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Codes as Fractals and Noncommutative Spaces

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Abstract We consider the CSS algorithm relating self-orthogonal classical linear codes to q-ary quantum stabilizer codes and we show that to such a pair of a classical and a quantum code one can associate geometric spaces constructed using methods from noncommutative geometry, arising from rational noncommutative tori and finite abelian group actions on Cuntz algebras and fractals associated to the classical codes.

Keywords Error-correcting codes · Noncommutative spaces · Fractals · CSS algorithm · Noncommutative tori · Cuntz algebras

Mathematics Subject Classification 94B27 · 58B34 · 81P45

1 Introduction

New methods derived from Noncommutative Geometry have been recently applied to the theory of error-correcting codes, to signal analysis, and to problems of information and communication.

In [22] and [23] the asymptotic bound of error correcting codes introduced in [21] is studied as a phase transition of certain quantum statistical mechanical systems constructed out of error-correcting codes and code parameters. This construction involves associating to error-correcting codes certain operator algebras (quantum systems built out of creation-annihilation operators associated to code words) and fractal spaces describing the set of infinite words in the language determined by the code. In the more recent paper [23], this approach is further developed to address the computability problem discussed in [20], and the asymptotic bound for error-correcting codes is related to the Kolmogorov complexity of codes and to a thermodynamical partition function that weights codes according to their complexities.

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In [3] the fractals associated to classical error-correcting codes, as described in [22], are related to a construction of spectral triples on Cantor sets of [25] and to a procedure to obtain crossed product constructions for such spectral triples.

In [24] constructions of wavelets on fractals are obtained from the noncommutative geometry of Cuntz–Krieger algebras. This approach builds upon the previous use of representations of Cuntz algebras as a way to construct and analyze wavelets on fractals ([4,9,16,17]), as well as on wavelet constructions on fractals obtained in [15]. The techniques developed in [24], applied to the operator algebras of codes considered in [22], provide corresponding wavelet constructions on the fractal associated to the code, which provide a new set of analytic methods to study the decoding procedure in terms of a wavelet representation.

In [19] the geometry of noncommutative tori and their quantized theta functions is related to signal analysis and the Gabor frame for modulation spaces.

Given this growing interest in the use of operator algebras and noncommutative geometry in coding theory, we have written this paper with the intent of familiarizing people working in coding theory with methods, concepts, and techniques from noncommutative geometry.

To this purpose, we concentrate on a well known construction in coding theory, namely the CSS algorithm relating self-orthogonal classical linear codes to q-ary quantum stabilizer codes, and we show how one can reformulate the data of the classical and quantum codes involved in the procedure and their relation in terms of geometric spaces that are well known in noncommutative geometry, such as rational noncommutative tori, Cuntz–Krieger algebras, crossed products of algebras by group actions and certain bundles over tori with fractal fibers.

The main new result in this paper is the geometric formulation of the CSS relation between classical and quantum codes in terms of noncommutative spaces.

In addition, we review some of the background notions from noncommutative geometry and some of the results of [22], for the use of readers with a coding theory background; we also briefly review the CSS algorithm for the use of readers with a possible noncommutative geometry background, who may be less familiar with the coding theory side.

More precisely, the paper is structured as follows. In the rest of this introductory section we review some basic facts about the CSS algorithm relating self-orthogonal classical linear codes to q-ary quantum stabilizer codes.

We then show in Sect. 2 that the construction of q-ary quantum stabilizer codes can be naturally expressed in terms of the geometry of rational noncommutative tori. We also show that, if the q-ary quantum stabilizer codes is obtained from a classical self-orthogonal linear code via the CSS algorithm, then some properties of the classical code can be seen in the resulting algebra, such as a filtration that corresponds to the Hamming weight.

In Sect. 3, we recall some results of [22] on associating to a classical code C a fractal Λ_C and an operator algebra, a Cuntz algebra $\mathcal{O}_C$ or a Toeplitz algebra $\mathcal{T}_C$. We give an explicit example of a very simple code for which one can completely visualize the associated fractal space. Since we are dealing only with linear codes here, unlike in the more general setting of [22], we can enrich these spaces and algebras with group actions coming from the linear structure of the code. We show that one obtains in this way a crossed product algebra that has the Rokhlin property.

We then show that the fractals of classical codes can be embedded, compatibly with the group actions in a disconnection of a torus and that the geometric construction via rational noncommutative tori obtained in the previous section can be pulled back to the fractal Λ_C via this embedding and the projection from the disconnection to the torus giving rise to a quotient space by the group action which is a fibration over a torus with fiber a fractal. We also show how one can use a crossed product algebra defined by the action of $(\mathbb{Z}/p\mathbb{Z})^2$ on the disconnection of the torus T^2 to obtain a noncommutative space with the property that all the noncommutative spaces associated to individual classical codes via the group action on the associated fractal Λ_C can be embedded inside (powers of) this universal one. This gives a common space inside which to compare noncommutative spaces of different codes and relate their properties. We also give a reinterpretation of the weight polynomial of a linear code in terms of subfractals of Λ_C and multiplicities of embeddings of the corresponding Toeplitz algebras.

1.1 Classical Linear Codes

We recall briefly the general setting of classical codes, following [29]. An alphabet is a finite set $\mathfrak{A}$ of cardinality $q \geq 2$. A classical code is a subset $C \subset \mathfrak{A}^n$. Elements of C are code words, identified with n-tuples $x = (a_1, \dots, a_n)$ in $\mathfrak{A}^n$.

We set $k = k(C) = \log_q \#C$ and $\lfloor k \rfloor$ the integer part of k. The code rate or *transmission rate* of the code is the ratio $R = k/n$.

The Hamming distance between two code words $x = (a_i)$ and $y = (b_i)$ is given by $d(x, y) = \#\{i \,|\, a_i \neq b_i\}$. The *minimum distance* $d = d(C)$ of the code is given by $d(C) = \min\{d(x, y) \,|\, x, y \in C,\ x \neq y\}$. The *relative minimum distance* of the code is the ratio $\delta = d/n$.

A classical code C with these parameters is called an $[n, k, d]_q$ code.

The most important class of codes, in the classical setting, is given by the *linear codes*. In this class, the alphabet is given by the elements of a finite field $\mathfrak{A} = \mathbb{F}_q$ of cardinality $q = p^r$ and characteristic $p > 0$. The code is linear if $C \subset \mathbb{F}_q^n$ is an $\mathbb{F}_q$-linear subspace of the vector space $\mathbb{F}_q^n$. In particular $k = \lfloor k \rfloor$ is an integer for linear codes and is the dimension of C as a vector space.

Given an $\mathbb{F}_q$-bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{F}_q^n$, a code $C \subset \mathbb{F}_q^n$ is *self-orthogonal* if, for all code words $x, y \in C$ one has $\langle x, y \rangle = 0$. The dual code $C^\perp$ is given by the set of vectors v in $\mathbb{F}_q^n$ satisfying $\langle v, x \rangle = 0$ for all $x \in C$. Thus, a self-orthogonal code satisfies $C \subseteq C^\perp$.

1.2 Quantum Stabilizer Codes

A qbit is a vector in the finite dimensional Hilbert space $\mathbb{C}^2$. Quantum codes as in [28] have been typically constructed over qbit spaces $(\mathbb{C}^2)^{\otimes n}$. These are referred to as binary quantum codes. However, more recently nonbinary quantum codes have also been constructed [5,27], especially in relation to classical codes associated to algebraic curves.

In this more general setting of nonbinary quantum codes, one considers a vector $\mathbb{C}^q$ representing the states of a q-ary system. A *q-ary quantum code* of *length n* and *size k* is then a k-dimensional $\mathbb{C}$-linear subspace of $\mathbb{C}^{q^n} = (\mathbb{C}^q)^{\otimes n}$. A quantum error is a linear map $E \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{q^n})$. For a quantum error of the form $E = E_1 \otimes \cdots \otimes E_n$, the weight is $w(E) = \#\{i \,|\, E_i \neq id\}$. A quantum error E is *detectable* by a quantum code Q if $P_Q E P_Q = \lambda_E P_Q$, where P_Q is the orthogonal projection onto $Q \subset \mathbb{C}^{q^n}$ and $\lambda_E \in \mathbb{C}$ is a constant depending only on E. The *minimum distance* of a quantum code Q is

$$d_Q = \max\{d \,|\, E \text{ is detectable } \ \forall E = E_1 \otimes \cdots \otimes E_n \text{ with } w(E) \leq d - 1\}. \tag{1.1}$$

A quantum code with these parameters is called a $[[n, k, d]]_q$ quantum code.

We recall the following notation and basic facts following [1]. Let $q = p^m$ and consider, as above, the field $\mathbb{F}_q$. Viewed as an $\mathbb{F}_p$-vector space, it can be identified, after choosing a basis, with $\mathbb{F}_p^m$. Thus, given an element $x \in \mathbb{F}_q^n$, $x = (a_1, \dots, a_n)$, we can identify the coefficients $a_i \in \mathbb{F}_q$ with vectors $a_i = (a_{i1}, \dots, a_{im})$ with a_{ij} in $\mathbb{F}_p$. These in turn can then be thought of as elements of $\mathbb{Z}/p\mathbb{Z}$, that is, as integer numbers $0 \leq a_{ij} \leq p - 1$. Thus, given a linear operator $L \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^p)$, such that $L^p = id$, we can consider the integer powers $L^{a_{ij}}$.

In particular, consider the two operators T and R on $\mathbb{C}^p$ given by the matrices

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{1.2}$$

$$R = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \xi & 0 & \cdots & 0 & 0 \\ 0 & 0 & \xi^2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & \xi^{p-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \xi^{p-1} \end{pmatrix}, \tag{1.3}$$

where $\xi = \exp(2\pi i/p)$. These have the properties that

$$T^p = R^p = id \quad \text{and} \quad TR = \xi RT, \tag{1.4}$$

which also imply the relations

$$T^k R^\ell = \xi^{k\ell} R^\ell T^k \quad \text{and} \quad (T^k R^\ell)(T^r R^s) = \xi^{-r\ell} T^{r+k} R^{s+\ell} = \xi^{sk-r\ell} (T^r R^s)(T^k R^\ell). \tag{1.5}$$

Moreover, the operators $T^k R^\ell$ form an orthonormal basis of $M_p(\mathbb{C}) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^p)$ with respect to the inner product $\langle A, B \rangle = \mathrm{Tr}(A^* B)$.

Consider then linear maps $E = E_1 \otimes \cdots \otimes E_n$ in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{q^n})$, with $q = p^m$, where the factors E_i are of the form $E_i = T_x R_y$, where x and y are elements in $\mathbb{F}_q$, which we write as vectors $x = (a_1, \ldots, a_m)$, $y = (b_1, \ldots, b_m)$ with coefficients a_i and b_i in $\mathbb{F}_p$, and we set $T_x = T^{a_1} \otimes \cdots \otimes T^{a_n}$ and $R_y = R^{b_1} \otimes \cdots \otimes R^{b_n}$, with the same conventions explained above and with T and R as in (1.2) and (1.3). Thus, for $v = (x_1, \ldots, x_n)$ and $w = (y_1, \ldots, y_n)$ vectors in $\mathbb{F}_q^n$, we can write a corresponding operator

$$E_{v,w} = T_{x_1} R_{y_1} \otimes \cdots \otimes T_{x_n} R_{y_n}. \tag{1.6}$$

The relations (1.4) and (1.5) imply that

$$E_{v,w} E_{v',w'} = \xi^{\langle v,w' \rangle - \langle w,v' \rangle} E_{v',w'} E_{v,w}, \tag{1.7}$$

where, for $v, w \in \mathbb{F}_q^n$, the bilinear form $\langle v, w \rangle$ is defined as

$$\langle v, w \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij}. \tag{1.8}$$

Similarly, one also has

$$E_{v,w} E_{v',w'} = \xi^{-\langle w,v' \rangle} E_{v+v',w+w'}, \tag{1.9}$$

and $E_{v,w}^p = id$ as a $p^{nm} \times p^{nm}$ matrix.

One then denotes by $\mathcal{E}$ (see [1]) the subgroup of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^{q^n})$ given by the invertible linear maps of the form

$$\mathcal{E} = \left\{ \xi^k E_{v,w} \mid v, w \in \mathbb{F}_q^n,\ 0 \le k \le p-1 \right\}. \tag{1.10}$$

It is a finite group of order p^{2mn+1}. The center $\mathcal{Z}$ of $\mathcal{E}$ is the subgroup $\{\xi^k\, id\}$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

A *quantum stabilizer code* is a quantum code that is obtained as joint eigenspace of all the linear transformations in a commutative subgroup of $\mathcal{E}$. Namely, let $\mathcal{S} \subset \mathcal{E}$ be a commutative subgroup with $\#\mathcal{S} = p^{r+1}$, and let $\chi : \mathcal{S} \to U(1)$ be a character that is trivial on $\mathcal{Z}$. Then the associated quantum stabilizer code $Q = Q_{\mathcal{S},\chi}$ is given by the linear subspace of $\mathbb{C}^{q^n}$

$$Q_{\mathcal{S},\chi} = \left\{ \psi \in \mathbb{C}^{q^n} \mid A\psi = \chi(A)\psi,\ \forall A \in \mathcal{S} \right\}. \tag{1.11}$$

The dimension of this vector space is p^{mn-r}, see [1].

1.3 Classical and Quantum Codes

A very interesting aspect of quantum stabilizer codes is that there is an efficient procedure to go back and forth between classical self-orthogonal linear codes and quantum stabilizer codes with a good control over the respective parameters. The procedure is explained in detail in [1] and we only recall it here briefly for what we will need to use later in the paper.

Given a quantum stabilizer code $Q = Q_{\mathcal{S},\chi}$ as above and an $\mathbb{F}_p$-linear automorphism $\varphi \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_p^m)$, the set

$$C = C_{Q,\varphi} = \{(v, \varphi^{-1}(w)) \mid E_{v,w} \in \mathcal{S}\} \tag{1.12}$$

is an $\mathbb{F}_p$-linear code of length $2n$, with $\#C = p^r$, where $\#\mathcal{S} = p^{r+1}$. It is self-orthogonal with respect to the bilinear form $\langle v, \varphi(w')\rangle - \langle v', \varphi(w)\rangle$, with $\langle v, w\rangle$ as in (1.8). The minimum distance d_Q of the quantum stabilizer code $Q_{\mathcal{S},\chi}$ is related to the classical code by $d_Q = d^\perp = d_{C^\perp \smallsetminus C} := \min \#\{i \mid v_i \neq 0 \text{ or } w_i \neq 0,\ (v, w) \in \mathbb{F}_q^{2n},\ (v, w) \in C^\perp \smallsetminus C\}$.

Conversely, given a classical linear self-orthogonal code in $\mathbb{F}_q^{2n}$, with $\#C = p^r$, the linear maps $E_{v,\varphi(w)}$, with (v, w) ranging over an $\mathbb{F}_p$-basis of C, together with the elements $\xi^k id$, generate a subgroup $\mathcal{S}$ of $\mathcal{E}$. The self-orthogonal condition implies by (1.9) that the subgroup $\mathcal{S}$ is abelian. By construction, it is of order $\#\mathcal{S} = p^{r+1}$. The associated quantum stabilizer codes $Q_{\mathcal{S},\chi}$ then have parameters $[[n, n - r/m, d^\perp]]_q$.

Notice how, in this construction, the field extension $\mathbb{F}_q$ of $\mathbb{F}_p$ is identified with the vector space $\mathbb{F}_p^m$, without keeping track of the field structure. The only choice in the data that can be arranged so as to remember the remaining structure is the automorphism φ. Namely, as shown in [1], that can be chosen so that the bilinear form becomes $\mathrm{Tr}(\langle v, w'\rangle - \langle v', w\rangle)$ with $\langle v, w\rangle = \sum_{i=1}^n v_i w_i$, with the product in the field $\mathbb{F}_q$ and $\mathrm{Tr} : \mathbb{F}_{p^m} \to \mathbb{F}_p$ the standard trace $\mathrm{Tr}(x) = \sum_{k=0}^{m-1} x^{p^k}$.

This procedure that constructs quantum stabilizer codes from classical self-orthogonal linear codes was further refined in [18], but for our purposes here this description suffices.

2 Quantum Codes and Rational Noncommutative Tori

In this section we show that the data of quantum stabilizer codes described above can also be described in terms of rational noncommutative tori.

2.1 Twisted Group Rings

We recall here also something about twisted group rings, which will be useful later. Given a discrete group G, the group ring $\mathbb{C}[G]$ admits a (reduced) C^*-completion $C_r^*(G)$ by taking the closure of $\mathbb{C}[G]$ in the operator norm of the algebra of bounded operators $\mathcal{B}(\ell^2(G))$, for the action of $\mathbb{C}[G]$ on $\ell^2(G)$ by $r_g f(g') = f(g'g)$. A multiplier $\sigma : G \times G \to U(1)$ is a 2-cocycle satisfying the conditions $\sigma(g, 1) = \sigma(1, g) = 1$ and $\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\sigma(g_2, g_3)$. The twisted group ring $\mathbb{C}[G, \sigma]$ is generated by the twisted translations $r_g^\sigma f(g') = f(g'g)\,\sigma(g', g)$. The properties of the multiplier ensure that the resulting algebra is still associative. The composition of twisted translations is given by $r_g^\sigma r_{g'}^\sigma = \sigma(g, g') r_{gg'}^\sigma$. The twisted (reduced) group C^*-algebra $C_r^*(G, \sigma)$ is the norm closure of $\mathbb{C}[G, \sigma]$ in $\mathcal{B}(\ell^2(G))$.

The following simple observation relates these general facts to the codes we recalled in the previous section.

Lemma 2.1 *For $q = p^m$, the matrix algebra $M_{q^n}(\mathbb{C})$ can be identified with the twisted group C^*-algebra $C^*((\mathbb{Z}/p\mathbb{Z})^{2mn}, \sigma)$, where the multiplier $\sigma : (\mathbb{Z}/p\mathbb{Z})^{2m} \times (\mathbb{Z}/p\mathbb{Z})^{2m} \to U(1)$ is given by*

$$\sigma((v, w), (v', w')) = \xi^{-\langle w, v'\rangle}, \tag{2.1}$$

with $\langle \cdot, \cdot \rangle$ defined as in (1.8) and with $\xi = \exp(2\pi i/p)$. This is, in turn, the C^*-algebra $C^*(\mathcal{E})$, with $\mathcal{E}$ as in (1.10), generated by the transformations $E_{v,w}$ of (1.6).

Proof The expression (2.1) defines a multiplier on $(\mathbb{Z}/p\mathbb{Z})^{2mn}$. In fact, $\sigma((v,w),(0,0)) = \sigma((0,0),(v,w)) = 1$ and

$$\begin{aligned}\sigma((v,w),(v',w'))\sigma((v+v',w+w'),(v'',w'')) &= \xi^{-\langle w,v'\rangle-\langle w,v''\rangle-\langle w',v''\rangle}\\ &= \sigma((v,w),(v'+v'',w'+w''))\sigma((v',w'),(v'',w'')).\end{aligned}$$

The twisted group C^*-algebra (which is the same as the twisted group ring in this finite dimensional case) $C^*((\mathbb{Z}/p\mathbb{Z})^{2mn},\sigma)$ then has generators $r^\sigma_{(v,w)}$ such that $r^\sigma_{(v,w)}r^\sigma_{(v',w')} = \xi^{-\langle w,v'\rangle}r^\sigma_{(v+v',w+w')}$. By direct comparison with (1.9), one sees that the identification $r^\sigma_{(v,w)} \mapsto E_{v,w}$ identifies $C^*((\mathbb{Z}/p\mathbb{Z})^{2mn},\sigma)$ with $C^*(\mathcal{E}/\mathcal{Z})$. In fact, notice that the relation (1.7) also follows from the twisted group ring relations since we obtain

$$r^\sigma_{(v,w)}r^\sigma_{(v',w')} = \sigma((v,w),(v',w'))\sigma((v',w'),(v,w))^{-1}r^\sigma_{(v',w')}r^\sigma_{(v,w)}$$

which then gives relation (1.7). The identification between $C^*(\mathcal{E}/\mathcal{Z})$ and $M_{q^n}(\mathbb{C})$ follows from the known fact that the transformations $E_{v,w}$ generate $\mathrm{End}_{\mathbb{C}}((\mathbb{C}^q)^{\otimes n})$. □

2.2 Rational Noncommutative Tori

The (rational or irrational) rotation algebras, also known as noncommutative tori, are the most widely studied examples of noncommutative spaces. As a C^*-algebra, the rotation algebra $\mathcal{A}_\theta$ is generated by two unitaries U and V, subject to the commutation relation

$$UV = \xi VU, \tag{2.2}$$

with $\xi = \exp(2\pi i\theta)$. In the rational case, $\theta \in \mathbb{Q}$, it is well known that these algebras are Morita equivalent to the commutative algebra of functions $C(\mathbb{T}^2)$ on the ordinary commutative torus $\mathbb{T}^2$, while in the irrational case $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$, the Morita equivalence classes correspond to the orbits of the action of $\mathrm{SL}_2(\mathbb{Z})$ on the real line by fractional linear transformations.

Let us look more closely at the rational case with $\xi = \exp(2\pi i/p)$. Then elements in the rotation algebra $\mathcal{A}_{1/p}$ are of the form

$$\mathcal{A}_{1/p} \ni a = \sum_{k,\ell} f_{k,\ell}(\mu,\lambda)\, T^k R^\ell, \tag{2.3}$$

where $f_{k,\ell}(\mu,\lambda)$ are continuous functions of $(\lambda,\mu) \in S^1 \times S^1 = \mathbb{T}^2$ and T and R are the matrices (1.2) and (1.3). The sum is a finite sum for $0 \le k, \ell \le p-1$ since $T^p = R^p = id$. In particular, the generators U and V are given, respectively, by $U = \mu T$ and $V = \lambda R$, with $\mu = \exp(2\pi i t)$ and $\lambda = \exp(2\pi i s)$ in S^1. To see this notice that the algebra $\mathcal{A}_{1/p}$ is generated by elements of the form

$$\sum_{k,\ell\in\mathbb{Z}} a_{k\ell} U^k V^\ell.$$

Since $T^p = R^p = id$, we can rewrite these as

$$\sum_{k,\ell\in\mathbb{Z}/p\mathbb{Z}}\ \sum_{k',\ell'\in\mathbb{Z}} a_{k+k'p,\ell+\ell'p}\mu^{k+k'p}\lambda^{\ell+\ell'p}T^kR^\ell = \sum_{k,\ell\in\mathbb{Z}/p\mathbb{Z}} f_{k,\ell}(\lambda,\mu)T^kR^\ell.$$

2.3 Quantum Codes and Vector Bundles

Recall (see [11], Proposition 12.2) that the rational noncommutative torus $\mathcal{A}_{n/m}$ is isomorphic to the algebra $\Gamma(T^2, \mathrm{End}(E_m))$ of sections of the endomorphism bundle of a rank m vector bundle E_m over the ordinary torus T^2, obtained as follows. Consider the trivial bundle over T^2 with fiber $M_m(\mathbb{C})$, with the action of $(\mathbb{Z}/m\mathbb{Z})^2$ given by

$$\tau_{1,0} : (\mu,\lambda,M) \mapsto (\mu, e^{-2\pi i n/m}\lambda, TMT^{-1}), \quad \tau_{0,1} : (\mu,\lambda,M) \mapsto (e^{2\pi i n/m}\mu, \lambda, RMR^{-1}).$$

The quotient by this action defines a non-trivial bundle over T^2, which we can view as the endomorphism bundle $\mathrm{End}(E_m)$ of a vector bundle E_m of rank m, with fiber $M_m(\mathbb{C})$. The algebra of sections $\Gamma(T^2, \mathrm{End}(E_m))$ is by construction the fixed point subalgebra of the algebra $C(T^2, M_m(\mathbb{C})) = C(T^2) \otimes M_m(\mathbb{C})$ of endomorphisms of the trivial bundle, under the action of $(\mathbb{Z}/m\mathbb{Z})^2$ described above. The above action gives on the algebra $C(T^2) \otimes M_m(\mathbb{C})$ the action

$$\begin{aligned} &\alpha_{1,0} : f(\mu, \lambda) \otimes M \mapsto f(\mu, e^{-2\pi i n/m}\lambda) \otimes TMT^{-1}, \\ &\alpha_{0,1} : f(\mu, \lambda) \otimes M \mapsto f(e^{2\pi i n/m}\mu, \lambda) \otimes RMR^{-1}. \end{aligned} \tag{2.4}$$

The fixed point subalgebra is then generated by the elements $\mu \otimes T$ and $\lambda \otimes R$, which satisfy the commutation relation of the generators U and V of the noncommutative torus, and is therefore isomorphic to $\mathcal{A}_{n/m}$. In particular, there is a C^*-algebra homomorphism $\mathcal{A}_{n/m} \to M_m(\mathbb{C})$ that sends the generators U and V to the matrices T and R.

We then use this description of the rational noncommutative tori to give a geometric interpretation of the data of quantum stabilizer codes.

Proposition 2.2 *Let E_p be the rank p bundle over T^2 such that $\mathcal{A}_{1/p} = \Gamma(T^2, \mathrm{End}(E_p))$. Then, for $q = p^m$, a q-ary quantum stabilizer code $Q_{\mathcal{S},\chi}$ of length n and size k corresponds to a subalgebra $\mathcal{A}_{\mathcal{S}} \subset \mathcal{A}_{1/p}^{\otimes r}$, with $r = nm$, and subbundle $\mathcal{F}_{\mathcal{S},\chi}$ of the external tensor product $E_p^{\boxtimes mn}$ over T^{2r}, on which the elements of the algebra $\mathcal{A}_{\mathcal{S}}$ act as scalars. Conversely, these data determine a q-ary quantum stabilizer code $Q_{\mathcal{S},\chi}$ of length n and size k.*

Proof Let us first consider the tensor product algebra $C(T^2, M_p(\mathbb{C}))^{\otimes r}$ where $r = mn$. We can write this also as $(C(T^2) \otimes M_p(\mathbb{C}))^{\otimes r} = C(T^{2r}) \otimes M_{q^n}(\mathbb{C}) = C(T^{2r}, M_{q^n}(\mathbb{C}))$, for $q = p^m$. This is therefore the algebra of endomorphisms of the trivial bundle with fiber $\mathbb{C}^{q^n}$ over the higher dimensional torus T^{2r}. The action of $(\mathbb{Z}/p\mathbb{Z})^2$ on $C(T^2, M_p(\mathbb{C}))$ given in (2.4) extends to an action of $(\mathbb{Z}/p\mathbb{Z})^{2r}$ on $C(T^{2r}, M_{q^n}(\mathbb{C}))$, which is given by

$$\alpha_{v,w} : f(\underline{\mu}, \underline{\lambda}) \otimes M \mapsto f(\xi^v \underline{\mu}, \xi^{-w} \underline{\lambda}) \otimes E_{v,w} M E_{v,w}^{-1}, \tag{2.5}$$

with $\underline{\mu} = (\underline{\mu}_1, \dots, \underline{\mu}_n) = (\mu_{11}, \dots, \mu_{1m}, \dots, \mu_{n1}, \dots, \mu_{nm})$ and similarly for $\underline{\lambda}$, where the notation $\xi^v \underline{\mu}$ means $\xi^v \underline{\mu} = (\xi^{a_{ij}} \mu_{ij})_{i=1,\dots,n; j=1,\dots,m}$, with $v = (x_1, \dots, x_n)$ and each $x_i = (a_{i1}, \dots, a_{im})$. The notation $\xi^{-w}\underline{\lambda}$ is analogous. We realize here the matrix algebra $M_{q^n}(\mathbb{C})$ as in Lemma 2.1, as the algebra $C^*(\mathcal{E}/\mathcal{Z}) = C^*((\mathbb{Z}/p\mathbb{Z})^{2mn}, \sigma)$ generated by elements $E_{v,w}$ as in (1.6).

The fixed point algebra of the action (2.5) defines the endomorphism algebra of a vector bundle on the torus T^{2r} of rank q^n. The external tensor product $E_1 \boxtimes E_2$ of two vector bundles V_1 and V_2, respectively over base spaces X_1 and X_2, is the vector bundle over $X_1 \times X_2$ given by $\pi_1^*(V_1) \otimes \pi_2^*(V_2)$, with π_1 and π_2 the projections of $X_1 \times X_2$ onto the two factors. We then see that the vector bundle on T^{2r} described above is, in fact, the r-times external tensor product of the bundle E_p on T^2, since the action (2.5) is the product of an action of the form (2.4) on each copy of $C(T^2, M_p(\mathbb{C}))$. Thus, the fixed point algebra is the algebra of endomorphisms $\Gamma(T^{2r}, E_p^{\boxtimes r})$.

The fixed point algebra of the action (2.5) on $C(T^{2r}, M_{q^n}(\mathbb{C}))$ is generated by elements of the form $\underline{\mu}(v) \otimes \underline{\lambda}(w) \otimes E_{v,w}$, where $\underline{\mu}(v, w)$ is the tensor product of those $\underline{\mu}(v)_{ij}$ for which $a_{ij} = 0$, and similarly for $\underline{\lambda}(w)$. Given the explicit form of the elements $E_{v,w}$ as in (1.6), we see that the fixed point algebra is equivalently generated by elements of the form $\mu_{ij} \otimes (1 \otimes \cdots \otimes T \otimes \cdots 1)$, with T in the (i, j)th coordinate of the tensor product, and $\lambda_{ij} \otimes (1 \otimes \cdots \otimes R \otimes \cdots \otimes 1)$, with R in the (i, j)th place. Thus, it is the r-fold tensor product $\mathcal{A}_{1/p}^{\otimes r}$ of the algebra $\mathcal{A}_{1/p}$ of the rational noncommutative torus.

Now suppose one is given a q-ary quantum stabilizer code of length n and size k. This means that we have a commutative subgroup $\mathcal{S}$ of $\mathcal{E}$ and a character $\chi : \mathcal{S} \to U(1)$ that is trivial on $\mathcal{Z}$ and such that the common eigenspace $Q_{\mathcal{S},\chi} \subset \mathbb{C}^{q^n}$ on which the operators $s \in \mathcal{S}$ act as $s\psi = \chi(s)\psi$ has complex dimension k.

The choice of the commutative subgroup $\mathcal{S}$ of $\mathcal{E}$ determines a commutative subalgebra $\mathcal{A}_{\mathcal{S}}$ of the algebra $\mathcal{A}_{1/p}^{\otimes r}$, which is the subalgebra generated by elements of the form $\underline{\mu}(v) \otimes \underline{\lambda}(w) \otimes E_{v,w}$ as above, with $E_{v,w} \in \mathcal{S}$. This is the commutative subalgebra of the endomorphism algebra $\Gamma(T^{2r}, E_p^{\boxtimes r})$, generated by the unitaries $\underline{\mu}(v) \otimes \underline{\lambda}(w) \otimes E_{v,w}$.

The common eigenspaces of the $E_{v,w} \in \mathcal{S}$ acting on $\mathbb{C}^{q^n}$ correspond to characters χ of $\mathcal{S}$. Thus, the eigenspace $Q_{\mathcal{S},\chi}$, for the character χ of the data of the q-ary quantum stabilizer code, determines a subbundle $\mathcal{F}_{\mathcal{S},\chi}$ of the bundle $E_p^{\boxtimes r}$ over T^{2r} with an action of the abelian subalgebra $\mathcal{A}_{\mathcal{S}}$ of $\mathcal{A}_{1/p}^{\otimes r}$ by endomorphisms. □

We can give a more explicit description of the algebra $\mathcal{A}_{\mathcal{S}}$ as follows.

Corollary 2.3 *The algebra* $\mathcal{A}_{\mathcal{S}} = C(X_{\mathcal{S}})$ *is the algebra of functions of a space* $X_{\mathcal{S}} = \cup_{\chi\in\hat{\mathcal{S}}} T_\chi$, *where* T_χ *is a quotient of the torus* T^{2r} *over which the bundle* $\mathcal{F}_{\mathcal{S},\chi}$ *descends to a direct sum* $\mathcal{L}^{\oplus k}_{\mathcal{S},\chi}$ *of k-copies of a line bundle.*

Proof The abelian subalgebra $\mathcal{A}_{\mathcal{S}}$ of $\mathcal{A}^{\otimes r}_{1/p}$ can be identified, via the Gelfand–Naimark correspondence, with the algebra of functions $C(X_{\mathcal{S}})$ on a compact Hausdorff topological space $X_{\mathcal{S}}$. To give an explicit description of the space $X_{\mathcal{S}}$ in relation to the torus T^{2r}, it is convenient to also view $\mathcal{A}_{\mathcal{S}}$ as the subalgebra of the abelian algebra $C(T^{2r}, \mathbb{C}[\mathcal{S}])$ generated by the elements $\underline{\mu}(v) \otimes \underline{\lambda}(w) \otimes E_{v,w}$ as above, with $E_{v,w} \in \mathcal{S}$. We write these elements in shorter notation as $\mu_s \otimes \lambda_s \otimes s$, for $s \in \mathcal{S}$. For varying $s \in \mathcal{S}$, the corresponding $\mu_s \otimes \lambda_s$ generate a subalgebra $C(T^{2r})$, which corresponds to a quotient space of T^{2r}.

By Pontrjagin duality, we can identify $\mathbb{C}[\mathcal{S}]$, which is the same as $C^*(\mathcal{S})$ since $\mathcal{S}$ is a finite (abelian) group, with $C(\hat{\mathcal{S}})$, for $\hat{\mathcal{S}}$ the character group. The isomorphism $C^*(\mathcal{S}) \simeq C(\hat{\mathcal{S}})$ is by Fourier transform. Since $\hat{\mathcal{S}}$ is also a finite (abelian) group, $C(\hat{\mathcal{S}}) = \oplus_{\chi\in\hat{\mathcal{S}}} \mathbb{C}_\chi$, where $\mathbb{C}_\chi$ is the 1-dimensional algebra of functions on the point $\chi \in \hat{\mathcal{S}}$. Thus, we have $C(T^{2r}, \mathbb{C}[\mathcal{S}]) = C(T^{2r} \times \hat{\mathcal{S}}) = \oplus_{\chi\in\hat{\mathcal{S}}} C(T^{2r}) \otimes \mathbb{C}_\chi$. The component in $C(T^{2r}) \otimes \mathbb{C}_\chi$ of the subalgebra $\mathcal{A}_{\mathcal{S}}$, which we denote by $\mathcal{A}_{\mathcal{S},\chi}$ is then generated by the elements of the form $\mu_s \otimes \lambda_s \otimes \hat{\delta}_s p_\chi$, where $\hat{\delta}_s \in C(\hat{\mathcal{S}})$ is the Fourier transform of the generator δ_s of $\mathbb{C}[\mathcal{S}]$, and p_χ is the projection onto the $\mathbb{C}_\chi$ component of $C(\hat{\mathcal{S}})$, where $\hat{\delta}_s p_\chi = \chi(s)$. Upon denoting by T_χ the quotient space of T^{2r} that corresponds to the subalgebra of $C(T^{2r})$ generated by the $\mu_s \otimes \lambda_s \otimes \hat{\delta}_s p_\chi$, we get $\mathcal{A}_{\mathcal{S}} = \oplus_{\chi\in\hat{\mathcal{S}}} C(T_\chi) \otimes \mathbb{C}_\chi$.

By construction, the subbundle $\mathcal{F}_{\mathcal{S},\chi}$ then restricts to T_χ as a direct sum $\mathcal{L}^{\oplus k}_{\mathcal{S},\chi}$ of k-copies of a line bundle $\mathcal{L}_{\mathcal{S},\chi}$, whose sections transform as $(\mu, \lambda, z) \mapsto (\mu_s\mu, \lambda_s\lambda, \chi(s)z)$.

2.4 Classical Codes and the Rational Noncommutative Torus

We show next how, in the case of a quantum stabilizer code obtained from a self-orthogonal classical linear code via the CSS algorithm, one can read some of the properties of the classical code in the algebra $\mathcal{A}_{\mathcal{S}}$.

Let C be a classical linear code $C \subset \mathbb{F}_q^n$ and let $Q_{\mathcal{S}_C,\chi}$ be a q-ary quantum stabilizer code obtained from C via the CSS algorithm recalled above. Recall that, for a code word $c \in C$ the Hamming weight $\varpi(c)$ is the number of non-zero coordinates of $c \in \mathbb{F}_q^n$.

Proposition 2.4 *The algebra* $\mathcal{A}_{\mathcal{S}} = C(X_{\mathcal{S}})$ *has a natural filtration by the Hamming weight of words in the classical code* C.

Proof Seen as a subalgebra of $C(T^{2r}) \otimes \mathbb{C}[\mathcal{S}]$, the commutative algebra $\mathcal{A}_{\mathcal{S}}$ is generated by elements of the form $\mu_s \otimes \lambda_s \otimes \delta_s$, where the μ_s and λ_s are defined as above as the μ_{ij} and λ_{ij}, respectively for the indices (i, j) for which $a_{ij} = 0$ and $b_{ij} = 0$ in the coordinates of (v, w), for $s = E_{v,w} \in \mathcal{S}$. Thus, we can write the algebra as $\mathcal{A}_{\mathcal{S}} = \oplus_{s\in\mathcal{S}} C(T_s) \otimes \delta_s$, where $C(T_s)$ is the subalgebra of $C(T^{2r})$ generated by the μ_s and λ_s as above. The spaces T_s are quotients of T^{2r} of dimension equal to $2r - \varpi(v, w)$, where $\varpi(v, w)$ is the Hamming weight of the word (v, w). Under multiplication in the algebra, the products of a generator of the form $\mu_s \otimes \lambda_s \otimes \delta_s$ and a generator of the form $\mu_{s'} \otimes \lambda_{s'} \otimes \delta_{s'}$ are (strictly) contained among the set of generators of the form $\mu_{s+s'} \otimes \lambda_{s+s'} \otimes \delta_{s+s'}$, hence $C(T_s) \otimes \delta_s \cdot C(T_{s'}) \otimes \delta_{s'} \subset C(T_{s+s'}) \otimes \delta_{s+s'}$, so that the filtration by the Hamming weight is compatible with the algebra structure on $\mathcal{A}_{\mathcal{S}}$. □

3 Algebras and Spaces of Classical and Quantum Codes

In this section we modify the previous setting to describe a noncommutative space where the pairs of a classical linear code and the corresponding quantum stabilizer code can be embedded as subspaces in a uniform way. This

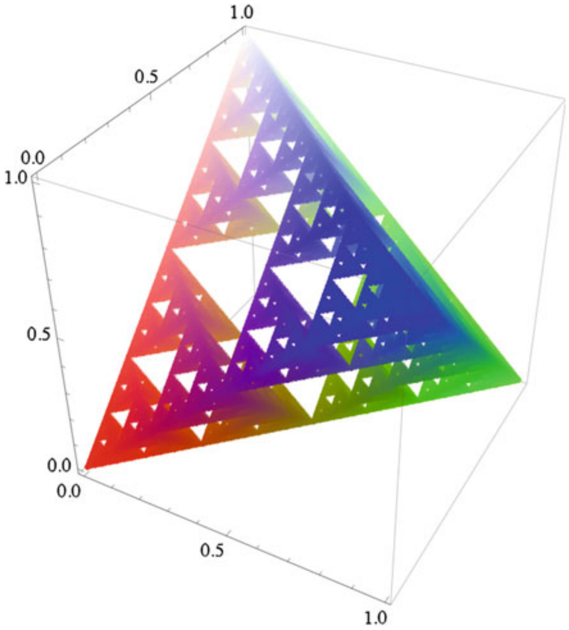

Fig. 1 The fractal associated to the code C of (3.1)

is based on a modification of the previous construction, where the rational noncommutative tori, obtained from endomorphism algebras of vector bundles over tori, are replaced by spaces obtained as bundles over tori with fiber a Cantor set. These are obtained by considering the fractals and the operator algebras associated to classical codes as in [22].

3.1 Classical Codes and Fractals

As shown in [22], to a classical (not necessarily linear) code $C \subset \mathfrak{A}^n$, one can associate a fractal Λ_C by identifying the alphabet $\mathfrak{A}$ with $\#\mathfrak{A} = q$ with the digits of the q-ary expansion of numbers in the interval $[0, 1]$, so that infinite sequence of code words $x_0 x_1 x_2 \dots$ determine a subset Λ_C of point in the cube $[0, 1]^n$. This subset is typically a Sierpinski fractal. The parameters of the code are related to the Hausdorff dimension of Λ_C and to the Hausdorff dimension of its intersections with translates of coordinate hyperplanes (see [22]).

To see concretely the fractal structure associated to a code, consider the simple example of the $[3, 2, 2]_2$ code C given by

$$C = \begin{cases} (0, 0, 0) \\ (0, 1, 1) \\ (1, 0, 1) \\ (1, 1, 0) \end{cases} \tag{3.1}$$

In this case, the corresponding fractal is the Sierpinski gasket illustrated in Fig. 1.

3.2 Spectral Triples on Fractals

As we mentioned in the introduction, the fractals associated to classical error-correcting codes were recently related ([3]) to spectral triples on Cantor sets to crossed product constructions for such spectral triples.

A spectral triple is a notion introduced in noncommutative geometry ([8]) as a generalization of the classical notion of a Riemannian spin manifold. The basic data are an involutive algebra $\mathcal{A}$ (which generalizes the algebra of smooth functions), with a representation by bounded operators on a Hilbert space $\mathcal{H}$, and a self-adjoint (unbounded) operator D on $\mathcal{H}$ with compact resolvent, which satisfies a compatibility condition with the algebra, given by the requirement that the commutators $[D, a]$ are bounded operators on $\mathcal{H}$. An ordinary compact Riemannian spin manifold M is described as a spectral triple by the data $\mathcal{A} = \mathcal{C}^\infty(M)$, $\mathcal{H} = L^2(M, S)$, with S the spinor bundle, and D the Dirac operator. The Riemannian metric can be reconstructed from these data. However, the advantage of the spectral triple formalism is that other kind of spaces like fractals, quantum groups, noncommutative tori, can be treated as smooth manifolds from the point of view of noncommutative geometry.

It is shown in §7.1 of [3] that the generating function

$$G_C(t) = \sum_N s_C(N) t^N$$

for the language associated to a code C, with $s_C(N) = \#\{w = w_1 \dots w_N \mid w_i \in C\}$, can be identified with the zeta function $\zeta_D(s)$ of the Dirac operator D on a natural spectral triple defined over the fractal Λ_C.

Some of the actions of G_C on Λ_C described here are especially suitable for the crossed product construction, as shown for instance in the recent paper [10]. This means that one can regard the crossed product $C(\Lambda_C) \rtimes G_C$ as a spectral triple (a noncommutative manifold) and apply to it methods of noncommutative Riemannian geometry.

There are several interesting constructions of noncommutative geometry, often in the form of spectral triples, applied to fractal spaces, such as those obtained in [6,7,12,13]. These can be applied to spaces such as the code fractal Λ_C, or the subfractals $\Lambda_{C,\ell,\pi}$ considered in [22], which we will discuss more in the following sections, or the quotients $(\Lambda_C \times Q^*_{\mathcal{S},\chi})/\mathcal{S}$ we introduce below. The spectral triples constructed in this way capture some of the information theoretic properties of both the codes, as was shown in [3].

3.3 Noncommutative Spaces and Quantum Statistical Systems from Codes

In [22] it was also suggested to consider operator algebras associated to a classical code $C \subset \mathfrak{A}^n$, in the form of a Toeplitz algebra $\mathcal{T}_C$ generated by isometries S_a for $a \in C$, $S_a^* S_a = 1$, with mutually orthogonal ranges, and the Cuntz algebra $\mathcal{O}_C$, which is the quotient of $\mathcal{T}_C$ obtained by imposing the additional relation $\sum_{a \in C} S_a S_a^* = 1$. The Cuntz algebra $\mathcal{O}_C$ has a natural representation as bounded operators on the Hilbert space $L^2(\Lambda_C, d\mu)$ with the Hausdorff measure of dimension $\dim_H(\Lambda_C)$, where the generators S_a act as

$$S_a f(x) = \chi_{\sigma_a(\Lambda_C)}(x) \Phi_a(\sigma(x))^{-1/2} f(\sigma(x)).$$

Here x is an infinite sequence of code words, $x = (x_1, \dots, x_n)$ with each $x_i = x_{i0} \cdots x_{in} \cdots, x_{ij} \in \mathfrak{A}, (x_{1j}, \dots, x_{nj}) \in C$. The map σ_a on Λ_C is given by $\sigma_a(x) = (a_1 x_1, \dots, a_n x_n)$, for $a = (a_1, \dots, a_n) \in C$ and the map σ is the one-sided shift that removes the $(x_{10}, \dots, x_{n0})$ code word of x and returns the same infinite sequence of code words shifted one step to the left, starting with $(x_{11}, \dots, x_{n1})$. The function Φ_a is the Radon–Nikodym derivative of the Hausdorff measure, $\Phi_a(x) = d\mu \circ \sigma_a / d\mu$.

As shown in [22], the natural time evolution on the Toeplitz algebra $\mathcal{T}_C$ given by $\sigma_t(S_a) = q^{int} S_a$ defines a quantum statistical mechanical system that has as partition function $Z_C(\beta) = (1 - q^{(R-\beta)n})^{-1}$, with R the rate of the code C. This is the same as the structure function of the language Λ_C, so that the entropy of the language (which is the log of the radius of convergence) agrees with the rate of the code.

3.4 Linear Codes and Group Actions

In the case of linear codes, one can enrich the construction above with additional structure.

Let $C \subset \mathbb{F}_q^n$ be a linear code. Let G_C be the additive group generated by the basis vectors of C. Then G_C acts on the algebras $\mathcal{T}_C$ and $\mathcal{O}_C$ by $\gamma_a : S_b \mapsto S_{b+a}$. This action shuffles the indices of the generating isometries hence it preserves the relations. Thus, one can consider the algebras $\mathcal{T}_C \rtimes G_C$ and $\mathcal{O}_C \rtimes G_C$. These are generated by elements of the form $S_a \gamma_b$ with product $S_a \gamma_b S_{a'} \gamma_{b'} = S_a S_{b+a'} \gamma_{b+b'}$.

Actions of finite abelian groups on Cuntz algebras were studied extensively in operator algebras, in relation to the Rokhlin property.

3.5 The Rokhlin Property

Finite group actions on C^*-algebras that have the Rokhlin property have been widely studied in the context of classification problems for C^*-algebras. The Rokhlin property for an action α of a finite group G on a C^*-algebra A prescribes the existence, for any finite $F \subset A$ and any $\epsilon > 0$, of mutually orthogonal projections e_g in A, for $g \in G$, such that $\|\alpha_g(e_h) - e_{gh}\| < \epsilon$ for all $g, h \in G$; $\|e_h a - a e_g\| < \epsilon$, for all $g \in G$ and $a \in F$, and $\sum_{g \in G} e_g = 1$. The importance of the Rokhlin property lies in the fact that it ensures that the group actions are classifiable in terms of K theoretic invariants. The case of quasi-free actions of finite groups on Cuntz algebras was considered in [14].

Lemma 3.1 *The action of G_C on the Cuntz algebra $\mathcal{O}_C$ has the Rokhlin property.*

Proof According to [14], an action α of a topological group G on the Cuntz algebra $\mathcal{O}_n$ is quasi-free if α_g globally preserves the linear span $\mathcal{H}_n$ of the generators $\{S_i\}_{i=1,\dots,n}$ of the Cuntz algebra, for each $g \in G$. The action of G_C on $\mathcal{O}_C$ described above is quasi-free in this sense, since it has the effect of permuting the generators S_a of $\mathcal{O}_C$, so it leaves the corresponding space, which we denote by $\mathcal{H}_C$, invariant. One then sees directly from Proposition 5.6 and Example 5.7 of [14], that the action of G_C on $\mathcal{O}_C$ has the Rokhlin property. □

We also mention here that, according to Proposition 5.5 of [26], an action of a finite group G on a Cantor set has the Rokhlin property if and only if the action is free. Later in this section we relate the action of G_C on $\mathcal{O}_C$ to an action on the fractal Λ_C.

3.6 Twisted Crossed Products and Codes

One can twist the crossed product algebras $\mathcal{T}_C \rtimes G_C$ and $\mathcal{O}_C \rtimes G_C$ by the cocycle σ as in (2.1).

Lemma 3.2 *Let $C \subset \mathbb{F}_{p^{2m}}^n$ be a linear code with $\#C = q^k$, with $q = p^{2m}$. Then $G_C \subset (\mathbb{Z}/p\mathbb{Z})^{2mn}$ is $G_C \simeq (\mathbb{Z}/p\mathbb{Z})^{2mk}$ and the multiplier (2.1) defines twisted crossed product algebras $\mathcal{T}_C \rtimes_\sigma G_C$ and $\mathcal{O}_C \rtimes_\sigma G_C$.*

Proof The twisted crossed product algebras are generated by elements $S_{(a,b)}\gamma^\sigma_{(v,w)}$ with $(a, b) \in C$ and $(v, w) \in (\mathbb{Z}/p\mathbb{Z})^{2mk}$, with the product given by

$$S_{(a,b)}\gamma^\sigma_{(v,w)} S_{(a',b')}\gamma^\sigma_{(v',w')} = \sigma(v, v') S_{(a,b)} S_{(v+a',w+b')}\gamma^\sigma_{(v+v',w+w')}.$$

The associativity, as above, is ensured by the multiplier properties of σ. □

Lemma 3.3 *The (twisted) action of G_C on $\mathcal{O}_C$ preserves the maximal abelian subalgebra of $\mathcal{O}_C$ isomorphic to $C(\Lambda_C)$.*

Proof The action of G_C on the generators S_a of $\mathcal{O}_C$ is given by $\gamma_b S_a = S_{a+b}$. The subalgebra of $\mathcal{O}_C$ isomorphic to $C(\Lambda_C)$ is generated by the range projections $S_\alpha S_\alpha^*$, where S_α, for some multi-index $\alpha = (a_1, \dots, a_m)$, $a_i \in C$, is a finite product $S_\alpha = S_{a_1} \cdots S_{a_m}$ of generators. The range projection $S_\alpha S_\alpha^*$ corresponds to the projection in $C(\Lambda_C)$ given by the characteristic function of the subset $\Lambda_C(\alpha)$ of infinite sequences of code words in Λ_C that start with the word α.

The induced action γ of the group G_C on the fractal Λ_C is then determined by the action on $C(\Lambda_C)$ that maps the characteristic function $\chi_{\Lambda_C(\alpha)} = S_\alpha S_\alpha^*$ to the characteristic function $\chi_{\Lambda_C(\gamma_b(\alpha))} = \gamma_b(S_\alpha)\gamma_b(S_\alpha^*)$, where $\gamma_b(S_\alpha) = \gamma_b(S_{a_1}) \cdots \gamma_b(S_{a_m}) = S_{a_1+b} \cdots S_{a_m+b}$.

This implies that the induced action on the Cantor set is given by addition in each digit of the expansion: for $(x, y) \in \Lambda_C$ given by $(x, y) = (x_0 x_1 \dots x_N \dots, y_0 y_1 \dots y_N \dots)$ with $(x_i, y_i) \in C$, one gets $\gamma_{v,w}(x, y) = ((x_0 + v)(x_1 + v) \dots (x_N + v) \dots, (y_0 + w)(y_1 + w) \dots (y_N + w) \dots)$, with $(x_i + v, y_i + w) \in C$.

Thus, one obtains a subalgebra $C(\Lambda_C) \rtimes_\sigma G_C$ of $\mathcal{O}_C \rtimes_\sigma G_C$ of the twisted crossed product. Elements of this subalgebra can be written as

$$a = \sum_{(v,w)\in C} f_{(v,w)}(x, y)\, \gamma^\sigma_{(v,w)}, \tag{3.2}$$

for $f_{(v,w)} \in C(\Lambda_C)$ and $\gamma^\sigma_{(v,w)}$ as above, with

$$f_{(v,w)}(x, y)\gamma^\sigma_{(v,w)} f_{(v',w')}(x, y)\gamma^\sigma_{(v',w')} = \sigma((v, w), (v', w'))\, f_{(v,w)}(x, y) f_{(v',w')}(\alpha_{v,w}(x, y))\, \gamma^\sigma_{(v+v',w+w')}.$$

□

Consider then a quantum stabilizer code $Q = Q_{\mathcal{S},\chi}$, associated to a classical self-orthogonal linear code C in $\mathbb{F}_p^{2nm}$, with an $\mathbb{F}_p$-automorphism $\varphi \in \mathrm{Aut}(\mathbb{F}_p^m)$, so that $\mathcal{S} = \{\xi^k E_{v,\varphi(w)} \mid (v, w) \in C\}$ is an abelian subgroup of $\mathcal{E}$. Thus, $Q = Q_{C,\varphi}$. Because of the self-orthogonal condition, the cocycle $\sigma((v, w), (v', w')) = \xi^{-\langle w,v'\rangle}$ is trivial, so the crossed product algebras $\mathcal{O}_C \rtimes_\sigma G_C$ and $C(\Lambda_C) \rtimes_\sigma G_C$ are just the untwisted $\mathcal{O}_C \rtimes G_C$ and $C(\Lambda_C) \rtimes G_C$ with G_C the abelian group identified with the subgroup of $\mathcal{S} \subset \mathcal{E}$ with elements the $E_{v,\varphi(w)}$. The same holds for the related algebras $\mathcal{T}_C \rtimes_\sigma G_C$ which is $\mathcal{T}_C \rtimes G_C$.

3.7 Disconnection and Group Actions

Consider points of $T^2 = S^1 \times S^1$ as points in the square $Q^2 = [0, 1] \times [0, 1]$ with the boundary identifications that give T^2, where we write the points of $[0, 1]$ in terms of their p-ary digital expansion: $x = 0.x_1 x_2 x_3 \dots x_N \dots$, with $x_i \in \{0, \dots, p-1\}$. As in the decimal case, the expansion is a 1:1 representation on the irrational points and 2:1 on the rational points. Fixing the first N digits of the expansion determines a subinterval of $[0, 1]$ of length p^{-N}.

There is a totally disconnected compact topological space $T^2_{\mathbb{Q}}$, called the disconnection of T^2 at the rational points, which maps surjectively to T^2 with a map that is 1:1 over the irrational points and 2:1 over the rational points. As a topological space, it is the spectrum of a commutative C^*-algebra $C(T^2_{\mathbb{Q}})$, which is the smallest C^*-algebra containing $C(T^2)$ in which all the characteristic functions of intervals $[kp^{-N}, (k+1)p^{-N})$ with $k \in \{0, \dots, p-1\}$ and $N \geq 1$ are continuous functions.

Lemma 3.4 *The group $(\mathbb{Z}/p\mathbb{Z})^2$ acts on the disconnection $T^2_{\mathbb{Q}}$ by*

$$\gamma_{(k,\ell)}(x, y) = (\gamma_k(x_0)\gamma_k(x_1) \dots \gamma_k(x_N) \dots, \gamma_\ell(y_0)\gamma_\ell(y_1) \dots \gamma_\ell(y_N) \dots), \tag{3.3}$$

where, for $a \in \mathbb{Z}/p\mathbb{Z}$, $\gamma_b(a) = a + b$ in $\mathbb{Z}/p\mathbb{Z}$. One can then form a crossed product algebra $C(T^2_{\mathbb{Q}}) \rtimes_\sigma (\mathbb{Z}/p\mathbb{Z})^2$, with the action (3.3), and with the twisting given by the cocycle $\sigma((v, w), (v', w')) = \xi^{-\langle w,v'\rangle}$. □

Proof The action $(x_i, y_i) \mapsto (\gamma_k(x_i), \gamma_\ell(y_i))$ on the ith digit of the p-ary expansion of $(x, y) \in T^2_{\mathbb{Q}}$ has the effect of moving the product of intervals $[x_i p^{-i}, (x_i + 1)p^{-i}) \times [y_i p^{-i}, (y_i + 1)p^{-i})$ inside T^2 to $[(x_i + k \mod p)p^{-i}, (x_i + 1 + k \mod p)p^{-i}) \times [(y_i + k \mod p)p^{-i}, (y_i + 1 + k \mod p)p^{-i})$. While this is not a continuous function on T^2 it becomes continuous on the totally disconnected $T^2_{\mathbb{Q}}$. Thus, one can form the crossed product C^*-algebra with respect to this action. It is generated by elements of the form $\sum_{g\in(\mathbb{Z}/p\mathbb{Z})^2} h_g(\lambda, \mu) r^\sigma_g$, with $(\lambda, \mu) \in T^2_{\mathbb{Q}}$ and where $r^\sigma_{g_1} r^\sigma_{g_2} = \sigma(g_1, g_2) r^\sigma_{g_1 g_2}$ and $r^\sigma_g h(\lambda, \mu) = h(\gamma_g(\lambda, \mu)) r^\sigma_g$.

3.8 Cantor Set Bundles

We start with the geometric setting we have discussed above in Sect. 2 and we see how that gets modified when we also take into account the fractal geometry Λ_C associated to the classical code C.

We have seen that a q-ary quantum stabilizer code $Q_{\mathcal{S},\chi}$ of length n and size k identifies a commutative subalgebra $\mathcal{A}_{\mathcal{S}}$ of the endomorphism algebra $\Gamma(T^{2r}, \mathrm{End}(E_p^{\boxtimes r}))$ of a vector bundle $E_p^{\boxtimes r}$ over the torus T^{2r}, where $q = p^m$ and $r = nm$.

Proposition 3.5 *If $C \subset F_q^{2n}$ is a self-orthogonal linear code and $Q_{\mathcal{S}_C,\chi}$ the associated q-ary quantum code, the fractal Λ_C can be embedded in the disconnection $T_{\mathbb{Q}}^{2r}$. The pullback of the subbundle $\mathcal{F}_{\mathcal{S},\chi} \subset E_p^{\boxtimes r}$ to Λ_C via the projection $T_{\mathbb{Q}}^{2r} \to T^{2r}$ and its quotient by the action of $\mathcal{S}_C$ determine a fibration over a torus with fiber Λ_C.*

Proof We can pull back the bundle $E_p^{\boxtimes r}$ along the projection map $\pi : T_{\mathbb{Q}}^{2r} \to T^{2r}$ and further restrict it to Λ_C by pulling it back along the embedding $\iota : \Lambda_C \hookrightarrow T_{\mathbb{Q}}^{2r}$.

In fact, the fractal Λ_C can be realized as a subspace of the product $(T_{\mathbb{Q}}^{2})^n$, by identifying points of Λ_C, which are infinite sequences of code words $c = c_1 c_2 \ldots c_N \ldots$, with $c_i \in C \subset \mathbb{F}_q^{2n} \simeq \mathbb{F}_p^{2r}$, with points of $(T_{\mathbb{Q}}^{2})^r$, by writing each c_i as a pair of r-tuples of elements in $\mathbb{Z}/p\mathbb{Z}$, $c_i = (x_{i,1}, \ldots, x_{i,r}, y_{i,1}, \ldots, y_{i,r})$, hence identifying the pair (x_j, y_j) of sequences $x_j = x_{1,j} x_{2,j} \ldots x_{N,j} \ldots$ and $y_j = y_{1,j} y_{2,j} \ldots y_{N,j} \ldots$, $j = 1, \ldots, n$ with the p-ary expansion of a point in $T_{\mathbb{Q}}^{2}$, hence $(x, y) \in (T_{\mathbb{Q}}^{2})^n$, with $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Over Λ_C the induced vector bundle can be trivialized, so that $\iota^* \pi^* E_p^{\boxtimes r} \simeq \Lambda_C \times \mathbb{C}^{q^n}$. The subbundle $\mathcal{F}_{\mathcal{S},\chi}$ of $E_p^{\boxtimes r}$ identified by the q-ary quantum stabilizer code $Q_{\mathcal{S},\chi}$ in turn pulls back to a subbundle $\iota^* \pi^* \mathcal{F}_{\mathcal{S},\chi} \simeq \Lambda_C \times Q_{\mathcal{S},\chi}$.

We now assume that C is a self-orthogonal linear code and that $Q_{\mathcal{S},\chi}$ is the associated q-ary quantum code, under the CSS algorithm. When we take into account the action of G_C on the linear code C, we then have compatible actions

$$\begin{tikzcd}
\iota^* \pi^* E_p^{\boxtimes r} \arrow[r, "\Phi_{(v,w)}"] \arrow[d] & \iota^* \pi^* E_p^{\boxtimes r} \arrow[d] \\
\Lambda_C \arrow[r, "\gamma_{(v,w)}"] & \Lambda_C
\end{tikzcd}$$

where in the trivialization $\iota^* \pi^* E_p^{\boxtimes r} \simeq \Lambda_C \times \mathbb{C}^{q^n}$, the action on $\iota^* \pi^* E_p^{\boxtimes r}$ is given by $\Phi_{(v,w)} = (\gamma_{(v,w)}, E_{v,w})$. The action preserves the subbundle $\mathcal{F}_{\mathcal{S},\chi}$, where the induced action is through the character χ,

$$\Phi_{(v,w)} = (\gamma_{(v,w)}, \chi(E_{v,w})).$$

When taking the quotient with respect to this action, using the trivializations of the bundles, one obtains quotient spaces, respectively of the form $(\Lambda_C \times \mathbb{C}^{q^n})/\mathcal{S}_C$ and $(\Lambda_C \times Q_{\mathcal{S}_C,\chi})/\mathcal{S}_C$. These are, respectively, locally trivial fibrations over the quotients $\mathbb{C}^{q^n}/\mathcal{S}_C$ and $Q_{\mathcal{S}_C,\chi}/\mathcal{S}_C$. We focus in particular on the case of the subspace $Q_{\mathcal{S}_C,\chi}$. Because the quotient $Q_{\mathcal{S}_C,\chi}/\mathcal{S}_C$ is singular at the origin, it is preferable to remove this singular point and consider instead the quotient of $Q^*_{\mathcal{S}_C,\chi} := Q_{\mathcal{S}_C,\chi} \smallsetminus \{0\}$. The action of $\mathcal{S}_C$ is through the character χ, that is, as multiplication by $\chi(E_{v,w}) \in U(1) \subset \mathbb{C}^*$. Thus, one can further restrict to the unit vectors and obtain an action on a torus $T^{p^{nm-r}}$, with quotient still topologically a torus. The fibration then induced a fibration over this torus with fiber a fractal Λ_C. □

Variants of this construction may be useful to better take into account the dynamical properties of the action of G_C on the fractal Λ_C. We give another example below.

3.9 Crossed Product Algebras and Embeddings

One can also use the fact that the fractal Λ_C embeds inside the disconnection $T^{2r}_{\mathbb{Q}}$, in a way that is compatible with the action of G_C, to compare different crossed product algebras $C(\Lambda_C) \rtimes_\sigma G_C$ for different codes inside a common noncommutative space.

Lemma 3.6 *Let $\mathcal{A} = C(T^2_{\mathbb{Q}}) \rtimes_\sigma (\mathbb{Z}/p\mathbb{Z})^2$ be the twisted crossed product algebra of the action of $(\mathbb{Z}/p\mathbb{Z})^2$ on the disconnection $T^2_{\mathbb{Q}}$. For any classical linear code $C \subset \mathbb{F}_p^{2n}$, there is an algebra homomorphism $\mathcal{A}^{\otimes n} \to C(\Lambda_C) \rtimes_\sigma G_C$.*

Proof For $\#C = p^{2k}$, we have $G_C \simeq (\mathbb{Z}/p\mathbb{Z})^{2k}$. We regard this as a subgroup $G_C \subset (\mathbb{Z}/p\mathbb{Z})^{2n}$ of the group of translations of the whole space $\mathbb{F}_p^{2n}$, as the subgroup of translations that preserve the linear subspace C. The embedding $\Lambda_C \hookrightarrow T^{2n}_{\mathbb{Q}}$ described in Proposition 3.5 determines an algebra homomorphism $C(T^2_{\mathbb{Q}})^{\otimes n} \to C(\Lambda_C)$ given by restriction of functions to Λ_C.

We write $\alpha : G_C \hookrightarrow (\mathbb{Z}/p\mathbb{Z})^{2n}$ for the embedding as a subgroup and $\rho : C((T^2_{\mathbb{Q}})^n) \to C(\Lambda_C)$ for the restriction of functions $\rho(f)(x) = f(\iota(x))$, with $\iota : \Lambda_C \hookrightarrow (T^2_{\mathbb{Q}})^n$ the embedding of the fractal Λ_C in the disconnection $T^{2n}_{\mathbb{Q}}$. The algebra homomorphism $\rho : C(T^2_{\mathbb{Q}})^{\otimes n} \to C(\Lambda_C)$ is compatible with the action of translations, since we have $\gamma_{\alpha(a)}(\iota(x)) = \iota(\gamma_a(x))$, for all $x \in \Lambda_C$ and all $a \in G_C$. Thus, we have a morphism of the crossed product algebras $C(T^{2n}_{\mathbb{Q}}) \rtimes_\sigma (\mathbb{Z}/p\mathbb{Z})^{2n} \to C(\Lambda_C) \rtimes_\sigma G_C$. Finally, we identify $C(T^{2n}_{\mathbb{Q}}) \rtimes_\sigma (\mathbb{Z}/p\mathbb{Z})^{2n}$ with the tensor product $(C(T^2_{\mathbb{Q}}) \rtimes_\sigma (\mathbb{Z}/p\mathbb{Z})^2)^{\otimes n}$. □

The algebra homomorphisms $\mathcal{A}^{\otimes n} \to C(\Lambda_C) \rtimes_\sigma G_C$ are constructed as restriction maps, hence in terms of noncommutative spaces these correspond to embedding the noncommutative spaces associated to linear codes, whose algebras of coordinates are the $C(\Lambda_C) \rtimes_\sigma G_C$, into a common noncommutative space, whose algebra of coordinates is $\mathcal{A}^{\otimes n}$. The latter therefore can be thought of as a "universal family" for all the noncommutative spaces of linear codes $C \subset \mathbb{F}_p^{2n}$, where the total space corresponds to the "largest" code, namely $\mathbb{F}_p^{2n}$ itself, acted upon by all the translations $(\mathbb{Z}/p\mathbb{Z})^{2n}$. Moreover, the subfractals $\Lambda_{C,\ell,\pi}$ associated to linear subcodes C_π, which we discuss in the next subsection, determine further compatible specialization maps $C(\Lambda_C) \rtimes_\sigma G_C \to C(\Lambda_{C_\pi}) \rtimes_\sigma G_{C_\pi}$.

3.10 Minimum Distance, Subfractals and the Weight Polynomial

We conclude this section with an observation on how one can reinterpret the weight polynomial of a linear code in terms of subfractals of the code fractal, satisfying certain scaling (self-similarity) properties, or equivalently in terms of counting embeddings to associated Toeplitz algebras.

We first recall briefly the interpretation of the minimum distance d of a code C in terms of the fractal geometry of Λ_C, as given in [22]. Notice that here we use a slightly different notation from [22] and our Λ_C is the $\bar{S}_C$ of [22], so the statement is slightly different from the one formulated for S_C in that paper, and we write it out here explicitly for convenience.

For $\ell = 1, \dots, d$, let Π_ℓ be the set of ℓ-dimensional subspaces in $\mathbb{R}^n$ defined by intersections of $n - \ell$ hyperplanes, each of which is a translate of a coordinate hyperplane. For any given such linear space $\pi \in \Pi_\ell$, we denote by $\Lambda_{C,\ell,\pi} = \Lambda_C \cap \pi$. The geometry of this intersection varies with the choice of the linear space. When non-empty, its form changes drastically when ℓ increases. More precisely, one has the following ([22]).

Lemma 3.7 *Let $C \subset \mathfrak{A}^n$ be a code with minimum distance $d = \min\{d(x, y) \mid x \neq y \in C\}$, in the Hamming metric. For all $\ell < d$, the set $\Lambda_{C,\ell,\pi}$ has $\dim_H(\Lambda_{C,\ell,\pi}) = 0$ and is either empty or it consists of a single point, while for $\ell \geq d$ the set $\Lambda_{C,\ell,\pi}$, when non-empty, has an actual fractal structure of positive Hausdorff dimension.*

Proof The property that C has minimum distance d means that any pair of distinct points $x \neq y$ in C must have at least d coordinates that do not coincide, since $d(x, y) = \#\{i \mid x_i \neq y_i\}$. Thus, in particular, this means that no two points of the code lie on the same π, for any π as above of dimension $\ell \le d-1$, while there exist at least one π in Π_d which contains at least two points of C. In terms of the iterative construction of the fractal S_C, this means the following. For a given $\pi \in \Pi_\ell$ with $\ell \le d-1$, if the intersection $C \cap \pi$ is non-empty it must consist of a single point. Thus, when restricted to a linear space $\pi \in \Pi_\ell$ with $\ell \le d-1$, at the first step the induced construction of $\Lambda_{C,\ell,\pi}$ consists of replacing the single unit cube of dimension ℓ, $Q^\ell = Q^n \cap \pi$, with a single copy of a scaled cube of volume $q^{-\ell}$, successively iterating the same procedure. This produces a single family of nested cubes of volumes $q^{-\ell N}$ with intersection a single vertex point. The Hausdorff dimension is clearly zero. When $\ell = d$ one knows there exists a choice of $\pi \in \Pi_d$ for which $C \cap \pi$ contains at least two points. Then the induced iterative construction of the set $\Lambda_{C,\ell,\pi}$ starts by replacing the cube $Q^d = Q^n \cap \pi$ with $\#(C \cap \pi)$ copies of the same cube scaled down to have volume q^{-d}. The construction is then iterated inside all the resulting $\#(C \cap \pi)$ cubes. Thus, one obtains a set of positive Hausdorff dimension $\dim_H(\Lambda_{C,\ell,\pi})$, since we have a positive solution $s > 0$ to the scaling equation $\#(C \cap \pi) \cdot q^{-\ell s} = 1$.

Thus, as observed in [22], the parameter d of the code C can be regarded as the threshold value of ℓ where the sets $\Lambda_{C,\ell,\pi}$ jump from being trivial to being genuinely fractal objects.

For example, consider the code C of Fig. 1 and (3.1). The translates of coordinate hyperplanes intersect C in the following way: $C \cap \{x_1 = 0\} = \{(0,0,0),(0,1,1)\}$, $C \cap \{x_1 = 1\} = \{(1,0,1),(1,1,0)\}$, $C \cap \{x_2 = 0\} = \{(0,0,0),(1,0,1)\}$, $C \cap \{x_2 = 1\} = \{(0,1,1),(1,1,0)\}$, $C \cap \{x_3 = 0\} = \{(0,0,0),(1,1,0)\}$ and $C \cap \{x_3 = 1\} = \{(0,1,1),(1,0,1)\}$, so that all the corresponding $\Lambda_{C,2,\pi}$ have positive Hausdorff dimension. On the other hand, for $\ell = 1$, all the intersections of C with an intersection of two of the above hyperplanes consist of at most one point.

In the case of linear codes, the Hamming distance $d(x,y) = \#\{i \mid a_i \neq b_i\} = \#\{i \mid a_i - b_i \neq 0\} = d(x-y,0)$, so that the minimum distance is measured by $d(C) = \min\{d(x,0) \mid x \in C,\ x \neq 0\}$. The Hamming weight of $x \in C$ is the number of non-zero components of x. Thus, the minimum distance is also the minimum Hamming weight, $d(C) = \min\{w(x) \mid x \in C,\ x \neq 0\}$.

Thus, to describe the minimum distance as in Lemma 3.7, it suffices to consider those $\pi \in \Pi_\ell$ that are intersections of coordinate hyperplanes, hence $\mathbb{F}_q$-linear subspaces in $\mathbb{F}_q^n$, instead of considering also their translates. This identifies subfractals $\Lambda_{C,\ell,\pi}$ associated to $C_\pi = C \cap \pi$, where the C_π are also linear codes. We write $\Pi_\ell^0 \subset \Pi_\ell$ for the set of linear subspaces π given by intersections of ℓ coordinate hyperplanes.

In the example of (3.1), there are three such subfractals for $\ell = d = 2$, which correspond to the intersections with the three coordinate hyperplanes, $C_1 = \{(0,0,0),(0,1,1)\}$, $C_2 = \{(0,0,0),(1,0,1)\}$, and $C_3 = \{(0,0,0),(1,1,0)\}$.

The Toeplitz algebras $\mathcal{T}_C$ are functorial with respect to injective maps of sets $f : C \to C'$, with the corresponding morphism of algebras mapping $S_a \mapsto S_{f(a)}$. The Cuntz algebras are only functorial with respect to bijections.

Thus, for each set $\Lambda_{C,\ell,\pi}$ of positive Hausdorff dimension, corresponding to an intersection $C_\pi = C \cap \pi$ with $\#(C \cap \pi) > 1$, we have an injective morphism of the corresponding Toeplitz algebras $T_\pi : \mathcal{T}_{C_\pi} \to \mathcal{T}_C$ associated to the inclusion $C_\pi \subset C$. Moreover, if π and π' are two elements in Π_ℓ, with $\ell \ge d$, such that $\#C_\pi = \#C_{\pi'} > 1$, we have an isomorphism of the corresponding algebras $\mathcal{T}_{C_\pi} \simeq \mathcal{T}_{C_{\pi'}}$.

In the example of $[3,2,2]_2$ code of (3.1), the algebras $\mathcal{T}_{C_\pi}$ for all the translates of the coordinate hyperplanes $\pi \in \Pi_2$ are isomorphic, and one correspondingly has six different embeddings of this as a subalgebra of $\mathcal{T}_C$. While, if one counts only those that also correspond to linear codes, one has only three, coming from the intersections of C with the three coordinate hyperplanes, as above.

For a linear code C, one can consider the associated *weight polynomial* of the code C. We recall here briefly the definition and properties, see [2]. The basic observation is that, for a linear code, The weight polynomial is given by

$$\mathcal{A}(x,y) = \sum_{i=1}^{n} \mathcal{A}_i x^{n-i} y^i, \quad \text{with} \quad \mathcal{A}_i = \#\{x \in C \mid w(x) = i\}. \tag{3.4}$$

In the example of the code C of (3.1), the weight polynomial is $\mathcal{A}(x,y) = x^3 + 3xy^2$.

One can then easily see the following interpretation of the coefficients of the weight polynomial.

Lemma 3.8 *For a linear code C, the coefficient $\mathcal{A}_i$ of the weight polynomial $\mathcal{A}(x, y)$ is given by*

$$\mathcal{A}_i = \# \cup_{\pi \in \Pi^0_{n-i}} (C_\pi \smallsetminus \{0\}).$$

These linear subcodes C_π correspond to subfractals $\Lambda_{C,n-i,\pi}$ of Λ_C with scaling equation $\#(C \cap \pi) q^{-(n-i)s} = 1$.

Proof Any point $x \in C$ with $w(x) = i$ lies on an intersection of coordinate hyperplanes $\pi \in \Pi^0_{n-i}$. Thus, $\mathcal{A}_i$ counts the number of nonzero $x \in C$ that lie in some $\pi \in \Pi^0_{n-i}$, that is, $\mathcal{A}_i = \#\{x \neq 0 \in C \mid \exists \pi \in \Pi^0_{n-i} : x \in \pi\} = \#\{x \neq 0 \in C \mid x \in cup_{\Pi^0_{n-i}} \pi\}$. Moreover, if $w(x) = i$ so that $x \in \pi$, for some $\pi \in \Pi^0_{n-i}$, the intersection C_π is not contained in any $\pi' \in \Pi^0_{n-i-1}$, since $x \notin \pi'$, so that $\Lambda_{C,n-i,\pi}$ is obtained by scaling $\#C_\pi$ copies of the cube Q^{n-i} of volume $q^{-(n-i)}$, so that the scaling equation is as stated. □

Thus, one can view the weight polynomial of the code as a generating function for the multiplicities of the embeddings $\mathcal{T}_{C_\pi} \to \mathcal{T}_C$ for linear subcodes with $\pi \in \Pi^0_\ell$ giving rise to nontrivial subfractals.

As seen in [22] the Toeplitz algebra $\mathcal{T}_C$ and the Cuntz algebra $\mathcal{O}_C$ associated to a classical code C have representations on the Hilbert space $L^2(\Lambda_C, d\mu_H)$ and a time evolution $\sigma_t(S_a) = q^{itn} S_a$, whose critical temperature KMS state recovers integration in the Hausdorff measure of dimension $\dim_H(\Lambda_C)$ on the fractal Λ_C. The embeddings $\mathcal{T}_{C_\pi} \to \mathcal{T}_C$ therefore inherit an action on the same Hilbert space and the induced time evolution. The critical temperature KMS state for the time evolution on the subalgebra then recovers the integration in the Hausdorff measure of dimension $\dim_H(\Lambda_{C,\ell,\pi})$ on the subfractal $\Lambda_{C,\ell,\pi}$.

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