d$_0$x
p
0
1
D$_0$Δ^1
0
2
2
x
0
p
1
s$_1$x
p
s$_1$
0
1
0
1
s$_1$p
collapse

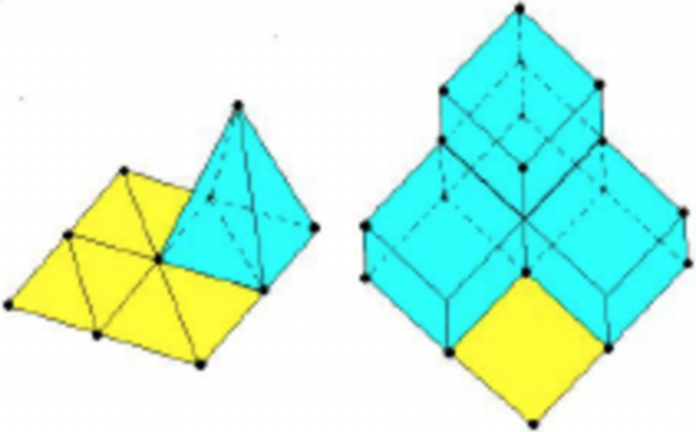
- Nerve: simplicial sets from categories
  - category Cat of small categories with functors as morphisms, nerve functor $\mathcal{N} : \mathrm{Cat} \to \Delta\mathcal{S}$ to the category of simplicial sets $\Delta\mathcal{S} = \mathrm{Func}(\Delta^{op}, \mathcal{S})$
  - for a small category $\mathcal{C}$ the nerve $\mathcal{N}(\mathcal{C})$ has a 0-simplex (vertex) for each object of $\mathcal{C}$, a 1-simplex (edge) for each morphism, a 2-simplex for each composition of two morphishs, a k-simplex for every chain of k composable morphisms
  - face maps: composition of two adjacent morphisms at the i-th place of a k-chain $d_i : \mathcal{N}_k(\mathcal{C}) \to \mathcal{N}_{k-1}(\mathcal{C})$ and degeneracies are insertions of the identity morphism at an object in the chain

• Products: product of simplexes is not a simplex but can be decomposed as a union of simplexes



Cubes behave better than simplexes with respect to products

Simplicial and cubical complexes



- Cubical sets in topology
  - $\mathcal{I}$ unit interval as combinatorial structure consisting of two vertices and an edge connecting them
  - $|\mathcal{I}| = [0,1]$ geometric realization: unit interval as topological space (subspace of $\mathbb{R}$)
  - $\mathcal{I}^n$ for the n-cube as combinatorial structure and $|\mathcal{I}^n| = [0,1]^n$ its geometric realization
  - $\mathcal{I}^0$ a single point
  - face maps $\delta_i^a : \mathcal{I}^n \to \mathcal{I}^{n+1}$, for $a \in \{0,1\}$ and $i = 1, \dots, n$

$$\delta_i^a(t_1, \dots, t_n) = (t_1, \dots, t_{i-1}, a, t_i, \dots, t_n)$$

  - degeneracy maps $s_i : \mathcal{I}^n \to \mathcal{I}^{n-1}$

$$s_i(t_1, \dots, t_n) = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$$

- cubical relations for $i < j$

$$\delta_j^b \circ \delta_i^a = \delta_i^a \circ \delta_{j-1}^b \quad \text{and} \quad s_i \circ s_j = s_{j-1} \circ s_i$$

and relations

$$\delta_i^a \circ s_{j-1} = s_j \circ \delta_i^a \quad i < j$$

$$s_j \circ \delta_i^a = 1 \quad i = j$$

$$\delta_{i-1}^a \circ s_j = s_j \circ \delta_i^a \quad i > j$$

- Cube category: $\mathfrak{C}$ has objects $\mathcal{I}^n$ for $n \geq 0$ and morphisms generated by the face and degeneracy maps δ_i^a and s_i
- Cubical set: functor $C : \mathfrak{C}^{op} \to \mathcal{S}$ to the category of sets.
- notation: $C_n := C(\mathcal{I}^n)$

- variant of the cube category $\mathfrak{C}_c$ with additional degeneracy maps $\gamma_i : \mathcal{I}^n \to \mathcal{I}^{n-1}$ called connections

$$\gamma_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \max\{t_i, t_{i+1}\}, t_{i+2}, \ldots, t_n)$$

satisfying relations

$$\gamma_i \gamma_j = \gamma_j \gamma_{i+1}, \; i \leq j; \quad s_j \gamma_i = \begin{cases} \gamma_i s_{j+1} & i < j \\ s_i^2 = s_i s_{i+1} & i = j \\ \gamma_{i-1} s_j & i > j \end{cases}$$

$$\gamma_j \delta_i^a = \begin{cases} \delta_i^a \gamma_{j-1} & i < j \\ 1 & i = j, j+1, \; a = 0 \\ \delta_j^a s_j & i = j, j+1, \; a = 1 \\ \delta_{i-1}^a \gamma_j & i > j+1. \end{cases}$$

- role of degeneracy maps: maps s_i identify opposite faces of a cube, additional degeneracies γ_i identify adjacent faces

- cubical set with connection: functor $C : \mathfrak{C}_c^{op} \to \mathcal{S}$ to the category of sets
- category of cubical sets has these functors as objects and natural transformations as morphisms
- so morphisms given by collection $\alpha = (\alpha_n)$ of morphisms $\alpha_n : C_n \to C'_n$ satisfying compatibilities $\alpha \circ \delta_i^a = \delta_i^a \circ \alpha$ and $\alpha \circ s_i = s_i \circ \alpha$ (and in the case with connection $\alpha \circ \gamma_i = \gamma_i \circ \alpha$)
- cubical nerve $\mathcal{N}_{\mathfrak{C}}\mathcal{C}$ of a category $\mathcal{C}$ is the cubical set with

$$(\mathcal{N}_{\mathfrak{C}}\mathcal{C})_n = \mathrm{Fun}(\mathcal{I}^n, \mathcal{C})$$

  with $\mathcal{I}^n$ the *n*-cube seen as a category with objects the vertices and morphisms generated by the 1-faces (edges), and $\mathrm{Fun}(\mathcal{I}^n, \mathcal{C})$ is the set of functors from $\mathcal{I}^n$ to $\mathcal{C}$
- when working with cubical sets with connection homotopy equivalent to simplicial nerve
- R. Antolini, *Geometric realisations of cubical sets with connections, and classifying spaces of categories*, Appl. Categ. Structures 10 (2002), no. 5, 481–494.

Simplexes and Probabiities

- X finite set with cardinality $N+1=\#X$
- Probability measure P on X: to all $x\in X$ assign $P_x\geq 0$ with normalization $\sum_{x\in X} P_x = 1$
- Set $\mathcal{P}(X)$ of all probability measures on X is a copy of the simplex Δ_N

$$\mathcal{P}(X) = \{(P_x)_{x\in X}\,|, P_x\geq 0 \text{ and } \sum_{x\in X} P_x = 1\} \simeq \Delta_N,$$

where vertices of simplex Δ_N are probabilities $P=\delta^{(x)}$ with $\delta^{(x)}_x = 1$ and $\delta^{(x)}_y = 0$ for all $y\neq x$

- choose a bijection $X\simeq \{0,\ldots,N\}$ then

$$\mathcal{P}(X) = \Delta_N = \{\underline{x}=(x_i)_{i=0}^N \in \mathbb{R}_+^{N+1}\,|\, \sum_{i=0}^N x_i = 1\}$$

Partial ordering

- transformations of probability distributions: transformations of $\mathbb{R}^{N+1}$ preserving positivity and ℓ_1-norm of non-negative vectors
- for $\underline{x} \in \mathbb{R}^{N+1}$ let $\underline{x}^{\downarrow}$ be the vector obtained from $\underline{x}$ by a permutation of coordinates so that in non-increasing order

$$x_0^{\downarrow} \geq x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_N^{\downarrow}$$

- for $\underline{x}, \underline{y} \in \mathbb{R}^{N+1}$ let $\underline{y} \succeq \underline{x}$ iff

$$\sum_{i=0}^{k} x_i^{\downarrow} \leq \sum_{i=0}^{k} y_i^{\downarrow}, \quad \forall 0 \leq k \leq N$$

where for probabilities the last one is

$$\sum_{i=0}^{N} x_i^{\downarrow} = \sum_{i=0}^{N} y_i^{\downarrow} = 1$$

- have $\underline{x} \succeq \underline{x}$ and if $\underline{x} \succeq \underline{y}$ and $\underline{y} \succeq \underline{z}$ then $\underline{x} \succeq \underline{z}$
- don't have $\underline{y} \succeq \underline{x}$ and $\underline{x} \succeq \underline{y}$ implies $\underline{y} = \underline{x}$ (only up to a permutation), but when written in the form $\underline{y}^{\downarrow}$ and $\underline{x}^{\downarrow}$ implies equal
- so get partial ordering on these
- there are pairs $\underline{x}, \underline{y}$ for which neither $\underline{x} \succeq \underline{y}$ nor $\underline{y} \succeq \underline{x}$
- there is a smallest element given by the uniform distribution

$$\underline{x}_{(N+1)} = (\frac{1}{N+1}, \ldots, \frac{1}{N+1})$$

- largest: $N+1$ probabilities with one 1 entry and all other entries 0

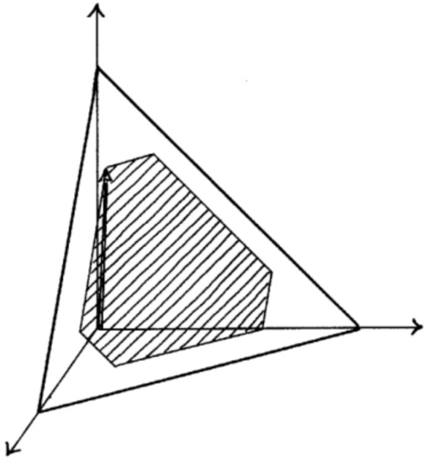
- Shur convex functions $\underline{x} \succeq \underline{y} \Rightarrow f(\underline{x}) \geq f(\underline{y})$
- examples: elementary symmetric functions

$$s_0(\underline{x}) = 1, \quad s_1(\underline{x}) = x_1 + \cdots + x_N,$$

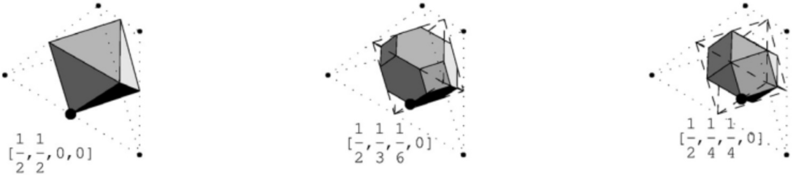
$$s_2(\underline{x}) = \sum_{i<j} x_i x_j, \quad \cdots, \quad s_N(\underline{x}) = x_1 \cdots x_N.$$

- set of vectors majorized by a given one is a convex set

$$\underline{x} \succeq \underline{y} \text{ and } \underline{x} \succeq \underline{z} \quad \Rightarrow \quad \underline{x} \succeq \lambda \underline{y} + (1-\lambda)\underline{z} \ \forall \lambda \in [0,1]$$



convex set of probabilities $\underline{x}$ inside Δ_3 majorized by a given vector: pure points of the convex set permutations of entries of the vector (from Ingemar Bengtsson, Karol Zyczkowski "The geometry of quantum states", Cambridge 2007)



examples inside Δ_4: probabilities majorized by given vector (from Ingemar Bengtsson, Karol Zyczkowski "The geometry of quantum states", Cambridge 2007)

Stochastic Matrices

- $S = (S_{ij}) \in M_{N\times N}(\mathbb{C})$ with $S_{ij} \geq 0$ and $\sum_{i=1}^N S_{ij} = 1$ (columns are probability measures)
- $S_{ij} \geq 0$: preserves positivity; $\sum_{i=1}^N S_{ij} = 1$: preserves ℓ_1-norm
- bistochastic if $\sum_{j=1}^N S_{ij} = 1$ as well
- bistochastic matrices also fix the point $\underline{x}_{(N)}$: they cause contraction of the probability simplex toward the center
- Hardy–Littlewood–Pólya: $\underline{x} \succeq \underline{y} \Rightarrow \exists\, S$ bistochastic with $\underline{y} = S\underline{x}$
- bistochastic matrices form a semigroup; only ones with a bistochastic inverse are permutation matrices
- Birkhoff theorem: set of $N \times N$ bistochastic matrices is a convex polytope with pure points the $N!$ permutation matrces

Perron-Frobenius Theorem (strictly positive case)

- $A = (A_{ij})$ with $A_{ij} > 0$
- $\exists$ positive eigenvalue $\lambda_A = \rho(A)$ (spectral radius) with all other eigenvalues $|\lambda| < \lambda_A$
- eigenvalue λ_A is simple
- eigenvector $\underline{v}_A = (v_{A,i})$, with $A\,\underline{v}_A = \lambda_A \underline{v}_A$, entries $v_{A,i} > 0$
- same for transpose A^τ: left eigenvector $\underline{w}_A$ with $\underline{w}_A\, A = \lambda_A \underline{v}_A$ and with $w_{A,i} > 0$ and with

$$\underline{w}_A^\tau \cdot \underline{v}_A = 1$$

- these are only eigenvectors with positive entries (up to scalar multiples)
- limit

$$\lim_{k\to\infty} \frac{A^k}{\lambda_A^k} = \underline{v}_A \cdot \underline{w}_A^\tau.$$

- column sums

$$\min_i \sum_j A_{ij} \le \lambda_A \le \max_i \sum_j A_{ij}$$

Perron-Frobenius Theorem (non-negative case)

- $A = (A_{ij})$ with $A_{ij} \geq 0$
- assume irreducible: $\forall i, j \; \exists m \geq 1$ with $(A^m)_{ij} > 0$
- equivalent: associate to A a graph G_A with N vertices and oriented edge from i to j iff $A_{ij} > 0$, then A irreducible means that G_A is *strongly connected*: given any two vertices there is an oriented path from the first to the second
- period h_A of A is greatest common divisor of the m such that $(A^m)_{ii} > 0$ (independent of i if A irreducible)

- Perron-Frobenius: $\exists$ eigenvalue $\lambda_A = \rho(A) > 0$ simple
- $\exists$ left/right eigenvectors $\underline{w}_A, \underline{v}_A$ with positive entries
- these are the only eigenvectors that are positive (up to scalar multiples)
- there are h_A complex eigenvalues λ with $|\lambda| = \lambda_A$, each $\lambda = \lambda_A \zeta$ (ζ root of 1) and simple
- column sums

$$\min_i \sum_j A_{ij} \leq \lambda_A \leq \max_i \sum_j A_{ij}$$

Stochastic matrices

- S stochastic matrix, columns sum estimate of Perron-Frobenius theorem shows $\lambda_S = 1$
- Perron-Frobenius eigenvector is an invariant measure $S\underline{v}_S = \underline{v}_S$
- stochastic matrices as linear maps = Markov maps
- Markov chain = sequence of Markov maps

Note: can also consider stochastic $N \times M$ matrices with $S_{ij} \geq 0$ and $\sum_{j=1}^{M} S_{ij} = 1$ for $i = 1, \ldots, N$

Categories of finite probabilities

- Category of finite probability spaces with stochastic matrices $\mathcal{FP}$
- objects $\mathrm{Obj}(\mathcal{FP})$ are pairs (X, P) of a finite set X with a probability measure P
- morphisms $S \in \mathrm{Hom}_{\mathcal{FP}}((X, P), (Y, Q))$ are stochastic $(\#Y \times \#X)$-matrices S with
  - $S_{yx} \geq 0$, for all $x \in X$, $y \in Y$
  - $\sum_{y \in Y} S_{yx} = 1$ for all $x \in X$
  - the probability measures are related by $Q = S\, P$

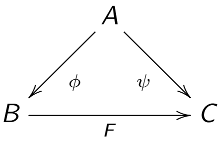
Note: objects (X, P) of $\mathcal{FP}$ can be thought of as fuzzy sets, with P_x value at $x \in X$ of the membership function of the fuzzy set

- Category of stochastic maps FinStoch (Baez–Fritz)
- objects in $\mathrm{Obj}(\mathtt{FinStoch})$ are finite sets X
- morphisms $\mathrm{Hom}_{\mathtt{FinStoch}}(X, Y)$ are stochastic maps (assign a probability on Y to each point in X): stochastic matrices

$$S = (S_{yx}) \quad S_{yx} \geq 0, \quad \sum_{y \in Y} S_{yx} = 1, \, \forall x \in X.$$

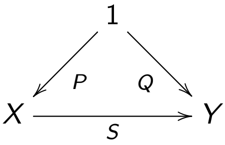
Under Category

- category $\mathcal{C}$ and object $A \in \mathrm{Obj}(\mathcal{C})$
- under category $A \downarrow \mathcal{C}$ (also denoted $A/\mathcal{C}$):
  - objects $\mathrm{Obj}(A \downarrow \mathcal{C})$ are morphisms $\phi : A \to B$ of $\mathcal{C}$ with source A
  - morphisms $F \in \mathrm{Hom}_{A\downarrow\mathcal{C}}(\phi : A \to B, \psi : A \to C)$ are commuting triangles



Categories $\mathcal{FP}$ and FinStoch

- $\mathcal{FP}$ is the under category $1 \downarrow \texttt{FinStoch}$
- stochastic maps $P : 1 \to X$ are probability distributions on X
- so objects of $1 \downarrow \texttt{FinStoch}$ are pairs (X, P)
- triangles



are stochastic matrices S with $SP = Q$

Category of finite probability measures $\mathtt{FinProb}$

- objects in $\mathrm{Obj}(\mathtt{FinProb})$ are pairs (X, P) of a finite set with a probability measure
- morphisms $f \in \mathrm{Hom}_{\mathtt{FinProb}}((X, P), (Y, Q))$ are functions $f : X \to Y$ that are measure preserving

$$Q_y = \sum_{x \in f^{-1}(y)} P_x.$$

- relation between $\mathtt{FinProb}$ and $\mathcal{FP}$: subcategory $\mathtt{FinProb}$ where for each x there is a unique $y = y(x)$ such that $S_{yx} > 0$: then stochastic matrix must have $S_{yx} = 1$, so S is a (single valued) function $f : X \to Y$ that is measure preserving

Pointed Sets and $\mathcal{FP}$

- category $\mathcal{FS}_*$ of finite pointed sets: objects (X, x_0) finite sets with a base point; morphisms $f : (X, x_0) \to (Y, y_0)$ functions between sets $f : X \to Y$ that preserve base points, $f(x_0) = y_0$
- $\mathcal{FS}_*$ as subcategory of $\mathcal{FP}$: objects (X, δ_{x_0}) and morphisms stochastic maps $S : (X, \delta_{x_0}) \to (Y, \delta_{y_0})$ with $S_{yx} = \chi_{f^{-1}(y)}(x)$, with χ the indicator function,

$$\delta_{y_0,y} = \sum_x S_{yx} \delta_{x_0,x} = \sum_{x \in f^{-1}(y)} \delta_{x_0,x}$$

Category $\mathcal{S}_*$ of pointed sets (not necessarily finite) similarly defined

Note: can think of finite probability (X, P) as a formal convex combination of pointed sets

$$(X, P) = \sum_{x \in X} P_x \, (X, x)$$

Category of probabilistic pointed sets $\mathcal{PS}_*$

- objects in $\mathrm{Obj}(\mathcal{PS}_*)$ are convex combinations of pointed sets

$$\Lambda X = \sum_i \lambda_i (X_i, x_i),$$

  where $\Lambda = (\lambda_i)$ wtih $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$ and $X = \{(X_i, x_i)\}$ a finite collection of pointed sets

- morphisms $\Phi \in \mathrm{Mor}_{\mathcal{PS}_*}(\Lambda X, \Lambda' X')$ are pairs $\Phi = (S, F)$
  1. S is a stochastic map with $S\Lambda = \Lambda'$
  2. $F = (F_{ji})$ is a collection of probabilistic pointed maps $F_{ji} : (X_i, x_i) \to (X'_j, x'_j)$

- probabilistic pointed map F_{ji} is a finite set $\{F_{ji,a}\}$ of pointed maps $F_{ji,a} : (X_i, x_i) \to (X'_j, x'_j)$ together with a set of probabilities $\mu_a^{(ji)}$ with $\sum_a \mu_a^{(ji)} = S_{ji}$

- composition of morphisms $\Phi = (S, F) : \Lambda X \to \Lambda' X'$ and $\Phi' = (S', F') : \Lambda' X' \to \Sigma Y$ given by

$$\Phi' \circ \Phi = (S' \circ S, F' \circ F)$$

- $S' \circ S$ product of stochastic matrices
- $F' \circ F = \{(F' \circ F)_{ki}\}$ with set $(F' \circ F)_{ki} = \{F'_{kj,a} \circ F_{ji,b}\}$ and probabilities $\mu_a^{(kj)} \mu_b^{(ji)}$
- with $\sum_{a,b,j} \mu_a^{(kj)} \mu_b^{(ji)} = \sum_j S'_{kj} S_{ji} = (S' \circ S)_{ki}$
- probability associated to set $(F' \circ F)_{ki}$ in $F' \circ F$ is $(S' \circ S)_{ki}$

Embedding of category $\mathcal{FP}$ in $\mathcal{PS}_*$

- mapping $\Lambda = (\lambda_i)$ to the set $\Lambda\star = \sum_i \lambda_i(\{\star_i\}, \star_i)$
- morphisms $S\Lambda = \Lambda'$ to $\Phi = (S, \mathbf{1})$ with $\mathbf{1} = \{1_{ji}\}$ with probabilities S_{ji}

• forgetful functor from $\mathcal{PS}_*$ to $\mathcal{FP}$ maps ΛX to finite probability Λ and morphism $\Phi = (S, F)$ to stochastic matrix S

Probabilistic categories $\mathcal{PC}$

- from pointed sets $\mathcal{S}_*$ to probabilistic pointed sets $\mathcal{PS}_*$
- more general similar procedure starting from a category $\mathcal{C}$
- $\mathcal{PC}$ category with objects in $\mathrm{Obj}(\mathcal{PC})$ given by formal finite convex combinations

$$\Lambda C = \sum_i \lambda_i C_i,$$

  with $\Lambda = (\lambda_i)$ with $\sum_i \lambda_i = 1$ and $C_i \in \mathrm{Obj}(\mathcal{C})$
- morphisms $\Phi : \Lambda C \to \Lambda' C'$, $\Phi \in \mathrm{Hom}_{\mathcal{PC}}(\Lambda C, \Lambda' C')$ given by pairs $\Phi = (S, F)$
  - S a stochastic matrix with $S\Lambda = \Lambda'$
  - $F = \{F_{ab,r}\}$ finite collection of morphisms $F_{ab,r} : C_b \to C'_a$ with assigned probabilities μ_r^{ab}
  - probabilities satisfy $\sum_r \mu_r^{ab} = S_{ab}$

interpret F as mapping of C_a to C'_b by choosing randomly in the set $\{F_{ab,r}\}$ with probability μ_r^{ab} of choosing $F_{ab,r}$

Wreath product interpretation

- construction of $\mathcal{PC}$ from $\mathcal{C}$ can be seen as a wreath product $\mathcal{FP} \wr \mathcal{C}$ of the category $\mathcal{C}$ with the category $\mathcal{FP}$ of finite probabilities
- it has the effect of rendering the category $\mathcal{C}$ probabilistic (objects are a mixture of objects and morphisms are randomly chosen with a stochastic map of the mixtures)

Zero Object

- Zero object $0 \in \mathrm{Obj}(\mathcal{C})$: for all $X \in \mathrm{Obj}(\mathcal{C})$ there is a unique morphism $0 \to X$ and a unique morphism $X \to 0$
- Zero objects are unique up to unique isomorphism

- points $\star$ are zero objects in the category $\mathcal{S}_*$ of pointed sets
- singletons $(\star, 1)$ are zero objects in the category $\mathcal{FP}$
  - unique morphism $\hat{Q} : (\{x\}, 1) \to (Y, Q)$ given by $\hat{Q}_{yx} = Q_y$
  - unique morphism $\hat{1} : (Y, Q) \to (\{x\}, 1)$ given by $\hat{1}_{xy} = 1$ for all $y \in Y$

$$1 = \sum_{y \in Y} Q_y = \sum_{y \in Y} \hat{1}_{xy} Q_y$$

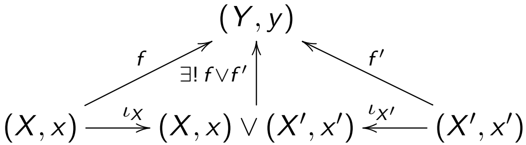
- singletons $X = (\{x\}, x)$ with $\Lambda = 1$ are zero objects in $\mathcal{PS}_*$
  - unique morphism $\Phi = (S, F) : (\{x\}, x) \to \sum_i \lambda_i (X_i, x_i)$ with $S = \hat{\Lambda}$ (unique morphism in $\mathcal{FP}$ from $(\{x\}, 1)$ to Λ) and $F = (F_i)$ with $F_i : x \mapsto x_i$ with probability λ_i
  - unique morphism $\Phi = (S, F) : \Lambda X \to (\{x\}, x)$ where $S = \hat{1}_\Lambda$ (unique morphism in $\mathcal{FP}$ from Λ to $(\{x\}, 1)$) and $F = (F_i)$ with $F_i : (X_i, x_i) \to (\{x\}, x)$ the constant function with probability 1
- same argument: if $\mathcal{C}$ has a zero object then $\mathcal{PC}$ has a zero object given by the zero object of $\mathcal{C}$ with $\Lambda = 1$

Categorical Sum (Coproduct)

- category of pointed sets $\mathcal{S}_*$: coproduct of (X, x) and (X', x')

$$(X, x) \vee (X', x') := (X \sqcup X'/x \sim x')$$

- universal property of coproduct: $\exists$ morphisms $\iota_X, \iota_{X'}$ such that for any given morphisms $f : (X, x) \to (Y, y)$ and $f' : (X', x') \to (Y, y)$



- category of probabilistic pointed sets $\mathcal{PS}_*$: coproduct of $\Lambda X = \sum_{i=1}^N \lambda_i (X_i, x_i)$ and $\Lambda' X' = \sum_{j=1}^M \lambda'_j (X'_j, x'_j)$

$$\Lambda X \amalg \Lambda' X' := \sum_{ij} \lambda_i \lambda'_j \, (X_i, x_i) \vee (Y_j, x'_j)$$

  with coproducts $(X_i, x_i) \vee (Y_j, x'_j)$ of pointed sets
- induces coproduct on $\mathcal{FP}$

$$(X, P) \amalg (X', P') = (X \times X', P \cdot P')$$

  statistically independent probabilities

universal property of coproduct in $\mathcal{PS}_*$

- morphisms $\Phi = (S, F) : \Lambda X \to \Sigma Y$ and $\Phi' = (S', F') : \Lambda' X' \to \Sigma Y$ with $\Sigma Y = \sum_k \sigma_k (Y_k, y_k)$
- $S\Lambda = \Sigma$ and $S'\Lambda' = \Sigma$
- $F = \{f_{ka,r}\}_{r=1}^N$ with probabilities $\sum_r \mu_r^{(ka)} = S_{ka}$ and $F' = \{f'_{ka',r'}\}_{r'=1}^M$ with probabilities $\sum_{r'} \mu_{r'}^{(ka)} = S'_{ka}$
- construct $F \vee F'$ as set $\{f_{ka,r} \vee f'_{ka',r'}\}$ of pointed maps from coproducts of pointed sets with probabilities $\sigma_k^{-1} \mu_r^{(ka)} \mu_{r'}^{(ka')}$ for $\sigma_k \neq 0$ and $M^{-1}\mu_r^{(ka)} + N^{-1}\mu_{r'}^{(ka')}$ for $\sigma_k = 0$
- then set $(\Phi \amalg_\lambda \Phi') = (S \amalg S', F \vee F')$ with

$$(S \amalg_\lambda S')_{k,(a,a')} = \begin{cases} \sigma_k^{-1} \cdot S_{k,a} \cdot S'_{k,a'} & \sigma_k \neq 0 \\ S_{k,a} + S'_{k,a'} & \sigma_k = 0. \end{cases}$$

- to see universality holds, need to construct morphisms $\Psi : \Lambda X \to \Lambda X \amalg \Lambda' X'$ and $\Psi' : \Lambda' X' \to \Lambda X \amalg \Lambda' X'$, such that, for any Φ, Φ' morphisms to ΣY diagram commutes

$$\begin{array}{ccccc} & & \Sigma Y & & \\ & \overset{\Phi}{\nearrow} & \uparrow {\scriptstyle \Phi \amalg \Phi'} & \overset{\Phi'}{\nwarrow} & \\ \Lambda X & \xrightarrow{\Psi} & \Lambda X \amalg \Lambda' X' & \xleftarrow{\Psi'} & \Lambda' X' . \end{array}$$

- take $\Psi = (\mathcal{I}, \mathcal{F})$ and $\Psi' = (\mathcal{I}', \mathcal{F}')$ with morphisms $\mathcal{I} \in \mathrm{Mor}_{\mathcal{FP}}(\Lambda, \Lambda \cdot \Lambda')$, $\mathcal{I}' \in \mathrm{Mor}_{\mathcal{FP}}(\Lambda', \Lambda \cdot \Lambda')$ given by

$$(\mathcal{I})_{(b,b'),a} = \delta_{ab}\, \lambda'_{b'} \quad \text{and} \quad (\mathcal{I}')_{(b,b'),a'} = \delta_{a'b'}\, \lambda_b .$$

- these satisfy $\mathcal{I}\Lambda = \Lambda \cdot \Lambda'$ and $\mathcal{I}'\Lambda' = \Lambda \cdot \Lambda'$
- probabilistic pointed maps $\mathcal{F} = (\mathcal{F}_{(b,b'),a} = \delta_{ab} \mathcal{F}_{b'b})$, $\mathcal{F}' = (\mathcal{F}'_{(b,b'),a'} = \delta_{a'b'} \mathcal{F}'_{bb'})$ inclusions of pointed sets

$$(X_b, x_b) \overset{\mathcal{F}_{b'b}}{\hookrightarrow} (X_b, x_b) \vee (X'_{b'}, x'_{b'}), \quad (X'_{b'}, x'_{b'}) \overset{\mathcal{F}'_{bb'}}{\hookrightarrow} (X_b, x_b) \vee (X'_{b'}, x'_{b'})$$

respectively chosen with probability $\lambda'_{b'}$ and λ_b

Coproduct in probabilistic categories $\mathcal{PC}$

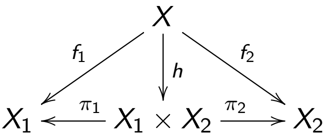
- similarly if $\mathcal{C}$ has coproduct then probabilistic category $\mathcal{PC}$ also has coproduct

$$\Lambda C \amalg \Lambda' C' = \sum_{i,j} \lambda_i \lambda'_j \;\; C_i \amalg_{\mathcal{C}} C'_j$$

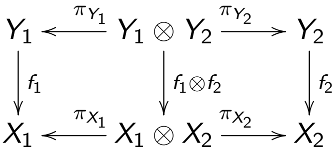
- it satisfies universal property: same argument as for $\mathcal{PS}_*$

Statistical independence: product or coproduct?

- categorical product: an object $X_1 \times X_2$ such that, for all morphisms $f_1 : X \to X_1$ and $f_2 : X \to X_2$ $\exists!$ morphism $h : X \to X_1 \times X_2$ with commuting



- category $\mathcal{FP}$ does *not* have a universal categorical product
- ...*but* tensor category with projections: $(\mathcal{C}, \otimes)$ with two natural transformations $\pi_i : \otimes \to \Pi_i$ with $\Pi_i(X_1, X_2) = X_i$ and for any morphisms $f_i : Y_i \to X_i$ commuting



- in tensor category with projections two morphisms $f_i : X \to X_i$ are *independent* if there exists a morphism $h : X \to X_1 \otimes X_2$ such that product diagram commutes
- in $\mathcal{FP}$ this notion of independence agrees with statistical independence

$$(X_1, P_1) \otimes (X_2, P_2) = (X_1 \times X_2, P_1 P_2)$$

- in previous construction of coproduct shown that product of statistically independent measures can also be interpreted as *coproduct* in the category $\mathcal{FP}$