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Spectral action models of gravity on packed swiss cheese cosmology

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Abstract

We present a model of (modified) gravity on spacetimes with fractal structure based on packing of spheres, which are (Euclidean) variants of the packed swiss cheese cosmology models. As the action functional for gravity we consider the spectral action of noncommutative geometry, and we compute its expansion on a space obtained as an Apollonian packing of three-dimensional spheres inside a four-dimensional ball. Using information from the zeta function of the Dirac operator of the spectral triple, we compute the leading terms in the asymptotic expansion of the spectral action. They consist of a zeta regularization of the divergent sum of the leading terms of the spectral actions of the individual spheres in the packing. This accounts for the contribution of points 1 and 3 in the dimension spectrum (as in the case of a 3-sphere). There is an additional term coming from the residue at the additional point in the real dimension spectrum that corresponds to the packing constant, as well as a series of fluctuations coming from log-periodic oscillations, created by the points of the dimension spectrum that are off the real line. These terms detect the fractality of the residue set of the sphere packing. We show that the presence of fractality influences the shape of the slow-roll potential for inflation, obtained from the spectral action. We also discuss the effect of truncating the fractal structure at a certain scale related to the energy scale in the spectral action.

Keywords: spectral action, modified gravity, Packed swiss cheese cosmology

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1. Introduction

1.1. Fractal structures in cosmology

The usual assumptions of isotropy and homogeneity of spacetime would require that the matter distribution scales uniformly in space. Large-scale violations of homogeneity were discussed, for instance, in [47], while the idea of a fractal distribution of matter, scaling with a fractal dimension $D \neq 3$, was suggested in [46]. More recently, a growing literature based on the analysis of redshift catalogs at the level of galaxies, clusters, and superclusters has collected considerable evidence for the presence of fractality and multifractality in cosmology. We refer the reader Labini et al's survey [53] for a detailed discussion; see also [28]. While there is no complete agreement on the resulting dimensionality, partly due to difficulties in the interpretation of redshift data in estimating co-moving distances, multifractal models in cosmology have been widely studied in recent years. Cosmological models exhibiting a fractal structure can be constructed, adapting the original 'swiss cheese model' of [47]. The resulting models are usually referred to as packed swiss cheese cosmology (PSCC); see [44] for a recent detailed survey. The main idea in the construction of swiss cheese models of cosmology is to have spacetimes that are locally inhomogenous but appear globally isotropic and that satisfy everywhere the Einstein equation. In the original construction of PSCC models, in a region defined by a standard Friedmann-Robertson-Walker (FRW) cosmology, several non-overlapping spheres are inscribed, inside which the mass is contracted to a smaller higher-density region, hence creating inhomogeneities. The solution inside the ball is patched to the external FRW solution along a surface with vanishing Weyl curvature tensor (which ensures isotropy is preserved). A swiss cheese-type model based on the Tolman metric was developed in [49, 50]. In packed swiss cheese cosmology models, a configuration of such spheres is chosen so that they are tangent to each other and arranged into a higher-dimensional version of the Apollonian packing of circles [29]. In a variant of this model, see the discussion in section 8 of [44], instead of compressing the matter inside each spherical region, at each stage of the construction process the matter is expanded to lie along the spherical shell, so that one ends up with a model of gravity interacting with matter, supported on the resulting fractal. The point of view we follow in this paper is similar to the latter: we consider spacetimes that are products of a time direction and a fractal arrangement of 3-spheres (or of other spherical space forms). We develop a model of gravity on such packed swiss cheese cosmology (PSCC) models using the spectral action as an action functional for gravity.

1.2. Spectral triples

In noncommutative geometry, the formalism of spectral triples extends ordinary Riemannian (and spin) geometry to noncommutative spaces [16]. This approach encodes the metric structure in the data of a triple $ST = (\mathcal{A}, \mathcal{H}, D)$ of an involutive algebra $\mathcal{A}$ (associative, but not necessarily commutative), with a (faithful) representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by bounded operators on a Hilbert space $\mathcal{H}$, and with the additional structure of a Dirac operator D, namely an unbounded, self-adjoint operator, densely defined on $\mathcal{H}$ with the properties:

1: $(I + D^2)^{-1/2}$ is a compact operator
2: for all $a \in \mathcal{A}$, the commutators $[D, \pi(a)]$ are densely defined and extend to bounded operators on $\mathcal{H}$.

The metric dimension of a spectral triple is defined as

$$\mathfrak{d}_{ST} := \inf\{p > 0 | \mathrm{tr}(I + D^2)^{-p/2} < \infty\}. \tag{1.1}$$

A spectral triple is said to be finitely summable if $\mathfrak{d}_{ST} < \infty$.

The notion of dimension for a spectral triple is more elaborate than just the metric dimension. Indeed, a more refined notion of dimension is given by the dimension spectrum, $\Sigma_{ST} \subset \mathbb{C}$. This is a set of complex numbers, defined as the set of poles of a family of zeta functions associated with the Dirac operator of the spectral triple. In the case where $\mathrm{Ker}\, D = 0$, the zeta function of the Dirac operator is given by $\zeta_D(s) = \mathrm{Tr}(|D|^{-s})$. Let $\delta(T) = [|D|, T]$ and let $\mathcal{B}$ denote the algebra generated by the $\delta^m(\pi(a))$ and $\delta^m([D, \pi(a)])$, for all $a \in \mathcal{A}$ and $m \in \mathbb{N}$. One considers additional zeta functions of the form $\zeta_{D,a}(s) = \mathrm{Tr}(a|D|^{-s})$, for arbitrary, $a \in \mathcal{A}$, and $\zeta_{D,b}(s) = \mathrm{Tr}(b|D|^{-s})$, for arbitrary $b \in \mathcal{B}$. The dimension spectrum is the set of poles of the functions $\zeta_{D,a}(s)$ and $\zeta_{D,b}(s)$. It typically includes other points, in addition to the metric dimension, and may include real non-integer points as well as complex points off the real line. Spectral triples associated with fractals typically have non-integer and non-real points in their dimension spectram. In the following, we will use the notation $\Sigma^+_{ST} := \Sigma_{ST} \cap \mathbb{R}_+$ for the part of the dimension spectrum contained in the non-negative real line. Geometrically, the dimension spectrum represents the set of dimensions in which the space manifests itself, when viewed as a noncommutative space. Even in the case of an ordinary manifold, the dimension spectrum contains additional points, besides the usual topological dimension. The non-negative dimension spectrum Σ^+_{ST}, in particular, describes the dimensions that contribute terms to the action functional for gravity, as we discuss in more detail in section 1.3 below, while the points of the dimension spectrum that lie off the real line contribute fluctuations in the form of log oscillatory terms, as we will see in section 3.1. We say that the dimension spectrum is *simple* if the poles are simple poles. Spectral triples with simple dimension spectram are sometimes referred to as 'regular'. However, the terminology 'regular spectral triple' is often used in the literature with a different meaning, related to 'smoothness' properties (see for instance [48]). Thus, in the following we will use the terminology 'simple dimension spectrum' to avoid confusion.

A compact spin Riemannian manifold M can be described by a spectral triple $ST_M = (C^\infty(M), L^2(M, \mathbb{S}), \not{D}_M)$, by taking $\mathcal{A} = C^\infty(M)$, the algebra of smooth functions, $\mathcal{H} = L^2(M, \mathbb{S})$, the Hilbert space of square-integrable spinors, and $D = \not{D}_M$, the Dirac operator, which is a self-adjoint square root of the (negative) Laplacian of the manifold. The metric dimension of ST_M agrees with the dimension of M, by Weyl's law for the Dirac spectrum. One can also recover the geodesic distance on M from ST_M: for any two points $x, y \in M$

$$d_{\mathrm{geo}}(x, y) = \sup\{|f(x) - f(y)| \,|\, \|[D, \pi(f)]\| \leqslant 1\}.$$

reconstruction theorem [17] shows that the manifold M itself can be reconstructed from the data of a commutative spectral triple that satisfies a list of additional axioms describing properties of the geometry such as orientability, Poincaré duality, etc. The non-negative dimension spectrum $\Sigma^+_{ST_M}$ consists of non-negative integers less than or equal to $\dim(M)$ (see section 2.9 for more details).

1.3. The spectral action as a model for (modified) gravity

The formalism of spectral triples plays a crucial role in the construction of models of gravity coupled to matter based on noncommutative geometry. The main ideas underlying the construction of these models can be summarized as follows:

- The spectral action is a natural action functional for gravity on any (commutative or noncommutative) space described by a finitely summable spectral triple.
- On an ordinary manifold, the asymptotic expansion of the spectral action recovers the usual Einstein–Hilbert action of gravity, with additional modified gravity terms (Weyl conformal gravity and Gauss–Bonnet gravity).
- In the case of an 'almost commutative geometry' (locally a product $M \times F$ of an ordinary manifold M and a finite noncommutative space), the model of gravity on $M \times F$ given by the spectral action describes gravity coupled to matter on M, with the matter content (fermions and bosons) completely determined by the geometry of the finite noncommutative space F.

We refer the reader to the detailed account of the construction of such models given in [11] and in chapter 1 of [19]. For a finitely summable spectral triple, the spectral action functional [8] is defined as

$$\mathcal{S}(\Lambda) = \mathrm{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \mathrm{Spec}(D)} \mathrm{Mult}(\lambda) f(\lambda/\Lambda)$$

where f is a non-negative even smooth approximation to a cutoff function and Λ is a positive real number. As Λ grows, more rescaled eigenvalues of the form λ/Λ escape the cutoff of f and the expression grows.

In the case of a finitely summable spectral triple with a dimension spectrum consisting of simple poles on the positive real line, the spectral action can be expanded asymptotically for large Λ [8]. The asymptotic expansion relies on the Mellin transform relation between the zeta function of the Dirac operator and the heat kernel. The asymptotic expansion of the spectral action is then of the form

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\beta \in \Sigma^+_{ST}} f_\beta \, \Lambda^\beta \, {\textstyle\fint} |D|^{-\beta} \; + \; f(0)\zeta_D(0), \tag{1.2}$$

where $f_\beta = \int_0^\infty f(v) v^{\beta-1}\, \mathrm{d}v$ are the momenta of f, the summation is over the points of the non-negative dimension spectrum Σ^+_{ST}, and the coefficients are residues of the zeta function,

$${\textstyle\fint} |D|^{-\beta} = \frac{1}{2} \mathrm{Res}_{s=\beta}\, \zeta_D(s), \tag{1.3}$$

representing the noncommutative integration in dimension β.

In the case of a four-dimensional manifold M, one can write the asymptotic expansion in the form [9]

$$\mathrm{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4,$$

where the f_i are momenta of the cutoff function f, with $f_0 = f(0)$ and $f_k = \int_0^\infty f(v) v^{k-1}\, \mathrm{d}v$. Physically, the coefficients a_0, a_2 and a_4 correspond, respectively, to the cosmological term, the Einstein–Hilbert term, and the modified gravity terms (Weyl curvature and Gauss–Bonnet) of the gravity action functional. In the case of an almost-commutative geometry, the asymptotic expansion of the spectral action delivers additional bosonic terms, including Yang–Mills terms for the gauge bosons, and kinetic and interaction terms for Higgs bosons, and (non-minimal) coupling of matter to gravity (with the Higgs conformally coupled to gravity). The fermionic terms in the action functional for gravity coupled to matter come from an additional term not included in the spectral action, which accounts for the kinetic terms of the fermions and the boson-fermion interaction terms, see [11, 19]. For the purpose of the present paper, we are only interested in the gravitational terms, though couplings to matter could also be included, by taking a product of the geometries we will be discussing with a finite noncommutative geometry.

We will see in the next section that, in the case of the packed swiss cheese cosmology, the spectral action has new contributions that arise from an additional point in the dimension spectrum that reflects the fractality of the model, as well as log-periodic oscillations contributed by the points of the dimension spectrum that are off the real line.

1.4. Summary of results

The main new results in this papers are structured as follows.

In section 2.1 we obtain an estimate, in the form of an upper bound, on the exponent of convergence of the zeta function $\zeta_{\mathcal{L}}(s)$ of the length spectrum of an Apollonian packing $\mathcal{P}$ of 3-sphere (proposition 2.2); we describe the spectral triple of $\mathcal{P}$ (definition 2.3); and we compute the zeta function $\zeta_{\mathcal{D}_{\mathcal{P}}}(s)$ of the Dirac operator of the spectral triple (proposition 2.6) in terms of the zeta function of the unit 3-sphere and the zeta function $\zeta_{\mathcal{L}}(s)$ of the length spectrum. We then discuss the structure of the dimension spectrum (lemma 2.7).

In section 3, we use the results on the zeta function to obtain an expansion of the spectral action functional. In section 3.1 we discuss how the heat kernel expansion, and consequently the expansion of the spectral action, is altered by the presence of complex points of the dimension spectrum off the real line. For the case of a fractal geometry with exact self-similarity realized by a single contraction ratio, we obtain an explicit form of the log-oscillatory terms coming from the non-real points of the dimension spectrum, in the form of a Fourier series that converges to a smooth function (proposition 3.1). In section 3.2 we discuss approximations by truncation of the Fourier series of the oscillatory terms We then identify a set of four analytic conditions on the zeta function $\zeta_{\mathcal{L}}(s)$ of the length spectrum of the Apollonian packing (definition 3.3), which ensure that the spectral action has an expansion where the oscillatory terms can be approximated by a series of contributions from length spectra (fractal strings) with exact self-similarity. The contribution from the real points of the dimension spectrum yields gravitational terms as in the case of a three-dimensional geometry, with an additional term coming from the only real pole of $\zeta_{\mathcal{L}}(s)$ at its exponent of convergence (proposition 3.5). We also compute the form of the expansion of the spectral action when taking a geometry that is a product of the Apollonian arrangement $\mathcal{P}$ of 3-spheres with a compactified time axis (proposition 3.6).

In section 4 we investigate the effect on the spectral action functional of a truncation of the fractal structure at a certain energy-dependent scale. We obtain estimates on the size of the error term and its dependence on the energy Λ (propositions 4.4 and 4.6).

In section 5 we construct another model of fractal space, which allows for the presence of 'cosmic topology.' This is obtained by taking a Sierpiński fractal arrangement of spherical dodecahedra and then simultaneously closing up all of them via the action of the icosahedral group, obtaining a fractal arrangement of Poincaré homology spheres (usually referred to as dodecahedral spaces in the cosmic topology literature). This is a simpler fractal than the Apollonian sphere packing, since it has exact self-similarity with a single contraction ratio $(2 + \phi)^{-1}$, where ϕ is the golden ratio. In this case we can compute more explicitly the new terms that arise in the expansion of the spectral action, including the oscillatory terms (propositions 5.2 and 5.3 and corollary 5.4).

In section 6 we compute the effect of the additional terms in the spectral action expansion on the shape of the slow-roll potential obtained by perturbing the Dirac operator by a scalar field (propositions 6.1 and 6.2).

2. Spectral triples and zeta functions for packed swiss cheese cosmology

2.1. Apollonian packings of D-dimensional spheres

Higher-dimensional generalizations of the Apollonian packings of circles in the plane, consisting of 'packings' of $(D-1)$-dimensional hyperspheres S^{D-1} inside a D-dimensional space $\mathbb{R}^D$, have been variously studied, for instance in [26, 30, 36, 40, 43, 51]. We recall here some useful facts, following [30].

A Descartes configuration in D dimensions consists of $D+2$ mutually tangent $(D-1)$-dimensional (hyper)spheres. We write S_a^{D-1} for a sphere of radius a. The curvature $c = 1/a$ is positive for the orientation of S_a^{D-1} with an outward-pointing normal vector and negative for the opposite orientation. The curvatures of the spheres in a Descartes configuration satisfy the quadratic Soddy–Gosset relation

$$\left(\sum_{k=1}^{D+2} \frac{1}{a_k}\right)^2 = D \sum_{k=1}^{D+2} \left(\frac{1}{a_k}\right)^2. \tag{2.1}$$

This relation can be formulated in matrix terms as $c^t Q_D c = 0$, with $c = (1/a_1, \ldots, 1/a_{D+2})$ the vector of curvatures, and Q_n the quadratic form determined by the matrix

$$Q_D = I_{D+2} - D^{-1}\, 1_{D+2}\, 1_{D+2}^t,$$

where $1_{d+2}^t = (1, 1, \ldots, 1)$ and i_{d+2} is the identity matrix. the augmented curvature-center coordinates of a sphere s_a^{d-1} with center $x = (x_1, \ldots, x_d)$ in $\mathcal{R}^d$ consist of a $(d+2)$-vector

$$w = \left(\frac{\|x\|^2 - a^2}{a}, \frac{1}{a}, \frac{1}{a}x_1, \ldots, \frac{1}{a}x_D\right),$$

where the first coordinate describes the curvature of the sphere obtained from the given one by inversion in the unit sphere. The reason for the first coordinate is so that one can unambiguously extend the augmented curvature-center coordinates to include the special case of degenerate spheres with zero curvature (hyperplanes). Given a Descartes configuration of spheres, one assigns to it a $(D+2) \times (D+2)$ matrix $\mathcal{W}$ whose jth row is the vector of augmented curvature-center coordinates of the jth sphere in the configuration. The space $\mathcal{M}_D$ of all possible Descartes configuration in D dimensions is then identified with the space of all solutions $\mathcal{W}$ to the equation

$$\mathcal{W}^t\, Q_D\, \mathcal{W} = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2\, I_D \end{pmatrix}. \tag{2.2}$$

The space of solutions $\mathcal{M}_D$ is endowed with a left and a right action of the Lorentz group $O(D+1, 1)$.

The D-dimensional Apollonian group $\mathcal{G}_D$ is the group generated by the $(D+2) \times (D+2)$ matrices S_j of the form

$$S_j = I_{D+2} + \frac{2}{D-1} e_j\, 1_{D+2}^t - \frac{2D}{D-1} e_j e_j^t,$$

with I_{D+2} the identity, e_j the jth standard coordinate vector, and 1_{D+2} the vector with all coordinates equal to one.

It is shown in [30] that in dimension $D \geqslant 4$ the Apollonian group $\mathcal{G}_D$ is no longer a discrete subgroup of $\mathrm{GL}(D+2, \mathbb{R})$ and its orbits on $\mathcal{M}_D$ no longer correspond to sphere packings. However, the dual Apollonian group $\mathcal{G}_D^\perp$ is a discrete subgroup of $\mathrm{GL}(D+2, \mathbb{R})$, and the Apollonian packings of $(D-1)$-dimensional spheres we will be considering here are obtained, as in theorem 4.3 of [30], as orbits of the dual Apollonian group on $\mathcal{M}_D$. The dual Apollonian group $\mathcal{G}_D^{\perp}$ is generated by reflections $S_j^{\perp}$ of the form

$$S_j^{\perp} = I_{D+2} + 2\, 1_{D+2} e_j^t - 4\, e_j e_j^t, \tag{2.3}$$

with e_j the jth unit coordinate vector. The matrix $S_j^{\perp}$ implements inversion with respect to the jth sphere. The Apollonian packing is obtained by iteratively adding new Descartes configurations of spheres obtained from an initial configuration by iteratively applying inversions with respect to some of the spheres. When $D \neq 3$ the only relations in the dual Apollonian group $\mathcal{G}_D^{\perp}$ are $(S_j^{\perp})^2 = 1$. Thus, the spheres added at the nth iterative step of the construction of the Apollonian packing are in correspondence with all the possible reduced sequences

$$S_{j_1}^{\perp} S_{j_2}^{\perp} \cdots S_{j_n}^{\perp}, \quad j_k \neq j_{k+1}, \, \forall k,$$

acting on the point $\mathcal{W} \in \mathcal{M}_D$ that corresponds to the initial Descartes configuration. Clearly, there are $(D+2)(D+1)^{n-1}$ such sequences, hence the number of spheres in the nth level of the iterative construction is

$$N_n := \#\{S_{a_{n,k}}^{D-1}: \quad \text{fixed } n\} = (D+2)(D+1)^{n-1}.$$

In the following, we will focus on the case $D = 4$, of Apollonian packings of 3-spheres.

2.2. Lengths, packing constant, and zeta function

We proceed as in [12, 13] to associate a spectral triple to an Apollonian packing $\mathcal{P}_D$ of $(D-1)$-spheres in a D-dimensional space. As above, let

$$\mathcal{L}_D(\mathcal{P}_D) = \{a_{n,k},\, n \in \mathbb{N},\, 1 \leqslant k \leqslant (D+2)(D+1)^{n-1}\} \tag{2.4}$$

be the list (with multiplicities) of the radii $a_{n,k}$ of the $(D+2)(D+1)^{n-1}$ spheres $S_{a_{n,k}}^{D-1}$ that are added in the nth stage of the iterative construction of the packing.

The *packing constant* (or exponent of the packing) $\sigma_D(\mathcal{P}_D)$ of a packing $\mathcal{P}_D$ of $(D-1)$-spheres is defined as the exponent of convergence of the series

$$\sum_{n\in\mathbb{N}} \sum_{k=1}^{(D+2)(D+1)^{n-1}} a_{n,k}^s,$$

that is,

$$\begin{aligned}\sigma_D(\mathcal{P}_D) &= \sup\{s \in \mathbb{R}_+^* : \sum_{n\in\mathbb{N}} \sum_{k=1}^{(D+2)(D+1)^{n-1}} a_{n,k}^s = \infty\} \\ &= \inf\{s \in \mathbb{R}_+^* : \sum_{n\in\mathbb{N}} \sum_{k=1}^{(D+2)(D+1)^{n-1}} a_{n,k}^s < \infty\}.\end{aligned} \tag{2.5}$$

For $s > \sigma_D(\mathcal{P}_D)$, one defines the zeta function $\zeta_{\mathcal{L}_D}(s)$ as the sum of the series

$$\zeta_{\mathcal{L}_D}(s) = \sum_{n\in\mathbb{N}} \sum_{k=1}^{(D+2)(D+1)^{n-1}} a_{n,k}^s. \tag{2.6}$$

The zeta functions $\zeta_{\mathcal{L}_D}(s)$, like the more general zeta functions of fractal strings considered in [37], need not in general have analytic continuation to meromorphic function on the whole complex plane, but there are a *screen* $\mathcal{S}$, namely a curve of the form $S(t) + it$, with $S: \mathbb{R} \to (-\infty, \sigma_D(\mathcal{P}_D)]$, and a *window* $\mathcal{W}$ consisting of the region to the right of the screen curve $\mathcal{S}$ in the complex plane, where $\zeta_{\mathcal{L}_D}(s)$ has analytic continuation. We refer the reader to [37] for a more detailed account of screens and windows for zeta functions of fractal strings.

2.3. Packing constant and Hausdorff dimension

The residual set of an Apollonian circle packing consists of the complement of the union of all the open balls consisting of the interiors of the circles in the packing. It was shown in [4] that the packing constant σ_2, defined as in equation (2.5), is equal to the Hausdorff dimension of the residual set of the circle packing. In the higher-dimensional setting the problem of estimating the Hausdorff dimension of the residual set of a packing of $(D-1)$-dimensional spheres is much more involved, but there are some general estimates, obtained in [38] and [32].

Consider the infimum of the packing constants over all packings $\mathcal{P}_D$,

$$\sigma_D = \inf_{\mathcal{P}_D} \sigma_D(\mathcal{P}_D).$$

Assuming all the spheres $S^{D-1}_{a_{n,k}}$ in the packing are contained in the unit ball B^D, and denoting by $B^D_{a_{n,k}}$ the D-dimensional ball with $\partial B^D_{a_{n,k}} = S^{D-1}_{a_{n,k}}$, the residual set of the packing is given by

$$\mathcal{R}(\mathcal{P}_D) = B^D \backslash \cup_{n,k} B^D_{a_{n,k}}.$$

Let $\dim_H(\mathcal{R}(\mathcal{P}_D))$ denote the Hausdorff dimension of the residual set and

$$\delta_D = \inf_{\mathcal{P}_D} \dim_H(\mathcal{R}(\mathcal{P}_D))$$

denote the infimum over all packings of the Hausdorff dimensions. The upper entropy dimension $h^+(\mathcal{R}(\mathcal{P}_D))$ of the residual set $\mathcal{R}(\mathcal{P}_D)$ is defined as

$$h^+(\mathcal{R}(\mathcal{P}_D)) = \limsup_{\epsilon\to 0} -\frac{\log N_\epsilon(\mathcal{R}(\mathcal{P}_D))}{\log \epsilon},$$

where for a set X, the number $N_\epsilon(X)$ counts the smallest number of sets of diameter less than 2ϵ that cover X. The lower entropy dimension is defined similarly, with a liminf instead of limsup. It is known that the entropy dimension provides an upper bound for the Hausdorff dimension. Then we have the following estimates ([38] and [32]).

Proposition 2.1. *The radii $a_{n,k}$ of a packing $\mathcal{P}_D$ satisfy $\sum_{n,k} a^D_{n,k} = 1$ and $\sum_{n,k} a^{D-1}_{n,k} = \infty$, hence $D-1 < \sigma_D(\mathcal{P}_D) \leqslant D$. The infima satisy $\delta_D \leqslant \sigma_D$, and for individual packings $\dim_H(\mathcal{R}(\mathcal{P}_D)) \leqslant h^+(\mathcal{R}(\mathcal{P}_D)) = \sigma_D(\mathcal{P}_D)$.*

The identity $\sum_{n,k} a^D_{n,k} = 1$ follows from the packing property, namely the requirement that the residual set $\mathcal{R}(\mathcal{P}_D)$ in the D-dimensional unit ball has zero D-dimensional volume. The value $\dim_H(\mathcal{P}_D)$ is not known exactly. Some estimates have been obtained, with various methods, in [26, 43, 51]. We provide a simple rough estimate in section 2.4 below, for the specific case of 3-spheres.

2.4. Dimension estimate

Let $\mathcal{P} = \mathcal{P}_4$ be an Apollonian packing of three-dimensional spheres $S^3_{a_{n,k}}$. We compute here a rough approximation to the packing constant $\sigma_4(\mathcal{P})$ of the Apollonian packing, defined as in equation (2.5).

Proposition 2.2. *By replacing the collection of radii $\{a_{n,k}\}$ in the nth level of the Apollonian packing $\mathcal{P}$ of 3-spheres with a single value $a_n = N_n/\gamma_n$, where γ_n/N_n is the average curvature in the nth level, one obtains an approximate estimate of the packing constant,*

$$\sigma_{4,av}(\mathcal{P}) \sim 3.85193\ldots$$

Proof. As discussed above, the number of 3-spheres in the nth level of the packing $\mathcal{P}$ is given by the number of reduced sequences in the generators of the group $\mathcal{G}_n$, namely

$$N_n := \#\{S^3_{a_{n,k}}: \quad \text{fixed } n\} = (D+2)(D+1)^{n-1}|_{D=4} = 6 \cdot 5^{n-1}. \tag{2.7}$$

Let γ_n denote the sum of the curvatures of the spheres in the nth level,

$$\gamma_n = \sum_{k=1}^{6\cdot 5^{n-1}} \frac{1}{a_{n,k}}. \tag{2.8}$$

As shown in theorem 2 of [40], the generating function of the $\gamma_n = \gamma_n(s)$ is

$$G_{D=4}(u) = \frac{(1-x)(1-4x)u}{1-\frac{22}{3}x-5x^2}, \tag{2.9}$$

where $u = \gamma_0$ is the sum of the curvatures of the $D+2=6$ spheres in a Descartes configuration that gives the level-zero seed of the recursive construction. We obtain from this an estimate of the metric dimension by replacing the curvatures $1/a_{n,k}$ with their averages over levels. We denote the resulting approximation to the dimension by $\sigma_{4,av}(\mathcal{P})$. This is given by

$$\sigma_{4,av}(\mathcal{P}) = \lim_{n\to\infty} \frac{\log(6\cdot 5^{n-1})}{\log\left(\frac{\gamma_n}{6\cdot 5^{n-1}}\right)}.$$

We expand equation (2.9) in a power series. Since the specific value of u does not influence the large n behavior in the limit above, we look at the values for $u = 1$, and we obtain

$$G_{D=4} = \sum_{n=1}^{\infty} \gamma_n(1)x^n,$$

$$\gamma_n(1) = \frac{(11+\sqrt{166})^n(-64+9\sqrt{166}) + (11-\sqrt{166})^n(64+9\sqrt{166})}{3^n\cdot 10\cdot\sqrt{166}}.$$

This then gives $\sigma_{4,av}(\mathcal{P}) \sim 3.85193\ldots$ as stated. □

2.5. A spectral triple on the Cayley graph of the dual Apollonian group

Let $\mathcal{T}_D$ denote the Cayley graph of the dual Apollonian group $\mathcal{G}_D^\perp$. Since for $D \neq 3$ the group $\mathcal{G}_D^\perp$ is generated by the $D+2$ reflections $S_j^\perp$ of equaiton (2.3), with the only relations of the form $(S_j^\perp)^2 = 1$, the Cayley graph $\mathcal{T}_D$ is an infinite tree with all vertices of valence $D+2$. We endow the tree $\mathcal{T}_D$ with the structure of a finitely summable tree, in the sense of section 7 of [12], by choosing a base vertex v_0 and endowing all the $N_n = (D+2)(D+1)^{n-1}$ edges at a distance of n steps from v_0 with lengths $\ell(e_{n,k}) = a_{n,k}$, equal to the radii of the spheres in the nth level of the sphere packing. Then, as in theorem 7.10 of [12], one obtains a finitely summable spectral triple

$$ST_{\mathcal{T}_D} = (\mathcal{A}_{\mathcal{T}_D}, \mathcal{H}_{\mathcal{T}_D}, \mathcal{D}_{\mathcal{T}_D}) = \oplus_{e \in E(\mathcal{T}_D)} (\mathcal{A}_{\mathcal{T}_D}, \mathcal{H}_{\ell(e)}, D_{\ell(e)} + \frac{\pi}{2\ell(e)} I).$$

The involutive subalgebra $\mathcal{A}_{\mathcal{T}_D}$ of the C^*-algebra $C(\mathcal{T}_D)$ is determined, as in [12], by the condition that $f \in \mathcal{A}_{\mathcal{T}_D}$ has $[\mathcal{D}_{\mathcal{T}_D}, \pi(f)]$ densely defined and bounded, where $\pi\colon C(\mathcal{T}_D) \to \mathcal{B}(\mathcal{H}_{\mathcal{T}_D})$ is the representation by bounded operators on the Hilbert space of the triple. The pairs $(\mathcal{H}_{\ell(e)}, D_{\ell(e)})$ are constructed as in the 'interval spectral triple' of section 3 of [12], with $\mathcal{H}_\alpha = L^2([-\alpha, \alpha], \mu)$ with the normalized Lebesgue measure μ andD_α with eigenvectors the basis elements $e_m = \exp(i\pi m x/\alpha)$ with eigenvalue $\pi m/\alpha$. The Dirac operator $\mathcal{D}_{\mathcal{T}_D}$ then has spectrum

$$\begin{aligned} Spec(\mathcal{D}_{\mathcal{T}_D}) &= \left\{ \frac{\pi(2\,m+1)}{2\ell(e)} \colon\ e \in E(\mathcal{T}_D),\, m \in \mathbb{Z}_+ \right\} \\ &= \left\{ \frac{\pi(2\,m+1)}{2a_{n,k}} \colon\ n \in \mathbb{N},\, 1 \leqslant k \leqslant (D+2)(D+1)^{n-1},\, m \in \mathbb{Z}_+ \right\}. \end{aligned}$$

The shift $\pi/2\ell$(e)I to the Dirac operator $D_{\ell(e)}$ is introduced in [12] to avoid a kernel, so that the zeta function $\zeta_{D_{\mathcal{T}_D}}(s) = \mathrm{Tr}(|\mathcal{D}_{\mathcal{T}_D}|^{-s})$ is well defined. The zeta function of the Dirac operator of the spectral triple $ST_{\mathcal{T}_D}$ is given by

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{T}_D}|^{-s}) = \frac{2^{s+1}}{\pi^s}(1 - 2^{-s})\zeta(s)\,\zeta_{\mathcal{L}_D}(s),$$

where $\zeta(s)$ is the Riemann zeta function (see section 7.1 of [12]). The exponent of summability of the spectral triple (the metric dimension) is equal to the packing constant of equation (2.5),

$$\mathfrak{d}_{ST_{\mathcal{T}_D}} = \sigma_D.$$

2.6. The spectral triple of a sphere packing

Assume an Apollonian packing $\mathcal{P}_D$ of $(D-1)$-dimensional spheres $S^{D-1}_{a_{n,k}}$ in $\mathbb{R}^D$. We modify the construction above by introducing the contribution of the $(D-1)$-spheres $S^{D-1}_{a_{n,k}}$ of the packing, through their respective spectral triples. We replace the data $(\mathcal{H}_{\ell(e_{n,k})}, D_{\ell(e_{n,k})})$ of the construction above, for an edge $e_{n,k}$ of length $\ell(e_{n,k}) = a_{n,k}$, with new data of the form $(\mathcal{H}_{S^{D-1}_{a_{n,k}}}, \mathcal{D}_{S^{D-1}_{a_{n,k}}})$, where $\mathcal{H}_{S^{D-1}_{a_{n,k}}} = L^2(S^{D-1}_{a_{n,k}}, \mathbb{S})$ is the Hilbert space of square integrable spinors on the $(D-1)$-sphere $S^{D-1}_{a_{n,k}}$, and $\mathcal{D}_{S^{D-1}_{a_{n,k}}}$ is the Dirac operator, with spectrum

$$Spec(D_{S^{D-1}_{a_{n,k}}}) = \{\lambda_{\ell,\pm} = \pm a_{n,k}^{-1}\left(\frac{D-1}{2} + \ell\right)\colon\ \ell \in \mathbb{Z}_+\}$$

and multiplicities

$$\mathrm{Mult}(\lambda_{\ell,\pm}) = 2^{[\frac{D-1}{2}]}\binom{\ell+D}{\ell}.$$

Definition 2.3. The spectral triple of the Apollonian packing

$$\mathcal{P}_D = \{S^{D-1}_{a_{n,k}}\colon\ n \in \mathbb{N},\, 1 \leqslant k \leqslant (D+2)(D+1)^{n-1}\},$$

is given by

$$(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{\mathcal{P}_D}, \mathcal{D}_{\mathcal{P}_D}) = \oplus_{e \in \mathcal{E}(\mathcal{T}_D)} (\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{S^{D-1}_{\ell(e)}}, \mathcal{D}_{S^{D-1}_{\ell(e)}}), \tag{2.10}$$

where $\mathcal{T}_D$ is the Cayley graph of $\mathcal{G}_D^{\perp}$, as above, with edge lengths $\ell(e_{n,k}) = a_{n,k}$, and the data $(\mathcal{H}_{S^{D-1}_{a_{n,k}}}, \mathcal{D}_{S^{D-1}_{a_{n,k}}})$ are defined as above. The involutive subalgebra $\mathcal{A}_{\mathcal{P}_D}$ consists of $f \in C(\mathcal{P}_D)$ with $[\mathcal{D}_{\mathcal{P}_D}, \pi(f)]$ densely defined and bounded.

The fact that this is indeed a spectral triple follows from the general results of [12] and [13]. In particular, the spectral action of the swiss cheese cosmology model is obtained by considering the case of a packing of three-dimensional spheres,

$$\mathrm{ST}_{PSC} := (\mathcal{A}_{\mathcal{P}_4}, \mathcal{H}_{\mathcal{P}_4}, \mathcal{D}_{\mathcal{P}_4}). \tag{2.11}$$

In order to compute the spectral action for the spectral triple of a packing of 3-spheres, we first recall some facts about the spectral action of a single 3-sphere.

2.7. The spectral action on the 3-sphere

We start by recalling some very simple and well-known facts about the round sphere S^3 and its spectral action functional. We will need these in the rest of this section as building blocks to construct the spectral triple and the spectral action for the packed swiss cheese cosmology.

The Dirac operator on the 3-sphere S^3 with the round metric of unit radius has spectrum $sectionpec(D_{S^3}) = \{n + \frac{1}{2}\}$ with spectral multiplicities $\mathrm{Mult}\left(n + \frac{1}{2}\right) = n(n+1)$. Hence the spectral action takes the form

$$\mathcal{S}_{S^3}(\Lambda) = \mathrm{Tr}(f(D_{S^3}/\Lambda)) = \sum_{n \in \mathbb{Z}} n(n+1) f\left(\left(n + \frac{1}{2}\right)/\Lambda\right). \tag{2.12}$$

Lemma 2.4. *The zeta function of the Dirac operator is of the form*

$$\zeta_{D_{S^3}}(s) = 2\zeta\left(s-2, \frac{3}{2}\right) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right), \tag{2.13}$$

where $\zeta(s, q)$ is the Hurwitz zeta function. The spectral triple ST^3_S has a simple dimension spectrum, with $\Sigma^+_{ST_{S^3}} = \{1, 3\}$. The asymptotic expansion of the spectral action is correspondingly of the form

$$\mathcal{S}_{S^3}(\Lambda) \sim \Lambda^3 f_3 - \frac{1}{4}\Lambda f_1 \tag{2.14}$$

Proof. The result immediately follows by writing

$$\mathrm{Tr}(|D_{S^3}|^{-s}) = \sum_{k \geqslant 0} 2(k+1)(k+2)\left(k + \frac{3}{2}\right)^{-s} = \sum_{k \geqslant 0} 2\left(k + \frac{3}{2}\right)^{-(s-2)} - \frac{1}{2}\sum_{k \geqslant 0}\left(k + \frac{3}{2}\right)^{-s}.$$

The Hurwitz zeta function $\zeta(s, q)$ has a simple pole at $s = 1$ with residue 1, hence $\zeta_{D_{S^3}}(s)$ has simple poles at $s = 1$ and $s = 3$, with residues $\mathrm{Res}_{s=1}\zeta_{D_{S^3}}(s) = -1/2$ and $\mathrm{Res}_{s=3}\zeta_{D_{S^3}}(s) = 2$ respectively. Then, applying equation (1.2), one obtains the spectral action expansion. In the constant term we have $\zeta_{D_{S^3}}(0) = 2\zeta(-2, 3/2) - \zeta(0, 3/2)/2 = 0$, since $\zeta(-2, 3/2) = -1/4$ and $\zeta(0, 3/2) = -1$. □

Corollary 2.5. *In the case of a 3-sphere S^3_a with the round metric of radius $a > 0$, the zeta function is of the form*

$$\zeta_{D_{S^3_a}}(s) = a^s\left(2\zeta(s-2, \frac{3}{2}) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right)\right), \tag{2.15}$$

and the asymptotic expansion of the spectral action is given by

$$\mathcal{S}_{S^3_a}(\Lambda) \sim (\Lambda a)^3 f_3 - \frac{1}{4}(\Lambda a) f_1. \tag{2.16}$$

Proof. The spectrum of the Dirac operator $D_{S^3_a}$ is a scaled copy $\frac{1}{a}(\frac{1}{2}+\mathbb{Z})$ of the spectrum of $D_{S^3_1}$, and the multiplicities coincide. Thus, we have

$$\begin{aligned}\mathrm{Tr}(|D_{S^3_a}|^{-s}) &= \sum_{n=1}^{\infty} 2n(n+1)\left(\frac{n+1/2}{a}\right)^{-s} = 2a^s \sum_{n=1}^{\infty} n(n+1)(n+1/2)^{-s} \\ &= 2a^s \sum_{n=1}^{\infty}(n+1/2)^2(n+1/2)^{-s} - \frac{a^s}{2}\sum_{n=1}^{\infty}(n+1/2)^{-s} \\ &= 2a^s \sum_{n=0}^{\infty}(n+3/2)^{-(s-2)} - \frac{a^s}{2}\sum_{n=0}^{\infty}(n+3/2)^{-s}\end{aligned}$$

When $\Re(s) > 3$ (the metric dimension of the 3-sphere), this simplifies to (2.15). □

A method for non-perturbative computations of the spectral action functional based on the Poisson summation formula was developed in [9], for sufficiently regular geometries for which the Dirac spectrum and the spectral multiplicities are explicitly known. In particular, the spectral action for the round sphere S^3 was computed in [9] using this method. The computation was generalized to spherical space forms, [54], and to three-dimensional tori and Bieberbach manifolds in [41, 42, 45]. The computation for the 3-sphere in [9] can be summarized quickly as follows. Let f be a rapidly decaying even function. The eigenvalues of D^3_S form an arithmetic progression, and there is a polynomial $P(u) = u^2 - \frac{1}{4}$ that interpolates the spectral multiplicities, $\mathrm{Mult}(\lambda) = P(\lambda)$. Thus, one can write the spectral action as

$$\mathcal{S}_{S^3}(\Lambda) = \sum_{n\in\mathbb{Z}} g\left(n + \frac{1}{2}\right),$$

where $g(u) = \left(u^2 - \frac{1}{4}\right) f(u/\Lambda)$ is also a rapidly decaying function. This is then the sum of the values of a rapidly decaying function on points of a lattice, which can be evaluated using the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} g(n + \frac{1}{2}) = \sum_{n\in\mathbb{Z}} (-1)^n \hat{g}(n),$$

where

$$\hat{g}(x) = \int_{\mathbb{R}} g(u) \mathrm{e}^{-2\pi \mathrm{i} x u} du = \int_{\mathbb{R}} \left(u^2 - \frac{1}{4}\right) f(u/\Lambda) \mathrm{e}^{-2\pi \mathrm{i} x u} du$$

is the Fourier transform

$$\hat{g}(x) = \Lambda^3 \int_{\mathbb{R}} v^2 f(v) e^{-2\pi i \Lambda x v} dv - \frac{1}{4}\Lambda \int_{\mathbb{R}} f(v) e^{-2\pi i \Lambda x v} dv,$$

after substituting $u = \Lambda v$. Let $\hat{f}^{(2)}$ denote the Fourier transform of $v^2 f(v)$, in the first term above. It is shown in [9] that the sum on the Fourier-transformed side can be very accurately approximated by the term with $n = 0$, yielding for any k

$$\mathrm{Tr}(f(D/\Lambda)) = \Lambda^3 \int_{\mathbb{R}} v^2 f(v) \mathrm{d}v - \frac{1}{4}\Lambda \int_{\mathbb{R}} f(v) \mathrm{d}v + O(\Lambda^{-k}).$$

In the case of the round 3-sphere S_a^3 of radius a, we have

$$\mathrm{Tr}(f(\frac{D_{S_a^3}}{\Lambda})) = \mathrm{Tr}(f(\frac{D_{S_1^3}}{\Lambda a})),$$

and the approximation formula above extends to S_a^3, replacing Λ with Λa, so one obtains

$$\mathrm{Tr}(f(D_{S_a^3}/\Lambda)) = (\Lambda a)^3 \int_{\mathbb{R}} v^2 f(v) \mathrm{d}v - \frac{1}{4}(\Lambda a) \int_{\mathbb{R}} f(v) \mathrm{d}v + O((\Lambda a)^{-K}), \tag{2.17}$$

for arbitrary $K \in \mathbb{N}$, which agrees with the expression (2.16), with the error term as in [9].

2.8. Zeta function of a 3-sphere packing

We focus here on the case of a packing $\mathcal{P} = \mathcal{P}_4$ of 3-spheres, where at the nth iterative step in the construction one has $6 \cdot 5^{n-1}$ spheres, with radii $a_{n,k}$ with $k = 1, \ldots, 6 \cdot 5^{n-1}$, starting with an initial Descartes configuration of 6 mutually tangent 3-spheres. As above, let $\mathcal{L} = \mathcal{L}_4 = \{a_{n,k} \mid n \in \mathbb{N}, k \in \{1, \ldots, 6 \cdot 5^{n-1}\}\}$ be the length spectrum of the radii of all the 3-spheres in the packing. We consider the associated zeta function (2.6) for $D = 4$, which we denote simply by $\zeta_{\mathcal{L}}(s)$,

$$\zeta_{\mathcal{L}}(s) := \sum_{n \in \mathbb{N}} \sum_{k=1}^{6 \cdot 5^{n-1}} a_{n,k}^s. \tag{2.18}$$

Proposition 2.6. *Let $\sigma_4(\mathcal{P})$ be the packing constant of an Apollonian packing $\mathcal{P}$ of three-dimensional spheres, as in equation (2.5). For $s > \sigma_4(\mathcal{P})$, the zeta function of the Dirac operator $\mathcal{D}_{\mathcal{P}}$ of the spectral triple $\mathrm{ST}_{\mathcal{P}SC}$ of equation (2.11) is given by*

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \left(2\zeta\left(s-2, \frac{3}{2}\right) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right)\right)\zeta_{\mathcal{L}}(s), \tag{2.19}$$

where $\zeta(s, q)$ is the Hurwitz zeta function and $\zeta_{\mathcal{L}}(s)$ is as in equation (2.18).

Proof. Since $0 \notin \mathrm{Spec}(D_{S_a^3})$, $D_{S_a^3}$ is invertible and so is the Dirac operator $\mathcal{D}_{\mathcal{P}}$ for the spectral triple $\mathrm{ST}_{\mathcal{P}SC}$. The metric dimension is then given by $\inf\{\beta > 0 | \mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-\beta}) < \infty\}$, where the zeta function is given by

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n=1}^{\infty} \sum_{k=1}^{6 \cdot 5^{n-1}} \mathrm{Tr}(|D_{S_{a_{n,k}}^3}|^{-s}).$$

Each term in this sum can be computed as in equation (2.15). We can then evaluate the zeta function of the spectral triple $\mathrm{ST}_{\mathcal{P}SC}$, using the fact that the contribution of each sphere $S_{a_{n,k}}^3$ is of the form $\mathrm{Tr}(|D_{S^3_{a_{n,k}}}|^{-s}) = a^s_{n,k}\left(2\zeta\left(s-2, \frac{3}{2}\right) - \frac{1}{2}\zeta(s, \frac{3}{2})\right)$. We obtain

$$\mathrm{Tr}(|D|^{-s}) = \sum_{k=0}^{\infty} \mathrm{Tr}(|D_{S^3_{a_{n,k}}}|^{-s}) = \left(2\zeta\left(s-2, \frac{3}{2}\right) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right)\right)\sum_{k=0}^{\infty} a^s_{n,k},$$

for $s > \sigma$, with $\sigma = \max\{3, 1, \sigma_4(\mathcal{P})\} = \sigma_4(\mathcal{P})$, where $\sigma_D(\mathcal{P}_D)$, as in equation (2.5), is the packing constant of $\mathcal{P}_D$, the exponent of convergence of the series $\sum a^s_{n,k}$. We know from section 2.3 that $3 \leqslant \sigma_4 \leqslant 4$, hence $\max\{3, 1, \sigma_4(\mathcal{P})\} = \sigma_4(\mathcal{P})$. □

2.9. Dimension spectrum

The definition of dimension spectrum we are using in this paper is the same as in [19]. It is slightly different from other versions in the literature, see [21] and [33]. In particular, note that the dimension spectrum Σ_M for an ordinary smooth manifold M of dimension $n = \dim M$ is given by the set $\Sigma_M = \Sigma_M^+ = \{0, 1, 2, \cdots, n\}$, according to Example 13.8 of [20], or by $\Sigma_M = \{m \in \mathbb{Z}: m \leqslant n\}$, according to [18], p 22, and proposition A.2 of [33]. The leading terms in the asymptotic expansion of the spectral action, which correspond to the gravitational terms in the action functional, arise from the points in $\Sigma_M^+ = \Sigma_M \cap \mathbb{R}_+$, which are the same in all cases. Hence for our purposes the slight discrepancy between different versions of the notion of dimension spectrum adopted in the literature does not affect the results.

In the following, we will focus on analyzing the poles in $\mathbb{R}_+^*$ and off the real line of the zeta function $\zeta_{\mathcal{D}_\mathcal{P}}(s) = \mathrm{Tr}(|\mathcal{D}_\mathcal{P}|^{-s})$. While these poles certainly contribute points to the dimension spectrum, there may, in principle, be additional poles coming from other zeta functions $\zeta_{b,\mathcal{D}_\mathcal{P}}(s) = \mathrm{Tr}(b|\mathcal{D}_\mathcal{P}|^{-s})$, for algebra elements $b \in \mathcal{B}$ not equal to the identity. In the case of smooth manifolds, it is known (see for instance proposition A.2 of [33]) that these zeta functions do not contribute additional poles. While there is no general result for arbitrary spectral triples, in the case of the spectral triple of a fractal geometry it is often suggested that the subalgebra of 'smooth functions' should consist of functions that are supported on finitely many levels of the fractal construction (for example, in the case of a Cantor set, that would mean locally constant functions). If the fractal is built out of pieces that are smooth manifolds (as in the case of a sphere packing) then one should also require that the functions are smooth on each smooth component. While this choice of smooth subalgebra does not necessarily have, in general, the same good analytic properties as the algebra of smooth functions on a smooth manifold, it is a natural choice in this setting. In this case, the fact that the additional zeta functions $\zeta_{b,\mathcal{D}_\mathcal{P}}(s)$ do not contribute new poles can then be reduced to the known case of manifolds. When we discuss perturbations of the Dirac operator by a scalar field, to obtain a slow-roll potential for inflationary models, we will assume that the scalar fields also live in this smooth subalgebra.

The result of proposition 2.6 then shows that the dimension spectrum of the spectral triple ST_{PSC} is given by the following set.

Lemma 2.7. *The dimension spectrum Σ_{PSC} of the spectral triple ST_{PSC} consists of the union of the dimension spectrum of the 3-sphere, a single other real point $\sigma_4(\mathcal{P})$, and a countable collection of points off the real line, lying in the window $\mathcal{W}$ where $\zeta_\mathcal{L}(s)$ has analytic continuation.*

In general it is difficult to characterize precisely the positions of the poles that are off the real line, except in the case of self-similar fractals. We will discuss how to obtain some control of the contributions of these points to the expansion of the spectral action in the following section.

3. Spectral action for packed swiss cheese cosmology

In this section we use the results of the previous section on the zeta function of the Dirac operator on the packed swiss cheese cosmology in order to study how the spectral action is affected by the presence of fractality. In particular, under some restrictive assumptions on the analytic properties of the zeta function $\zeta_{\mathcal{L}}(s)$ of the Apollonian packing, and using the relation between the heat kernel and the zeta function and results on the asymptotic expansion of the heat kernel, we will obtain an expansion of the spectral action that contains the familiar gravitational terms of a three-dimensional sphere, but also has additional terms determined by the residue of the zeta function at the packing constant, and a Fourier series of additional oscillatory terms coming from fluctuations produced by the presence of poles of the zeta function located off the real line.

3.1. Zeta function, heat kernel, and spectral action on fractals

An asymptotic expansion for the spectral action, in the sense of [8], is known to exist (see theorem 1.145 of [19]) whenever there is a small-time asymptotic expansion for the heat kernel of the corresponding Dirac operator. In the case of an ordinary manifold, or an almost-commutative geometry, the heat kernel expansion is known from classical results on pseudo-differential operators. For more general spaces, like fractal geometries, there are no analogous theorems that hold with the same level of generality, although several results on the heat kernel expansion on fractals are available, see for instance the detailed survey given in [22]. For some general results about Laplacians on fractals and heat kernels we also refer the reader to [34, 52].

Specifically in relation to the asymptotic expansion of the spectral action, cases where the zeta function has poles off the real line, which contribute log-oscillatory terms to the spectral action, were studied in [23] and [24].

The main new feature that arises in the case of fractal geometries is, as we have seen in the previous section, the presence of poles of the zeta function that are off the real line. In the case of the geometry of the Apollonian packings of 3-spheres we consider in this paper, those poles correspond to the poles off the real line of the zeta function $\zeta_{\mathcal{L}}(s)$ of the length spectrum $\mathcal{L} = \mathcal{L}(\mathcal{P})$ of the packing.

As discussed in sections 1–3 of [37], for general zeta functions of fractal strings $\mathcal{L}$ the distribution of the non-real poles can be very complicated. In the best possible case, which corresponds to fractals with a self-similar structure where the contraction ratios are all integer powers of a fixed scale $0 < r < 1$ (lattice case) the non-real poles lie, periodically spaced, on finitely many vertical lines. In cases with self-similar structure, but where the contraction ratios do not satisfy the lattice condition (non-lattice case), the poles off the real line have a quasi-periodic behavior and are approximated by a sequence of lattice strings.

In the case of a length spectrum with exact self-similarity realized by a single contraction ratio r, the poles off the real line lie on the vertical line with $\Re(s) = \sigma$, which is the Hausdorff dimension, and with periodic spacings of length $\frac{2\pi}{\log(1/r)}$, namely $s = \sigma + \frac{2\pi i m}{\log(1/r)}$ with $m \in \mathbb{Z}$. We will discuss in section 5.1 an example of this kind, which is relevant to our cosmological models. In such cases with exact self-similarity, it is known (see section 4 of

[22]) that the contribution of the off-real poles to the heat-kernel asymptotic consists of a series of log-oscillatory terms We have the following model case for this situation.

Proposition 3.1. *Let* X *be a fractal geometry with a Dirac operator* D_{X} *of the associated spectral triple with the following property: the eigenvalues of* $|D_X|$ *grow exponentially like* b^{n}*, for some* $b > 1$*, and the spectral multiplicities also grow exponentially like* a^{n} *for some* $a > 1$*. Then the spectral action* $\mathcal{S}_X(\Lambda) = \mathrm{Tr}(f(D_X/\Lambda))$ *has an expansion for large* Λ *of the form*

$$\mathcal{S}_X(\Lambda) \sim \Lambda^{\sigma} \sum_{m\in\mathbb{Z}} \Lambda^{\frac{2\pi i m}{\log b}} f_{s_m} \tag{3.1}$$

where $s_m = \sigma + \frac{2\pi i m}{\log b}$ *and* $\sigma = \frac{\log a}{\log b}$*. The coefficients* f_{s_m} *are given by integrals*

$$f_{s_m} = \frac{1}{\log b}\int_0^{\infty} f(u) u^{s_m - 1} \mathrm{d}u.$$

For sufficiently rapidly decaying test functions f(u), the Fourier series $\sum_m \Lambda^{\frac{2\pi i m}{\log b}} f_{s_m}$ *converges uniformly to a smooth function* $\mathfrak{f}_{\sigma}(\theta)$ *of the circle variable* $\theta = \frac{\log\Lambda}{\log b}$ *modulo* $2\pi\mathbb{Z}$*.*

Proof. The zeta function has the form $\zeta_{D_X}(s) = \sum_n a^n b^{-sn} = (1 - ab^{-s})^{-1}$, with simple poles at $s = \frac{\log a}{\log b} + \frac{2\pi i m}{\log b}$, and with exponent of convergence $\sigma = \frac{\log a}{\log b}$. The trace of the heat kernel has the 'exponential form'

$$\mathrm{Tr}(\mathrm{e}^{-tD^2}) = \sum_n a^n \mathrm{e}^{-tb^{2n}} \tag{3.2}$$

for some constants a, b. Then it is known (see section 4.2 of [22]) that one has a small-time asymptotic of the form

$$\begin{aligned}\mathrm{Tr}(\mathrm{e}^{-tD^2}) &\sim \frac{t^{-\frac{\log a}{2\log b}}}{2\log b} \sum_{m\in\mathbb{Z}} \Gamma\left(\frac{\log a}{2\log b} + \frac{\pi i m}{\log b}\right) \exp\left(-\frac{\pi i m}{\log b}\log t\right) \\ &= \frac{1}{2\log b}\sum_m \Gamma(s_m/2) t^{-s_m/2}. \end{aligned}\tag{3.3}$$

Indeed, through the Mellin transform relation between the heat kernel and the zeta function

$$|D_X|^{-s} = \frac{1}{\Gamma(s/2)}\int_0^{\infty} \mathrm{e}^{-tD_X^2} t^{\frac{s}{2}-1}\,\mathrm{d}t,$$

this corresponds to

$$\zeta_{D_X}(s) = \mathrm{Tr}(|D_X|^{-s}) = \sum_m \frac{\Gamma(s_m/2)}{\Gamma(s/2)\cdot(s - s_m)\cdot\log b} + \text{holomorphic}$$

with poles at $s = s_m = \sigma + \frac{2\pi i m}{\log b}$ with residue $1/\log b$. To obtain an expansion for the spectral action, one proceeds as in theorem 1.145 of [19]. One considers a test function written as a Laplace transform as $k(u) = \int_0^{\infty} \mathrm{e}^{-xu} h(x)\mathrm{d}x$, so that $k(tD_X^2) = \int_0^{\infty} \mathrm{e}^{-xtD_X^2} h(x)\mathrm{d}x$. Using the expansion (3.3) one then has

$$k(tD_X^2) \sim \sum_m \frac{\Gamma(s_m/2)}{2\log b} t^{-s_m/2} \int_0^{\infty} x^{-s_m/2} h(x)\mathrm{d}x.$$

Since $\Re(s_m) = \sigma > 0$, we can write $x^{-s_m/2}$ as the Mellin transform

$$x^{-s_m/2} = \frac{1}{\Gamma(s_m/2)} \int_0^\infty \mathrm{e}^{-xv}\, v^{\frac{s_m}{2}-1}\, \mathrm{d}v,$$

and thus obtain

$$\mathrm{Tr}(k(tD_X^2)) \sim \sum_m \mathrm{Res}_{s=s_m} \zeta_{D_X}(s)\, t^{-s_m/2} \int_0^\infty k(v) v^{\frac{s_m}{2}-1} \mathrm{d}v.$$

Then taking $f(u) = k(u^2)$ we obtain

$$\int_0^\infty k(v) v^{\frac{s_m}{2}-1} \mathrm{d}v = 2 \int_0^\infty f(u) u^{s_m-1} \mathrm{d}u.$$

We then set $t = \Lambda^{-2}$ to obtain the form of the spectral action and the expansion

$$\mathcal{S}_X(\Lambda) \sim \Lambda^\sigma \sum_{m\in\mathbb{Z}} \Lambda^{\frac{2\pi i m}{\log b}} \left(\int_0^\infty f(u) u^{s_m-1} \mathrm{d}u\right) \mathrm{Res}_{s=s_m} \zeta_{D_X}(s),$$

which gives (3.1). Using the relation between the Mellin and Fourier transforms, we can rewrite the coefficients

$$f_{s_m} = \frac{1}{\log b} \int_0^\infty f(u) u^\sigma\, \mathrm{e}^{-2\pi \mathrm{i} m \frac{\log u}{\log b}} \frac{\mathrm{d}u}{u} = \int_{\mathbb{R}} F(\lambda) \mathrm{e}^{-2\pi \mathrm{i} m \lambda} d\lambda = 2\pi \hat{F}(-2\pi m),$$

where $\lambda = \frac{\log u}{\log b}$ and $F(\lambda) = f(b^\lambda) b^{\lambda\sigma}$, and $\hat{F}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}} F(\lambda) e^{i\xi\lambda} \mathrm{d}\lambda$ is the Fourier transform. Provided the test function f is sufficiently rapidly decaying, the function $F(\lambda)$ is also a rapidly decaying function (at $\lambda \to +\infty$ because of the behavior of f and at $\lambda \to -\infty$ because of the term $b^{\lambda\sigma}$). Then the Fourier transform $\hat{F}(\xi)$ is also rapidly decaying, hence the Fourier series $\sum_m \Lambda^{\frac{2\pi \mathrm{i} m}{\log b}} f_{s_m} = \sum_m f_{s_m} \mathrm{e}^{2\pi \mathrm{i} m\theta}$ converges to a smooth function $\mathfrak{f}_\sigma(\theta)$. □

More generally, in the case of exact self-similarity realized by a set of contraction ratios $\{r_1, \ldots, r_m\}$, the zeta function $\zeta_{\mathcal{L}}(s)$ has a denominator of the form $1 - \sum_j r_j^{-s}$. The exponent of convergence is the self-similarity dimension given by the real number σ satisfying the self-similarity equation $\sum_{j=1}^m r_j^{-\sigma} = 1$. If the scaling factors r_j satisfy the lattice condition, namely if the subgroup $\prod_{j=1}^m r_j^{\mathbb{Z}} \subset \mathbb{R}_+^*$ is discrete, then (see theorem 2.17 of [37]) the complex poles lie on finitely many vertical lines with $\Re(s) \leqslant \sigma$, and are periodically spaced with period $2\pi/\log(r^{-1})$, where r is the multiplicative generator of the scaling group, with $r_j = r^{n_j}$ for some integers n_j. In this lattice case, assuming all the poles are simple and there are no cancellations from a numerator of $\zeta_{\mathcal{L}}(s)$, one still obtains an asymptotic expansion of the form (3.3) with one log oscillatory series for each of the finitely many vertical lines containing the complex poles of $\zeta_{\mathcal{L}}(s)$.

In the case with exact self-similarity realized by a set of contraction ratios $\{r_1, \ldots, r_m\}$ that do *not* satisfy the lattice condition, it is no longer true that the complex poles lie on finitely many vertical lines. It is known (theorem 2.17 of [37]) that in this case there are no other poles on the line $\Re(s) = \sigma$ except the real pole $s = \sigma$, but there is a sequence of complex poles approaching the vertical line $\Re(s) = \sigma$ from the left. Moreover, all the complex poles are contained in a vertical strip $\sigma_0 \leqslant \Re(s) \leqslant \sigma$, for some $\sigma_0 \in \mathbb{R}$. Moreover, in this general non-lattice case, the complex poles can be approximated by the poles of an infinite family of lattice cases, with increasingly large oscillation periods (see section 3 of [37]), which in turn correspond to an infinite family of Fourier series of log-oscillatory terms.

Remark 3.2. In the case of the $D = 2$ Apollonian circle packings, there are known results that characterize the presence of self-similarity [15]: these packings correspond to quadratic irrationalities, via a continued fractions argument. However, analogous results for the higher-dimensional Apollonian packings, characterizing the presence of exact self-similarity, are not presently known.

3.2. Approximations and expansion

In more general situations, even for nice cases of fractal geometries with exact self-similarity, we do not have such explicit control over the oscillatory terms as in the case of proposition 3.1, where one has a single scale factor for self-similarity. In particular, in cases of self-similar geometries that do not satisfy the lattice conditions, the oscillatory terms can only be described via a sequence of approximations. Thus, we need to introduce some choices of approximations in the description of the log-oscillatory contributions to the spectral action coming from the poles of the zeta function that are off the real line.

A first very rough approximation, which we will occasionally use in the following, consists of replacing the smooth function $\mathfrak{f}_\sigma(\theta)$ in the expansion $\mathcal{S}_X(\Lambda) \sim \Lambda^\sigma\, \mathfrak{f}_\sigma(\theta(\Lambda))$ of proposition 3.1 with its average value on the circle. This corresponds to selecting only the zero-order Fourier coefficient

$$\frac{1}{2\pi}\int_0^{2\pi} \mathfrak{f}_\sigma(\theta)\mathrm{d}\theta = f_\sigma = \frac{1}{\log(b)}\int_0^\infty f(u)u^{\sigma-1}\mathrm{d}u = \mathrm{Res}_{s=\sigma}\zeta_{D_X}(s)\cdot\int_0^\infty f(u)u^{\sigma-1}\mathrm{d}u.$$

This corresponds to only counting the contribution of the pole $s = \sigma$ on the real line and neglecting the contributions of the poles that lie off the real line.

In a similar way, one can decide to approximate the function $\mathfrak{f}_\sigma(\theta(\Lambda))$ by truncating the Fourier series at a higher order. In the case of a fractal geometry with the non-lattice property, where there is an infinite sequence of lattice approximations (see section 3 of [37]) to the off-real poles of the zeta function, these give rise to terms with increasingly long oscillation periods in the expansion of the spectral action. One can then choose to truncate the Fourier series at some fixed size $M = m/\log b$, so that oscillatory series with longer oscillation periods are truncated earlier and contribute less to the approximation.

Note that truncating the Fourier series in the spectral action expansion at some size $M = m/\log b$ can also be seen as truncating the series of oscillatory terms in the heat kernel expansion (3.3). The size of these terms is determined by the size of the value of the Gamma function $\Gamma\left(\frac{\log a}{2\log b} + \frac{\pi i m}{\log b}\right)$. The Gamma function decays exponentially fast along the vertical line $\Re(s) = \frac{\log a}{2\log b}$, so these oscillatory terms in the heat kernel expansion rapidly become very small in comparison to the contribution of the $m = 0$ term.

In all of these cases, when we introduce approximations to the oscillatory terms, the approximation we obtain for the spectral action is no longer really an asymptotic expansion in the sense of [31]. The usual meaning of asymptotic expansion implies that the function can be approximated around some value of the argument (or a limit value) up to arbitraril high order. For the purpose of building gravitational models, it will suffice to obtain an expansion of the spectral action up to order Λ^0 (including the oscillatory terms), and some sufficiently good approximation in cases where the oscillatory terms cannot be fully computed explicitly. For this reason, in the following we will simply use the terminology ‘expansion’ of the spectral action, rather than insisting on the stronger properties of a genuine asymptotic expansion.

3.3. Some analytic assumptions

As we pointed out in remark 3.2, unlike the Apollonian circles case, in dimension $D = 4$ we do not have a characterization of the presence of exact self-similarity in the sphere packing. However, in order to obtain a reasonably behaved model, with respect to the properties of the zeta function and the spectral action functional, we restrict our attention to a subset of all the possible Apollonian packings, identified by a set of requirements on the properties of the associated zeta function $\zeta_{\mathcal{L}}(s)$.

Definition 3.3. A packing $\mathcal{P}$ of three-dimensional spheres is *analytic* if it satisfies the following properties:

(1) The zeta function $\zeta_{\mathcal{L}}(s)$ of the packing $\mathcal{P}$ has analytic continuation to a meromorphic function on a region of the complex plane that contains the non-negative real axis.
(2) The analytic continuation $\zeta_{\mathcal{L}}(s)$ has only one pole on the non-negative real axis, located at $s = \sigma_4(\mathcal{P})$.
(3) The poles of $\zeta_{\mathcal{L}}(s)$ are simple.
(4) There is a family $\mathcal{L}_n$, $n \in \mathbb{N}$, of self-similar fractal strings with the lattice property, and with increasingly large periods, such that the complex poles of $\zeta_{\mathcal{L}}(s)$ are approximated by the complex poles of $\zeta_{\mathcal{L}_n}(s)$.

In terms of screens and windows, as in [37], the first condition above consists of the property that the screen function $S: \mathbb{R} \to (-\infty, \sigma_4(\mathcal{P})]$ satisfies $S(0) < 0$.

In the last condition, the period of a self-similar fractal string $\mathcal{L}_n$ with the lattice property is the length $\pi_n := \frac{2\pi}{-\log r_n}$ with the property that all the poles of $\zeta_{\mathcal{L}_n}(s)$ off the real line lie on finitely may vertical lines $\Re(s) = \sigma_j$ with periodic spacing by $\frac{2\pi}{-\log r_n}$. The approximation condition means that, for all $\epsilon > 0$, there exist an $n \in \mathbb{N}$ and an $R = R(\epsilon, n) > 0$, such that, within a vertical region of size at most R, the complex poles of $\zeta_{\mathcal{L}}(s)$ are within distance ϵ of the poles of $\zeta_{\mathcal{L}_n}(s)$. For more details see section 3.4.1 of [37], and see figure 3.6 of [37] for an explicit example of such an approximation.

3.4. Heuristics of analytic assumptions

At present, we do not have a characterization of the locus of packings satisfying the constraints listed in definition 3.3 (for example, in terms of a geometric locus in the configuration space $\mathcal{M}_D$ of Descartes configurations). We can, however, provide some heuristic explanation for the geometric meaning of the requirement that the zeta function $\zeta_{\mathcal{L}_D}(s)$ of the length spectrum $\mathcal{L}_D = \{a_{n,k}\}$ of an Apollonian packing $\mathcal{P}_D$ of $(D-1)$-dimensional spheres satisfies these properties.

Consider the possibility that a sphere packing has exact self-similarity. This would mean that there is a finite set $\{r_1, \ldots, r_m\}$ of scaling ratios, with the property that, for all n, k, the radii $a_{n,k} \in \mathbb{R}_+^*$ of the packing belong to the subgroup $\prod_{j=1}^m r_j^{\mathbb{Z}} \subset \mathbb{R}_+^*$. This subgroup will, in general, be dense in $\mathbb{R}_+^*$ (non-lattice case). In such cases, for the zeta function of (2.19), the factor $\zeta_{\mathcal{L}}(s)$ would have analytic continuation to a meromorphic function on all of $\mathbb{C}$ (see theorem 2.4 of [37]), hence the first condition of definition 3.3 would certainly be satisfied. Moreover, in such a case, the second condition would be satisfied by theorem 2.17 of [37]. The third condition would be satisfied, at least in the general case (again by theorem 2.17 of [37]). The last condition is obvious in the lattice case, and is a consequence of the approximation result of section 3 of [37] in the non-lattice case. In the non-lattice self-similar case, the self-similar strings $\mathcal{L}_n$ with the lattice conditions are constructed using Diophantine approximation (lemma 3.16 and theorem 3.18 of [37]).

Thus, one should think of the conditions of definition 3.3 as a generalization of the good conditions satisfied by the zeta function of a fractal with exact self-similarity. One can expect that they may be fulfilled by especially regular (especially symmetric) choices of Descartes configuration, although we leave a more precise mathematical investigation of this question to future work.

3.5. Spectral action expansion

For the rest of this section we make the assumptions that the packing $\mathcal{P}$ of three-dimensional spheres we are considering satisfies the three conditions listed in definition 3.3.

Under the assumptions of definition 3.3, the result of lemma 2.7 on the dimension spectrum $\Sigma_{ST_{PSC}}$ can be refined to the following form.

Proposition 3.4. *For a packing $\mathcal{P}$ of 3-spheres satisfying the properties of definition 3.3, the non-negative dimension spectrum of the spectral triple of the PSCC consists of the points*

$$\Sigma^+_{ST_{PSC}} = \{1, 3, \sigma_4(\mathcal{P})\}, \tag{3.4}$$

when the metric dimension of the spectral triple is $\mathfrak{d}_{PSC} = \sigma_4(\mathcal{P})$. The spectral triple has a simple dimension spectrum, which, in addition to the points of $\Sigma^+_{ST_{PSC}}$ on the real line, contains a countable family of points off the real line, contained in a horizontally bounded strip $\sigma_{min} \leqslant \Re(s) \leqslant \sigma_4(\mathcal{P})$, approximated by the non-real poles of a family $\mathcal{L}_n$ of self-similar fractal strings with the lattice property and with increasingly large oscillation periods.

Proof. Under the assumptions that the zeta function $\zeta_{\mathcal{L}}(s)$ has analytic continuation to a window including the positive real axis, we see that the zeta function $\zeta_{\mathcal{D}_\mathcal{P}}(s) = \mathrm{Tr}(|\mathcal{D}_\mathcal{P}|^{-s})$ of the Dirac operator of the spectral triple ST_{PSC} also has analytic continuation to a meromorphic function in the same region. Moreover, the assumption that the points $\{1, 3\}$ are not poles of $\zeta_{\mathcal{L}}(s)$ ensures that $\zeta_{\mathcal{D}_\mathcal{P}}(s)$ has simple poles at these points. It also has a simple pole at $s = \sigma_4(\mathcal{P})$ and at all the poles off the real line by the third assumption. Thus, the spectral triple has a simple dimension spectrum and the non-negative part of the dimension spectrum is given by (3.4). The last property about the poles off the real line follows from the last property of definition 3.3 and theorem 2.17 of [37]. □

Proposition 3.5. *Let $\mathcal{P}$ be a packing for which the assumptions listed above hold. Then the expansion of the spectral action for the spectral triple $ST_{PSC}(\mathcal{P})$ is of the form*

$$\mathrm{Tr}(f(\mathcal{D}_\mathcal{P}/\Lambda)) \sim \Lambda^3\,\zeta_{\mathcal{L}}(3) f_3 - \Lambda\,\frac{1}{4}\,\zeta_{\mathcal{L}}(1) f_1 + \Lambda^\sigma \left(\zeta(\sigma-2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2})\right)\mathcal{R}_\sigma f_\sigma + \mathcal{S}^{osc}_{\mathcal{P}}(\Lambda), \tag{3.5}$$

where $\sigma = \sigma_4(\mathcal{P})$ the packing constant, $\mathcal{R}_\sigma = \mathrm{Res}_{s=\sigma}\zeta_{\mathcal{L}}(s)$ the residue of the zeta function of the fractal string $\mathcal{L} = \mathcal{L}(\mathcal{P})$, and $f_\beta = \int_0^\infty v^{\beta-1} f(v)\mathrm{d}v$, the momenta of the test function, and $\mathcal{S}^{osc}_{\mathcal{P}}(\Lambda)$ is an oscillatory term involving the contributions of the points of the dimension spectrum that are off the real line. For $R > 0$, let $\mathcal{S}^{osc}_{\mathcal{P}}(\Lambda)_{\leqslant R}$ be the truncation of the oscillatory terms that only counts the contribution of the off-real poles with $|\Im(s)| \leqslant R$. Then the oscillatory term $\mathcal{S}^{osc}_{\mathcal{P}}(\Lambda)$ can be approximated by a sequence

$$\mathcal{S}_{\mathcal{P}}^{osc}(\Lambda)_{\leqslant R} \sim \sum_{j=0}^{N_n} \Lambda^{\sigma_{n,j}} \mathfrak{f}_{\sigma_{n,j}}(\theta_n(\Lambda)), \tag{3.6}$$

where $n \to \infty$ as $R \to \infty$, and where $\sigma_{n,j} = \Re(s_{n,j,m})$, for

$$\{s_{n,j,m} = \sigma_{n,j} + i(\alpha_{n,j} + \frac{2\pi m}{\log b_n})\}_{j=0,\dots,N_n, m\in\mathbb{Z}}$$

the set of non-real poles of the zeta functions $\zeta_{\mathcal{L}_n}(s)$, with $\sigma_{min} \leqslant \sigma_{n,j} \leqslant \sigma$ and periods $2\pi/\log b_n \to \infty$ and $n \to \infty$. The $\mathfrak{f}_{\sigma_{n,j}}(\theta_n(\Lambda))$ are smooth functions of the circle variable $\theta_n = \frac{\log\lambda}{\log b_n}$, with Fourier expansion $\mathfrak{f}_{\sigma_{n,j}}(\theta_n) = \sum_m f_{s_{n,m}} \mathrm{e}^{2\pi \mathrm{i} m\theta_n}$ with

$$f_{s_{n,j,m}} = (\zeta(s_{n,j,m} - 2, \frac{3}{2}) - \frac{1}{4}\zeta(s_{n,j,m}, \frac{3}{2}))\, \mathrm{Res}_{s=s_{n,j,m}} \zeta_{\mathcal{L}_n}(s) \int_0^\infty f(u)\, u^{s_{n,j,m}-1}\, \mathrm{d}u.$$

Proof. Under the three assumptions listed above on the zeta function $\zeta_{\mathcal{L}}(s)$, the residues at the points $s = 1$ and $s = 3$ of the dimension spectrum are given, respectively, by

$$\mathrm{Res}_{s=1}\zeta_{\mathcal{D}_\mathcal{P}}(s) = \frac{-1}{2}\mathrm{Res}_{s=1}\zeta(s, \frac{3}{2}) \cdot \zeta_{\mathcal{L}}(s) = -\frac{1}{2}\,\zeta_{\mathcal{L}}(1)$$

$$\mathrm{Res}_{s=3}\zeta_{\mathcal{D}_\mathcal{P}}(s) = 2\mathrm{Res}_{s=3}\zeta(s-2, \frac{3}{2}) \cdot \zeta_{\mathcal{L}}(s) = 2\,\zeta_{\mathcal{L}}(3).$$

Thus, the terms in the expansion of the spectral action

$$\mathrm{Tr}(f(\mathcal{D}_\mathcal{P}/\Lambda)) \sim \sum_{\beta\in\Sigma^+_{\mathrm{ST}_{PSC}}} f_\beta \Lambda^\beta ⨍|\mathcal{D}_\mathcal{P}|^{-\beta},$$

with $⨍|\mathcal{D}_\mathcal{P}|^{-\beta}$ the residues, as in equation (1.3), are given by

$$\begin{aligned}\mathrm{Tr}(f(\mathcal{D}_\mathcal{P}/\Lambda)) \sim{}& \Lambda^3\,\zeta_{\mathcal{L}}(3)\int_0^\infty v^2 f(v)\mathrm{d}v \\ &- \Lambda\,\frac{1}{4}\,\zeta_{\mathcal{L}}(1)\int_0^\infty f(v)\,\mathrm{d}v \\ &+ \Lambda^\sigma\,(\zeta(\sigma-2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2}))\mathrm{Res}_{s=\sigma}\zeta_{\mathcal{L}}(s)\int_0^\infty v^{\sigma-1}f(v)\mathrm{d}v,\end{aligned}$$

where $\sigma = \sigma_4(\mathcal{P})$. The approximate form of the oscillatory term is derived from the last property of definition 3.3 and from the form of the oscillatory terms of proposition 3.1. In the case of a self-similar string $\mathcal{L}_n$ with the lattice property, the poles of $\zeta_{\mathcal{L}_n}(s)$ off the real line consist of a finite union of sequences of the form $s_{n,j,m} = \sigma_{n,j} + i\left(\alpha_{n,j} + \frac{2\pi m}{\log b_n}\right)$, with the same period $\frac{2\pi i m}{\log b_n}$ and with $\alpha_{n,0} = 0$ and $\alpha_{n,j} \neq 0$ for $j > 0$, and with $\sigma_{n,0} = \sigma_n$ the self-similarity dimension, as shown in theorem 2.17 of [37]. The terms in the expansion of the spectral action that correspond to these poles are then approximated, for large n, by a finite sum of terms as in proposition 3.1. □

For the purpose of this paper we will not give a more detailed analysis of the convergence of the approximation by the sequence $\sum_j \Lambda^{\sigma_{n,j}} \mathfrak{f}_{\sigma_{n,j}}(\theta_n(\Lambda))$. A more precise analytic discussion of the nature of the approximation in (3.6) will require a more detailed understanding of self-similar structures in higher-dimensional Apollonian sphere packings than is presently available, and will need to be addressed elsewhere. In terms of the expansion of the spectral action we are going to use in explicit gravitational models, we will truncate the series of oscillatory terms as discussed in section 3.2.

3.6. Zeta regularization

The expression (3.5) for the spectral action of the sphere packing should be regarded as a 'zeta regularized' form of the divergent series

$$\mathcal{S}_{\mathcal{P}}(\Lambda) = \sum_{k=0}^{\infty} \mathcal{S}_{S^3_{a_k}}(\Lambda)$$

that adds the contributions coming from the spectral actions of the individual spheres in the packing. Indeed, since the spectral action of an individual sphere is of the form

$$\mathcal{S}_{S^3_{a_{n,k}}}(\Lambda) = \Lambda^3\, a_{n,k}^3\, f_3 - \frac{1}{4}\Lambda\, a_{n,k}\, f_1 + O((\Lambda a_{n,k})^{-K}), \tag{3.7}$$

and points 1 and 3 are both smaller than the exponent of convergence $\sigma_4(\mathcal{P})$ of the series $\sum_{n,k} a_{n,k}^s$, the series

$$\Lambda^3 f_3 \sum_{n,k} a_{n,k}^3 - \frac{1}{4}\Lambda f_1 \sum_{n,k} a_{n,k} \tag{3.8}$$

is divergent and requires a suitable regularization. The spectral action (3.5) can be interpreted as such a regularization. Notice also that the error term $O((\Lambda a_{n,k})^{-K})$ is very small for a fixed radius $a_{n,k}$ and for sufficiently large Λ, but when the radii $a_{n,k}$ vary over the set $\mathcal{L}(\mathcal{P})$ of lengths of the packing $\mathcal{P}$ it becomes large for any given Λ, so that equation (3.7) cannot be extended directly to the whole packing. The term

$$\Lambda^3 f_3\, \zeta_{\mathcal{L}}(3) - \frac{1}{4}\Lambda f_1\, \zeta_{\mathcal{L}}(1) \tag{3.9}$$

in (3.5) is just a classical form of zeta regularization of the series (3.8), with the divergent $\sum a_{n,k}^3$ replaced by $\zeta_{\mathcal{L}}(3)$ and the divergent $\sum a_{n,k}$ replaced by $\zeta_{\mathcal{L}}(1)$. The additional term in (3.5), which depends on the residue of $\zeta_{\mathcal{L}}(s)$ at $s = \sigma_4(\mathcal{P})$, detects the presence of a fractal structure in the geometry. We discuss these issues further in section 4 below.

3.7. PSCC spectral action

In the previous subsection we computed the expansion of the spectral action for an Apollonian packing of 3-spheres, under some assumptions on the behavior of the associated zeta function. Here we consider an associated (Euclidean) spacetime model. This generalizes to the case of a packing of spheres the simpler case of a single sphere S^3, where the associated spacetime is just $\mathbb{R} \times S^3$, with the Euclidean time line $\mathbb{R}$ compactified to a circle S^1 of size β. We generalize the form of the spectral action of $S^1_\beta \times S^3_a$, by replacing the 3-sphere S^3_a with a packing $\mathcal{P}$ of 3-spheres $S^3_{a_{n,k}}$, and using the results in the previous section.

Proposition 3.6. *Let $\mathcal{P}$ be a packing of 3-spheres satisfying the three conditions of definition 3.3. Consider the product geometry $S^1_\beta \times \mathcal{P}$ of $\mathcal{P}$ with a circle of size β. Then the spectral action has expansion with leading terms of the form*

$$\mathcal{S}_{S^1_\beta\times\mathcal{P}}(\Lambda)\sim 2\beta\Bigg(\Lambda^4\,\zeta_{\mathcal{L}}(3)\mathfrak{h}_3\;-\;\Lambda^2\,\frac{1}{4}\,\zeta_{\mathcal{L}}(1)\,\mathfrak{h}_1$$
$$+\;\Lambda^{\sigma+1}\Big(\zeta(\sigma-2,\,\frac{3}{2})-\frac{1}{4}\zeta(\sigma,\,\frac{3}{2})\Big)\mathcal{R}_\sigma\,\mathfrak{h}_\sigma\Bigg)$$
$$+\;\mathcal{S}_{S^1_\beta\times\mathcal{P}}(\Lambda)^{osc} \tag{3.10}$$

where $\sigma=\sigma_4(\mathcal{P})$ is the packing constant (2.5), $\mathcal{R}_\sigma$ is the residue of $\zeta_{\mathcal{L}}(s)$ at $s=\sigma$, and

$$\mathfrak{h}_3\;:=\;\pi\int_0^\infty\;h(\rho^2)\rho^3\mathrm{d}\rho,\quad \mathfrak{h}_1\;:=\;2\pi\int_0^\infty\;h(\rho^2)\rho\mathrm{d}\rho, \tag{3.11}$$

$$\mathfrak{h}_\sigma\;=\;2\int_0^\infty\;h(\rho^2)\rho^\sigma\mathrm{d}\rho. \tag{3.12}$$

The oscillatory contributions from poles off the real line are approximated by a sequence

$$\mathcal{S}_{S^1_\beta\times\mathcal{P}}(\Lambda)^{osc}_{\leqslant R}\sim\sum_{j=0}^{N_n}\Lambda^{\sigma_{n,j}+1}\mathfrak{g}_{\sigma_{n,j}}(\theta_n(\Lambda)), \tag{3.13}$$

with $n\to\infty$ when $R\to\infty$, where $\sigma_{n,j}=\Re(s_{n,j,m})$, with $s_{n,j,m}$ the non-real poles of $\zeta_{\mathcal{L}_n}(s)$ and $\mathfrak{g}_{\sigma_{n,j}}(\theta_n(\Lambda))$ smooth functions with Fourier coefficients

$$(\zeta(s_{n,j,m}-2,\,\frac{3}{2})-\frac{1}{4}\zeta(s_{n,j,m},\,\frac{3}{2}))\mathrm{Res}_{s=s_{n,j,m}}\zeta_{\mathcal{L}_n}(s)\mathfrak{h}_{s_{n,j,m}},$$

with $\mathfrak{h}_{s_{n,j,m}}$ defined as in (3.12).

Proof. As observed in lemma 2 of [9], the spectral action for $S^1_\beta\times S^3_a$, with the Dirac operator

$$D_{S^1_\beta\times S^3_a}=\begin{pmatrix}0 & D_{S^3_a}\otimes 1+i\otimes D_{S^1_\beta}\\ D_{S^3_a}\otimes 1-i\otimes D_{S^1_\beta} & 0\end{pmatrix}$$

is of the form

$$\mathrm{Tr}(h(D^2_{S^1_\beta\times S^3_a}/\Lambda))\sim 2\beta\Lambda\mathrm{Tr}(\kappa(D^2_{S^3_a}/\Lambda)), \tag{3.14}$$

for a test function $h(x)$, and with the test function κ on the right-hand side satisfying $\kappa(x^2)=\int_{\mathbb{R}}h(x^2+y^2)\mathrm{d}y$. It then follows that the expansion of the spectral action on $S^1_\beta\times S^3_a$ is given by (see theorem 3 of [9])

$$\mathrm{Tr}(h(D^2_{S^1_\beta\times S^3_a}/\Lambda))\sim 2\,\beta\Big(\Lambda^4\,a^3\,\mathfrak{h}_3\;-\;\frac{1}{4}\,\Lambda^2\,a\,\mathfrak{h}_1\Big),$$

with the notation of (3.11). We now consider a similar situation, with the product geometry $S^1_\beta\times S^3_a$ replaced by $S^1_\beta\times\mathcal{P}$, where $\mathcal{P}$ is a packing of 3-spheres satisfying the conditions of definition 3.3. The Dirac operator of the product geometry is again of the form

$$D_{S^1_\beta\times\mathcal{P}}=\begin{pmatrix}0 & \mathcal{D}_{\mathcal{P}}\otimes 1+i\otimes D_{S^1_\beta}\\ \mathcal{D}_{\mathcal{P}}\otimes 1-i\otimes D_{S^1_\beta} & 0\end{pmatrix},$$

where $\mathcal{D}_{\mathcal{P}}$ is the Dirac operator of the spectral triple ST_{PSC} described in definition 2.3. The same argument as in lemma 2 of [9] shows that, as in equation (3.14)

$$\mathrm{Tr}(h(D^2_{S^1_\beta\times\mathcal{P}}/\Lambda)) \sim 2\beta\Lambda\mathrm{Tr}(\kappa(\mathcal{D}^2_{\mathcal{P}}/\Lambda)),$$

with the test functions h and κ as above. Using the result of proposition 3.5 we then obtain, as above, the expression (3.10), with $\mathfrak{h}_\sigma$ given by

$$\mathfrak{h}_\sigma = \int_{\mathbb{R}_+\times\mathbb{R}} x^{\sigma-1}\, h(x^2+y^2)\mathrm{d}x\,\mathrm{d}y = \int_0^\infty h(\rho^2)\rho^\sigma \mathrm{d}\rho \int_{-\pi/2}^{\pi/2} \cos(\theta)\,\mathrm{d}\theta$$
$$= 2\int_0^\infty h(\rho^2)\rho^\sigma \mathrm{d}\rho.$$

The structure of the oscillatory terms is obtained as in the previous Proposition. □

In cosmological models based on the spectral action (see [41, 42]), the parameter β is an artifact introduced by the choice of a compactification of the Euclidean time coordinate along a circle of size β. As discussed in section 3.1 of [42], the parameter β can be interpreted as an inverse temperature and related to the temperature of the cosmological horizon.

4. Fractality scale truncation

A realistic model of fractal structures in cosmology will necessarily involve a choice of scale at which fractality is cut off: while the Universe may involve a fractal structure at the scale of Galaxy superclusters and clusters, it does not appear fractal at our scales, hence the self-similarity property is expected to break down at some level. In a gravity model based on the spectral action, which already naturally involves a dependence on an energy scale Λ, it is natural to assume that the scale at which fractality breaks down will be in some way dependent on Λ. In the construction of the spectral triple of the PSCC model, discussed in section 2.6 above, we obtained a spectral action functional as a suitable kind of 'zeta regularization' of the divergent series

$$\sum_{n=0}^\infty\sum_{k=1}^{N_n} \mathcal{S}_{S^3_{a_{n,k}}}(\Lambda),$$

where the sum is over all the 3-spheres in the packing $\mathcal{P}$, with $a_{n,k}$ their radii, and with $N_n = 6\cdot 5^{n-1}$, the number of spheres in the nth level of the packing construction. Indeed, as we have seen in the previous section, the spectral action $\mathcal{S}_{\mathcal{P}}(\Lambda)$ involves a zeta regularization of the above series, given by

$$\left(\Lambda^3 f_3 \sum_{n,k} a^3_{n,k} - \frac{1}{4}\Lambda f_1 \sum_{n,k} a_{n,k}\right)^{reg} = \Lambda^3 f_3\ \zeta_{\mathcal{L}}(3) - \frac{1}{4}\Lambda f_1\ \zeta_{\mathcal{L}}(1)$$

and an additional term

$$\Lambda^\sigma\left(\zeta\left(\sigma-2, \frac{3}{2}\right) - \frac{1}{4}\zeta\left(\sigma, \frac{3}{2}\right)\right)\mathcal{R}_\sigma f_\sigma$$

involving the residue $\mathcal{R}_\sigma = \mathrm{Res}_{s=\sigma}\zeta_{\mathcal{L}}(s)$ at $\sigma = \sigma_4(\mathcal{P})$, which describes the fractality of the Apollonian packing.

4.1. Sphere counting function

In a model where fractality is truncated at a certain scale, one only considers the sphere packing $\mathcal{P}$ up to a certain size. This requires estimating the number

$$\mathcal{N}_\alpha(\mathcal{P}) = \#\{S^3_{a_{n,k}} \in \mathcal{P}\colon\ a_{n,k} \geqslant \alpha\} \tag{4.1}$$

of spheres in the given packing whose radii are of size at least α. In the case of Apollonian packings of circles and of 2-spheres it is known, by a result of [5], that the Hausdorff dimension of the residual set of the packing is equal to

$$\dim_H(\mathcal{R}(\mathcal{P})) = \lim_{\alpha\to 0} -\frac{\log \mathcal{N}_\alpha(\mathcal{P})}{\log \alpha},$$

so that, for $\alpha \to 0$, one has $\mathcal{N}_\alpha(\mathcal{P}) \sim \alpha^{-\dim_H(\mathcal{R}(\mathcal{P}))+o(1)}$, see also [3]. The stronger result $\mathcal{N}_\alpha(\mathcal{P}) \sim c_{\mathcal{P}}\alpha^{-\dim_H(\mathcal{R}(\mathcal{P}))}$ was proved in [35]. A general heuristic argument for the existence of a power law governing the behavior of the sphere-counting function for sphere packings in arbitrary dimension is given in [1]. Let $\delta(\mathcal{P})$ denote the exponent of the power law, so that, for $\alpha \to 0$

$$\mathcal{N}_\alpha(\mathcal{P}) \sim \alpha^{-\delta(\mathcal{P})+o(1)}. \tag{4.2}$$

In fact, the result of [5] shows, in the case of an Apollonian packing of circles, that $\delta(\mathcal{P})$ is equal to the packing constant $\sigma_2(\mathcal{P})$, which combined with the result of [4] then gives the identification with the Hausdorff dimension. The general argument of section 2 of [5] is independent of the dimension, and it shows that, in general, one has the estimate

$$\limsup_{\alpha\to 0}\ -\frac{\log \mathcal{N}_\alpha(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}). \tag{4.3}$$

Thus, if the sequence has a limit, the limit has to be the packing constant $\sigma_D(\mathcal{P})$.

4.2. Spectral triple with truncation of fractality scale

Thus, in a cosmological model where fractality is truncated at a certain size α, one would consider a spectral triple of the form

$$(\mathcal{A}_{\mathcal{P}_\alpha}, \bigoplus_{n,k\colon a_{n,k}\geqslant\alpha} \mathcal{H}_{S^3_{a_{n,k}}}, \bigoplus_{n,k\colon a_{n,k}\geqslant\alpha} D_{S^3_{a_{n,k}}}),$$

where $\mathcal{P}_\alpha \subset \mathcal{P}$ is the part of the packing that includes only those spheres $S^3_{a_{n,k}}$ with $a_{n,k} \geqslant \alpha$, and $\mathcal{A}_{\mathcal{P}_\alpha} \subset C(\mathcal{P}_\alpha)$ satisfies the bounded commutator condition with the Dirac operator. Correspondingly, in this case, which involves only finitely many spheres, the spectral action would be of the form

$$\mathcal{S}_{\mathcal{P}_\alpha}(\Lambda) = \sum_{n,k\colon a_{n,k}\geqslant\alpha} \mathcal{S}_{S^3_{a_{n,k}}}(\Lambda). \tag{4.4}$$

Lemma 4.1. *Let $\mathcal{P}$ be a packing of 3-spheres satisfying the properties of definition 3.3, and with the property that the function $F(\alpha) = -\log(\mathcal{N}_\alpha(\mathcal{P}))/\log(\alpha)$ has a limit for $\alpha \to 0$. Then the spectral action $\mathcal{S}_{\mathcal{P}_\alpha}(\Lambda)$ diverges at least like $\alpha^{-(\sigma_4(\mathcal{P})-1)+o(1)}$, when $\alpha \to 0$.*

Proof. Each sphere contributes to the spectral action a term of the form

$$\Lambda^3 f_3 \sum_{n,k} a^3_{n,k} - \frac{1}{4}\Lambda f_1 \sum_{n,k} a_{n,k} + O((\Lambda a_{n,k})^{-K}). \tag{4.5}$$

Using the power law (4.2) for $\alpha \to 0$ we estimate

$$\sum_{n,k} a_{n,k}^3 \geqslant \alpha^{3-\delta(\mathcal{P})+o(1)} \quad \text{and} \quad \sum_{n,k} a_{n,k} \geqslant \alpha^{1-\delta(\mathcal{P})+o(1)}.$$

By equation (4.3) and the hypothesis on the existence of the limit, we know that $\delta(\mathcal{P}) = \sigma_4(\mathcal{P})$, which we know satisfies $\sigma_4(\mathcal{P}) > 3$, so that the exponents above are negative. Thus, for a fixed Λ and for $\alpha \to 0$, the sum

$$\sum_{n,k:\, a_{n,k} \geqslant \alpha} \left(\Lambda^3 f_3 \sum_{n,k} a_{n,k}^3 - \frac{1}{4} \Lambda f_1 \sum_{n,k} a_{n,k} \right)$$

diverges at least like the dominant term $\alpha^{-(\delta(\mathcal{P})-1)+o(1)}$. □

4.3. Truncation estimates on the spectral action

We then expect that a good approximation to the spectral action $\mathcal{S}_{\mathcal{P}_\alpha}(\Lambda)$ of the PSCC will be obtained by truncating fractality at a certain Λ-dependent scale. We discuss here possible choices of a function $\alpha = \alpha(\Lambda)$ that retain the property of having a good control on the error term of the spectral action $\mathcal{S}_{\mathcal{P}_{\alpha(\Lambda)}}(\Lambda)$. We first recall how one obtains explicit estimates for the error term in equation (2.17) of the spectral action on a 3-sphere, for a particular class of test functions, following the argument of corollary 4 of [9].

Lemma 4.2. *In the case of a cutoff function of the form $f(x) = P(\pi x^2)\mathrm{e}^{-\pi x^2}$, where P is a polynomial of degree d, the error term $\epsilon(\Lambda a)$ in the spectral action*

$$\mathcal{S}_{S_a^3}(\Lambda) = (\Lambda a)^3 \int_{\mathbb{R}} v^2 f(v)\mathrm{d}v - \frac{1}{4}(\Lambda a) \int_{\mathbb{R}} f(v)\mathrm{d}v + \epsilon(\Lambda a)$$

satisfies the estimate

$$|\epsilon(\Lambda a)| \leqslant (\Lambda a)^3 (5 + 7d + d^2) C_Q \mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2}, \tag{4.6}$$

whenever $\Lambda a \geqslant \sqrt{(d+1)(1+\log(d+1))}$ and $\Lambda a \geqslant 1$. The coefficient C_{Q} is the sum of the absolute values of the coefficients of the polynomial Q, with $\hat{f}(x) = Q(\pi x^2)\mathrm{e}^{-\pi x^2}$.

Proof. As in corollary 4 of [9], we have $\hat{f}^{(2)}(x) = (x^2 Z_1(\pi x^2) + Z_2(\pi x^2))\mathrm{e}^{-\pi x^2}$, where $Z_1 = -Q + 2Q' - Q''$, and $Z_2 = \frac{1}{2\pi}(Q - Q')$, while generally $x^k e^{-x/2} \leqslant 1$, for $x \geqslant 3k(1 + \log k)$. For $n \neq 0$ one then has

$$\hat{f}(n\Lambda a) = Q(\pi(n\Lambda a)^2)\mathrm{e}^{-\frac{\pi}{2}(n\Lambda a)^2} \cdot \mathrm{e}^{-\frac{\pi}{2}(n\Lambda a)^2} \leqslant C_Q \mathrm{e}^{-\frac{\pi}{2}(n\Lambda a)^2},$$

because, by hypothesis, $\pi(n\Lambda a)^2 \geqslant 3d(1 + \log d)$. Since the decay is more rapid than simply exponential, we can see that

$$\begin{aligned} 2\sum_{n=1}^{\infty} \mathrm{e}^{-\frac{\pi}{2}(n\Lambda a)^2} &\leqslant 2\mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2} + 2\sum_{n=4}^{\infty} \mathrm{e}^{-\frac{\pi}{2}n(\Lambda a)^2} \leqslant 2\mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2} \\ &\quad + 2\frac{\mathrm{e}^{-\frac{3\pi}{2}(\Lambda a)^2}}{-1 + \mathrm{e}^{\frac{\pi}{2}(\Lambda a)^2}} \\ &\quad \leqslant 2\mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2} + 2\frac{\mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2}/23}{-1 + 4.8} \leqslant 2.023\mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2}, \end{aligned}$$

under the assumption that $\Lambda a \geqslant 1$. Thus, we have

$$\sum_{n\neq 0}\hat{f}(n\Lambda a) \leqslant 2.023 C_Q \mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2}$$

Similarly, for $\hat{f}^{(2)}$ we get

$$\sum_{n\neq 0}\hat{f}^{(2)}(n\Lambda a) \leqslant 2.023\left(2 + 3d + \frac{1}{\pi}d^2\right) C_Q \mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2}.$$

By looking at the series for the spectral action after applying the Poisson summation formula, we see that the above terms contribute to the error as

$$|\epsilon(\Lambda a)| \leqslant (\Lambda a)^3 2.023\left(2 + 3d + \frac{1}{\pi}d^2\right) C_Q \mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2} + \frac{2.023}{4}\Lambda a C_Q \mathrm{e}^{-\frac{\pi}{2}(\Lambda a)^2},$$

which can then be estimated from the above as in (4.6). □

We can then adapt the error estimates of lemma 4.2 to the PSCC model, by performing a truncation on the fractality scale, dependent on the energy scale Λ.

Remark 4.3. Under the assumption that $a_{n,k} \geqslant \alpha$, the error term in (4.5) is at most $O((\Lambda\alpha)^{-K})$. Thus, it is natural to consider a model where the cutoff of fractality should happen at a scale α related to Λ by the property that $\alpha(\Lambda)\cdot\Lambda$ grows like a positive power of Λ, so that one maintains a good control on the error term for large Λ.

Proposition 4.4. *Consider a truncated packing $\mathcal{P}_\alpha$ of 3-spheres $S^3_{a_{n,k}}$ with $a_{n,k} \geqslant \alpha$, where $\alpha = \alpha(\Lambda) = \Lambda^{-1+\gamma}$ for some $0 < \gamma < 1$. Then the spectral action, computed using a test function of the form $f(x) = P(\pi x^2)e^{-\pi x^2}$ with P a polynomial of degree d, satisfies*

$$\mathcal{S}_{\mathcal{P}_{\Lambda^{-1+\gamma}}}(\Lambda) = \left(\sum_{a_{n,k}\geqslant\Lambda^{-1+\gamma}} a_{n,k}^3\right)\Lambda^3 f_3 - \frac{1}{4}\left(\sum_{a_{n,k}\geqslant\Lambda^{-1+\gamma}} a_{n,k}\right)\Lambda f_1 + \epsilon(\Lambda)$$

where the error term satisfies

$$|\epsilon(\Lambda)| \leqslant \Lambda^{3+\sigma(1-\gamma)} a_{\max}^3 (5 + 7d + d^2) C_Q \mathrm{e}^{-\frac{\pi}{2}\Lambda^{2\gamma}},$$

where $a_{\max} = \max\{a_{n,k}\}$ is the largest radius in the packing $\mathcal{P}$, and σ is the packing constant.

Proof. The estimate follows immediately from the previous Lemma, since we have

$$\begin{aligned}|\epsilon(\Lambda)| &\leqslant \sum_{a_{n,k}\geqslant\Lambda^{-1+\gamma}} \Lambda^3 a_{n,k}^3 (5 + 7d + d^2) C_Q \mathrm{e}^{-\frac{\pi}{2}(\Lambda a_{n,k})^2}\\ &\leqslant \mathcal{N}_{\Lambda^{-1+\gamma}}(\mathcal{P})\Lambda^3 a_{\max}^3 (5 + 7d + d^2) C_Q \mathrm{e}^{-\frac{\pi}{2}\Lambda^{2\gamma}}\\ &= \Lambda^{3+\sigma(1-\gamma)} a_{\max}^3 (5 + 7d + d^2) C_Q \mathrm{e}^{-\frac{\pi}{2}\Lambda^{2\gamma}}.\end{aligned}$$

□

We consider a further possible way of truncating the spectral action, for spheres with smaller radii and the behavior of the error term. We start with the following error term estimate, for a single sphere.

Lemma 4.5. *For a test function of the form $f(x) = P(\pi x^2)e^{-\pi x^2}$, for some polynomial* P *of degree* d*, and for $M\Lambda a$ the truncation scale, consider the truncated sum for the spectral action,*

$$\mathcal{S}_{S_a^3, M}(\Lambda) = \sum_{\lambda < M\Lambda a} \mathrm{Mult}(\lambda) f\,(\frac{\lambda}{\Lambda a}) = \sum_{\lambda < M\Lambda a} \left(\lambda^2 - \frac{1}{4}\right) f\left(\frac{\lambda}{\Lambda a}\right).$$

Let $N = \inf\{n \in \frac{1}{2} + \mathbb{Z} | n \geqslant \max\{M\Lambda a, \frac{3}{2}\}\}$. Then, assuming that

$$\Lambda a \geqslant \sqrt{(d+1)(1+\log(d+1))} \quad \text{and} \quad \Lambda a \geqslant 1,$$

the error term satisfies

$$|\epsilon(\Lambda a)| = 2\sum_{\lambda \geqslant N} \left(\lambda^2 - \frac{1}{4}\right) f\left(\frac{\lambda}{\Lambda a}\right) \leqslant \frac{2}{\pi}(\Lambda a)^2 \left(1 + \frac{2}{\pi}(\Lambda a)^2\right) C_P \mathrm{e}^{-\frac{\pi}{2}M^2}.$$

Proof. Since $u^2 - \frac{1}{4}$ is a strictly positive quantity when evaluated at the integers, its absolute value is always less than u^2. Furthermore, all terms in the series are positive, so that we can work with u^2 instead of $u^2 - \frac{1}{4}$ and our conclusions will remain valid. As in lemma 4.2, we use the fact that $x^k \mathrm{e}^{-\frac{x}{2}} \leqslant 1$, for all $x \geqslant 3k(1 + \log k)$. Then, as long as $x \geqslant \sqrt{(d+1)(1+\log(d+1))}$, we know that $x^2 f(x) \leqslant \frac{1}{\pi} C_P \mathrm{e}^{-\frac{\pi}{2}x^2}$.

Let our point of truncation be $M\Lambda a$, so that our sum is over $u < M\Lambda a$. Let N be as in the statement. Then we have

$$\begin{aligned} |\epsilon(\Lambda a)| &= 2\sum_{\lambda \geqslant N} \left(\lambda^2 - \frac{1}{4}\right) f\left(\frac{\lambda}{\Lambda a}\right) \leqslant 2\sum_{\lambda \geqslant N} (\Lambda a)^2 \left(\frac{\lambda}{\Lambda a}\right)^2 f\left(\frac{\lambda}{\Lambda a}\right) \\ &\leqslant 2\sum_{\lambda \geqslant N} (\Lambda a)^2 \frac{1}{\pi} C_P \mathrm{e}^{-\frac{\pi}{2}\left(\frac{\lambda}{\Lambda a}\right)^2} \leqslant \frac{2}{\pi}(\Lambda a)^2 C_P \sum_{v=0}^{\infty} \mathrm{e}^{-\frac{\pi}{2}\frac{1}{(\Lambda a)^2}(N^2 + v)} \\ &= \frac{2}{\pi}(\Lambda a)^2 C_P \mathrm{e}^{-\frac{\pi}{2}\left(\frac{N}{\Lambda a}\right)^2} \cdot \frac{1}{1 - \mathrm{e}^{-\frac{\pi}{2}\frac{1}{(\Lambda a)^2}}} \leqslant \frac{2}{\pi}(\Lambda a)^2 C_P \mathrm{e}^{-\frac{\pi}{2}M^2} \cdot \frac{1}{1 - \mathrm{e}^{-\frac{\pi}{2}\frac{1}{(\Lambda a)^2}}}. \end{aligned}$$

Now we check that $\left(1 - \mathrm{e}^{-\frac{\pi}{2}\frac{1}{(\Lambda a)^2}}\right)^{-1} < 1 + \frac{2}{\pi}(\Lambda a)^2$. Define $g(x) = \frac{1}{1 - \mathrm{e}^{-1/x}}$. In the case $x \geqslant 1$, we have

$$\begin{aligned} \mathrm{e}^{-1/x} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{-n}}{n!} = 1 - x^{-1} + \frac{1}{2}x^{-2} - \sum_{n=2}^{\infty} \left(\frac{x^{1-2n}}{(2n-1)!} - \frac{x^{-2n}}{(2n)!}\right) \\ &= 1 - x^{-1} + \frac{1}{2}x^{-2} - \sum_{n=2}^{\infty} \left(x - \frac{1}{2n}\right) \frac{x^{-2n}}{(2n-1)!} \leqslant 1 - x^{-1} + \frac{1}{2}x^{-2}, \end{aligned}$$

since every term in the third sum is positive. In the case $0 < x < 1$, substituting $u = 1/x$, with $u > 1$, we have

$$g(x) = \frac{1}{1 - \mathrm{e}^{-u}} = \frac{1}{1 - \mathrm{e}^{-u}} \frac{1 + 2\mathrm{e}^{-u}}{1 + 2\mathrm{e}^{-u}} = \frac{1 + 2\mathrm{e}^{-u}}{1 + (\mathrm{e}^{-u} - 2\mathrm{e}^{-2u})} \leqslant 1 + 2\mathrm{e}^{-u}.$$

Compare this with $1 + 1/u$: the term $2\mathrm{e}^{-\mathrm{u}}$ decreases faster than $1/u$, and $2\mathrm{e}^{-1} < 1$, so for all $u > 1$, we have $1 + 2\mathrm{e}^{-u} < 1/u$. Thus, for all $0 < x < 1$ we have $g(x) \leqslant 1 + x$. Together, these two cases give the desired inequality

$$(1 - e^{-\frac{\pi}{2}\frac{1}{(\Lambda a)^2}})^{-1} < 1 + \frac{2}{\pi}(\Lambda a)^2,$$

which then gives the stated estimate for the error term $|\epsilon(\Lambda a)|$. □

For a given scale M, we consider a summation as above, of the form

$$\sum_{\frac{3}{2M} < \Lambda a_{n,k} < M} \sum_{\lambda=\frac{3}{2}}^{\lceil M\Lambda a_{n,k}\rceil - 1} 2(\lambda^2 - \frac{1}{4}) f\left(\frac{\lambda}{\Lambda a_k}\right).$$

We are interested now in the case of smaller spheres, with $a_{n,k} < \alpha$, where $\alpha = \alpha(\Lambda)$ was a previously chosen cutoff. Thus, we relate the scale M to Λ accordingly, by assuming that $M = M(\Lambda) = \alpha(\Lambda)\Lambda = \Lambda^\gamma$. This, in turn, gives the lower bound $\frac{3}{2M} = \frac{3}{2}\Lambda^{-\gamma}$, which means that we are considering spheres with $a_{n,k} \geqslant \tilde{\alpha}(\Lambda) = \frac{3}{2}\Lambda^{-(1+\gamma)}$.

Proposition 4.6. *For $\alpha(\Lambda) = \Lambda^{-1+\gamma}$ and $\tilde{\alpha}(\Lambda) = \frac{3}{2}\Lambda^{-(1+\gamma)}$, and for sufficiently large Λ, we have*

$$\mathcal{S}_{\mathcal{P}_{\tilde{\alpha}(\Lambda)}}(\Lambda) = \sum_{\frac{3}{2}\Lambda^{-\gamma} < \Lambda a_{n,k} < \Lambda^\gamma} \sum_{\lambda=\frac{3}{2}}^{\lceil \Lambda^{1+\gamma} a_{n,k}\rceil - 1} 2(\lambda^2 - \frac{1}{4}) f\left(\frac{\lambda}{\Lambda a_k}\right) + \epsilon(\Lambda)$$

where

$$|\epsilon(\Lambda)| \leqslant \left(\frac{3}{2}\right)^\sigma \frac{2}{\pi} C_P \Lambda^{2\gamma + (1+\gamma)\sigma} e^{-\frac{\pi}{2}\Lambda^{2\gamma}}.$$

Proof. For large Λ we can estimate the number $\mathcal{N}_{\tilde{\alpha}(\Lambda)}(\mathcal{P})$ of spheres with $a_{n,k} \geqslant \tilde{\alpha}(\Lambda)$, with $\tilde{\alpha}(\Lambda)^{-\sigma} = \left(\frac{3}{2}\right)^\sigma \Lambda^{(1+\gamma)\sigma}$, where $\sigma = \sigma_4(\mathcal{P})$ is the packing constant. For each sphere in the summation, we can apply the error term estimate of the previous lemma, with $\Lambda a_{n,k} \leqslant \Lambda^\gamma$, and we obtain

$$|\epsilon(\Lambda)| \leqslant \left(\frac{3}{2}\right)^\sigma \Lambda^{(1+\gamma)\sigma} \frac{2}{\pi} C_P \Lambda^{2\gamma} (1 + \frac{2}{\pi}\Lambda^{2\gamma}) e^{-\frac{\pi}{2}\Lambda^{2\gamma}}.$$

□

We can view the estimate of the error term of this sum as a way to estimate the effect of changing the cutoff scale from $\alpha(\Lambda) = \Lambda^{-1+\gamma}$ to $\tilde{\alpha}(\Lambda) = \frac{3}{2}\Lambda^{-(1+\gamma)}$.

5. Related fractal models

5.1. Fractal dodecahedra and cosmic topology

In [7, 41, 42], cosmological models based on the spectral action functional of gravity are constructed for (compactified, Euclidean) spacetimes of the form $S^1 \times Y$ where Y is either a spherical space form or a flat Bieberbach manifold, and it is shown that the spectral action detects the different cosmic topologies through the shape of an associated slow-roll inflation potential. In particular, it is shown in [7, 41, 54] that the spectral action for a spherical space form $Y = S^3/\Gamma$ is given by

$$\mathcal{S}_Y(\Lambda) = \frac{1}{\#\Gamma}\, \mathcal{S}_{S^3}(\Lambda),$$

independently of the spin structure (even though the Dirac spectrum itself changes for different spin structures). Of particular interest for cosmic topology is the case where Y is the Poincaré homology 3-sphere (dodecahedral space), which is regarded as one of the most promising candidates for a non-simply connected cosmic topology [14, 39].

We consider here a different possible model with fractal structure, where the building blocks are spherical dodecahedra, folded up to form Poincaré homology spheres, arranged in a fractal configuration that generalizes the Sierpiński fractal to dodecahedral geometry. Other similar constructions can be done using other solids, and adapted to the other spherical space form candidates for cosmic topologies. These fractals are much simpler in structure than the Apollonian sphere packing described in the previous section, as the successive levels of the construction are all obtained by repeatedly applying the same uniform contraction factor. This makes the computation of the Hausdorff dimension immediate, as well as its identification with the exponent of convergence of the relevant zeta function. Moreover, it is also immediately clear that an analytic continuation exists to a meromorphic function on the entire complex plane, hence all the properties can be checked more easily.

More precisely, let $Y_a = S_a^3/\mathcal{I}_{120}$ be the quotient of a round 3-sphere of radius a by the isometric action of the icosahedral group $\mathcal{I}_{120}$. The choice of a fundamental domain, given by a spherical dodecahedron, and the action of the group $\mathcal{I}_{120}$ determine a tiling of S_a^3 consisting of 120 dodecahedra. The quotient 3-manifold Y_a is a Poincaré homology sphere, of volume $Vol\,(Y_a) = Vol\,(S_a^3/\mathcal{I}_{120}) = \frac{\pi^2}{60}a^3$.

Consider now the following well-known construction of a Sierpiński-type fractal based on the dodecahedron. Starting with an initial (solid) regular dodecahedron, one replaces it with 20 new regular dodecahedra, contained inside the volume bounded by the initial one, each placed in the corner corresponding to one of the 20 vertices of the original dodecahedron. Each of the new dodecahedra is a copy of the original one scaled by a factor of $(2 + \phi)^{-1}$ where ϕ is the golden ratio. One keeps iterating this procedure on each of the dodecahedra in the new configuration. Let $\mathcal{P}_{Y,n}$ be the union of dodecahedra obtained at the nth step of the construction, where we simultaneously perform the identification of faces, in each dodecahedron, according to the action of $\mathcal{I}_{120}$, so that each is folded up into a Poincaré homology 3-sphere. Let $\mathcal{P}_Y$ denote the resulting limit, which in the set theoretic sense is given by the intersection $\mathcal{P}_Y = \bigcap_{n\geqslant 1} \mathcal{P}_{Y,n}$. The following fact then follows immediately.

Lemma 5.1. *The Hausdorff dimension of the resulting set is*

$$\dim_H \mathcal{P}_Y = \frac{\log(20)}{\log(2+\phi)} = 2.3296\ldots \tag{5.1}$$

This is equal to the exponent of convergence of the series

$$\zeta_{\mathcal{L}(\mathcal{P}_Y)}(s) = \sum_{n\geqslant 0} 20^n (2+\phi)^{-ns},$$

which has analytic continuation to a meromorphic function on $\mathbb{C}$,

$$\zeta_{\mathcal{L}(\mathcal{P}_Y)}(s) = \frac{1}{1 - 20(2+\phi)^{-s}}$$

with simple poles at the points

$$s_m = \frac{\log(20)}{\log(2+\phi)} + \frac{2\pi i m}{\log(2+\phi)}, \quad \text{with} \quad m \in \mathbb{Z}, \tag{5.2}$$

all with the same residue

$$\mathrm{Res}_{s=s_m} \zeta_{\mathcal{L}(\mathcal{P}_Y)}(s) = \frac{1}{\log(2+\phi)}. \tag{5.3}$$

We then construct a spectral triple for the configuration $\mathcal{P}_Y$ using the same procedure as in [12]. Let $(C(Y_a), \mathcal{H}_{Y_a}, D_{Y_a})$ be the spectral triple of Y_a with $\mathcal{H}_{Y_a} = L^2(Y_a, \mathbb{S})$ the square integrable spinors and D_{Y_a} the Dirac operator. See [41, 54] for a more detailed discussion of these data.

Proposition 5.2. *The spectral triple*

$$(\mathcal{A}_{\mathcal{P}_Y}, \mathcal{H}_{\mathcal{P}_Y}, \mathcal{D}_{\mathcal{P}_Y}) = (\mathcal{A}_{\mathcal{P}_Y}, \oplus_n \mathcal{H}_{Y_{a_n}}, \oplus_n D_{Y_{a_n}}),$$

with $\mathcal{A}_{\mathcal{P}_Y} \subset C(\mathcal{P}_Y)$ satisfying the bounded commutator condition, and with $a_n = a(2+\phi)^{-n}$, has zeta function

$$\zeta_{\mathcal{D}_{\mathcal{P}_Y}}(s) = \frac{a^s}{120}\left(2\zeta\left(s-2, \frac{3}{2}\right) - \frac{1}{2}\zeta\left(s, \frac{3}{2}\right)\right)\zeta_{\mathcal{L}(\mathcal{P}_Y)}(s). \tag{5.4}$$

The positive part of the dimension spectrum is $\Sigma^+ = \{1, \sigma, 3\}$, with $\sigma = \dim_H(\mathcal{P}_Y)$, while the full dimension spectrum Σ also contains the points (5.2) off the real line. The metric dimension of the spectral triple is 3.

Proof. The spectrum of the Dirac operator on the Poincaré homology sphere Y_a, with the correct multiplicities, can be computed explicitly using the method of [2] of generating functions, see [54]. In the case of the trivial spin structure, it is shown in [54] that one can separate the spectrum into 60 arithmetic progressions $\{\frac{3}{2} + k + 60j\}$ with multiplicities interpolated by 60 explicit polynomials $P_k\left(\frac{3}{2} + k + 60j\right) = \mathrm{Mult}(\frac{3}{2} + k + 60j)$, which satisfy

$$\sum_{k=0}^{59} P_k(u) = \frac{1}{2}u^2 - \frac{1}{8}.$$

This implies that

$$\mathcal{S}_{Y_a}(\Lambda) = \sum_{k=0}^{59}\sum_{j\in\mathbb{Z}} P_k\left(\frac{3}{2} + k + 60j\right) f\left(\frac{\frac{3}{2} + k + 60j}{\Lambda}\right) = \frac{1}{120}\mathcal{S}_{S_a^3}(\Lambda)$$

and that

$$\zeta_{D_{Y_a}}(s) = \sum_{k=0}^{59}\sum_{j\in\mathbb{Z}} P_k\left(\frac{3}{2} + k + 60j\right)\left|\frac{3}{2} + k + 60j\right|^{-s} = \frac{1}{120}\,\zeta_{D_{S_a^3}}(s),$$

hence equation (5.4) then follows as in proposition 2.6. □

Using the same technique that we used in the previous construction, we can then compute the leading terms in the expansion of the spectral action. In this case we have a completely explicit description of the poles off the real line, so we also obtain a more explicit description of the log oscillatory corrections to the spectral action, which in this model behaves exactly as proposition 3.1.

Proposition 5.3. *Let $\mathcal{P}_Y$ be the fractal arrangement described above, to which we assign a spectral triple as in proposition 5.2. The contribution to the expansion of the spectral action* $\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}_Y}/\Lambda))$ *coming from the points of the dimension spectrum that lie on the positive real line is given by*

$$(\Lambda a)^3 \frac{\zeta_{\mathcal{L}(\mathcal{P}_Y)}(3)}{120} f_3 - \Lambda a \frac{\zeta_{\mathcal{L}(\mathcal{P}_Y)}(1)}{120} f_1 + (\Lambda a)^{\sigma} \frac{\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2})}{120 \log(2+\phi)} f_{\sigma}, \tag{5.5}$$

where $\sigma = \dim_H(\mathcal{P}_Y)$, while the contribution to the expansion of the spectral action coming from the non-real points of the dimension spectrum is given by a Fourier series

$$\Lambda^{\sigma} \sum_{m \neq 0} \Lambda^{\frac{2\pi i m}{\log(2+\phi)}} \frac{\zeta\left(s_m - 2, \frac{3}{2}\right) - \frac{1}{4}\zeta\left(s_m, \frac{3}{2}\right)}{120 \log(2+\phi)} f_{s_m},$$

with

$$f_{s_m} = \int_0^{\infty} f(u) u^{s_m - 1} \, \mathrm{d}u.$$

The series converges absolutely to a smooth function of the angular variable $\theta = \frac{\log \Lambda}{\log(2+\phi)}$ mod $2\pi\mathbb{Z}$.

Proof. The argument is exactly as in proposition 3.5, where we now have $2 < \sigma < 3$ and the residue $\mathcal{R}_{\sigma}$ (as well as those at the poles off the real line) is given by

$$\mathcal{R}_{\sigma} = \mathrm{Res}_{s=\sigma} \zeta_{\mathcal{L}(\mathcal{P}_Y)}(s) = \frac{1}{\log(2+\phi)}.$$

The complex poles s_m of $\zeta_{\mathcal{L}}(s)$ lie within the region of absolute convergence of the series defining the Hurwitz zeta function, hence the size of the terms $|\zeta\left(s_m - 2, \frac{3}{2}\right) - \frac{1}{4}\zeta(s_m, \frac{3}{2})|$ is controlled by a term $|\zeta\left(\sigma - 2, \frac{3}{2}\right)| + \frac{1}{4}|\zeta(\sigma, \frac{3}{2})|$. The convergence of the Fourier series above is then controlled by the convergence of $\sum_m \Lambda^{\frac{2\pi i m}{\log(2+\phi)}} f_{s_m}$, which converges absolutely to a smooth function of the periodic angle variable θ, as shown in proposition 3.1. □

Corollary 5.4. *Consider a (Euclidean, compactified) spacetime model of the form $S^1_{\beta} \times \mathcal{P}_Y$, with $\mathcal{P}_Y$ the fractal arrangement as above, and with β the size of the compactification. Then the contribution to the expansion of the spectral action $\mathcal{S}_{S^1_{\beta} \times \mathcal{P}_Y}(\Lambda)$ coming from real points of the dimension spectrum is given by*

$$2\beta \left(\Lambda^4 \frac{a^3 \zeta_{\mathcal{L}(\mathcal{P}_Y)}(3)}{120} \mathfrak{h}_3 - \Lambda^2 \frac{a \zeta_{\mathcal{L}(\mathcal{P}_Y)}(1)}{120} \mathfrak{h}_1 + \Lambda^{\sigma+1} \frac{a^{\sigma}(\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2}))}{120 \log(2+\phi)} \mathfrak{h}_{\sigma} \right), \tag{5.6}$$

with $\sigma = \dim_H(\mathcal{P}_Y)$ and $\mathfrak{h}_1$, $\mathfrak{h}_3$, $\mathfrak{h}_\sigma$ as in equations (3.11) and (3.12). The contribution to the points of the dimension spectrum that are off the real line is a Fourier series

$$\Lambda^{\sigma+1}\frac{2\beta a^\sigma}{120\log(2+\phi)}\sum_{m\neq 0}(\Lambda a)^{\frac{2\pi i m}{\log(2+\phi)}}\left(\zeta\left(s_m-2,\frac{3}{2}\right)-\frac{1}{4}\zeta\left(s_m,\frac{3}{2}\right)\right)\mathfrak{h}_{s_m},$$

where the coefficients $\mathfrak{h}_{s_m}$ are given by

$$\mathfrak{h}_{s_m} = 2\int_0^\infty h(\rho^2)\rho^{s_m}\mathrm{d}\rho.$$

The series converges to a smooth function of $\theta = \frac{\log\Lambda}{\log(2+\phi)} \mod 2\pi$.

Proof. The result follows exactly as in proposition 3.6. □

6. Slow roll inflation in fractal universes

In the case of a compactified Euclidean spacetime of the form $S^1_\beta \times S^3_a$, it was shown in [9] that perturbations of the Dirac operator by a scalar field $D^2 \mapsto D^2 + \phi^2$ produce, in the calculation of the spectral action, a potential $V(\phi)$ for the scalar field, obtained as a combination of functions $\mathcal{V}(\phi^2/\Lambda^2)$ and $\mathcal{W}(\phi^2/\Lambda^2)$ with

$$\begin{aligned}\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2)) \sim \pi\Lambda^4\beta a^3\int_0^\infty uh(u)\mathrm{d}u - \frac{\pi}{2}\Lambda^2\beta a\int_0^\infty h(u)\mathrm{d}u\\ + \pi\Lambda^4\beta a^3\,\mathcal{V}(\phi^2/\Lambda^2) + \frac{1}{2}\Lambda^2\beta a\,\mathcal{W}(\phi^2/\Lambda^2),\end{aligned}$$

where the functions are of the form

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x)-h(u))\mathrm{d}u,\ \mathcal{W}(x) = \int_0^x h(u)\mathrm{d}u.$$

In [9] it was first observed that the potential obtained in this way has the typical shape of the slow-roll inflation potentials. Whether one can accommodate a satisfactory inflaton model within the spectral action paradigm is still debated. It was first proposed that the Higgs field might play the role of inflaton field, but the possibility of a Higgs-based inflation scenario in the noncommutative geometry model was ruled out as incompatible with the measured value of the top quark mass in [6], based on constraints coming from the CMB data. In the noncommutative geometry models of gravity coupled to matter, the Higgs sector arises as inner fluctuation of the Dirac operator in the non-commutative fiber directions of an almost-commutative geometry. By contrast, even in the pure gravity case, where there is no finite non-commutative geometry, one can introduce a scalar perturbation of the Dirac operator of the kind described above, see the discussion in section 1.4 of [7].

It was observed in [7, 41, 42], where the construction is generalized for spherical space forms and Bieberbach manifolds, that the shape of the resulting slow-roll inflation potential $V(\phi)$ distinguishes between (almost all of) the different possible topologies and determines detectable signatures of the cosmic topology in the slow-roll parameters (which in turn determine spectral index and tensor-to-scalar ratio) and in the form of the power spectra for the scalar and tensor fluctuations. The result is similar in the case of the spherical space forms $Y_a = S^3_a/\Gamma$, with $V(\phi)$ replaced by $(\#\Gamma)^{-1}V(\phi)$, see [7, 41]. In this section we discuss how the inflation potential changes in the case of $S^1_\beta \times \mathcal{P}$, where $\mathcal{P}$ is either an Apollonian packing of 3-spheres or a configuration based on the fractal dodecahedron and the Poincaré homology sphere. In particular, we show that, in models of inflation based on the spectral action functional, the shape of the inflation potential changes depending on the fractal structure, hence the potential to detect measurable effects of the presence and the type of fractal structure. In particular, it follows that, in a model of gravity based on the spectral action, the presence of fractality in the spacetime structure leaves a detectable signature in quantities, like the slow-roll parameters and the power spectra for the scalar and tensor fluctuations, that are in principle measurable in the CMB, modulo the problem of determining the unknown parameter β of the model, already discussed in [41, 42].

We perturb the Dirac operator D of a packing $\mathcal{P}$ of 3-spheres by a scalar field ϕ. Correspondingly, the spectral action is modified by terms that determine a potential for the scalar field. We discuss the effect of the real points of the dimension spectrum (proposition 6.1) and of the fluctuations coming from the points off the real line (proposition 6.2) on the shape of the potential.

Proposition 6.1. *Let $\mathcal{P}$ be either a packing of 3-spheres satisfying the three conditions of definition 3.3, or a configuration of Poincaré homology 3-spheres arranged according to the fractal dodecahedron construction of section 5.1. Consider the product geometry $S^1_\beta \times \mathcal{P}$ of $\mathcal{P}$ with a circle of size β. Then the spectral action satisfies*

$$\begin{aligned}\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2)) \sim 2\beta\Big(\Lambda^4\,\zeta_{\mathcal{L}}(3)\mathfrak{h}_3 - \Lambda^2\,\frac{1}{4}\,\zeta_{\mathcal{L}}(1)\,\mathfrak{h}_1\Big)\\ + 2\beta\Lambda^{\sigma+1}\Big(\zeta(\sigma-2,\frac{3}{2}) - \frac{1}{4}\zeta(\sigma,\frac{3}{2})\Big)\mathcal{R}_\sigma\,\mathfrak{h}_\sigma\\ + \pi\Lambda^4\beta\zeta_{\mathcal{L}}(3)\mathcal{V}(\phi^2/\Lambda^2) + \frac{\pi}{2}\Lambda^2\beta\zeta_{\mathcal{L}}(1)\mathcal{W}(\phi^2/\Lambda^2)\\ + 4\beta\Lambda^{\sigma+1}\Big(\zeta(\sigma-2,\frac{3}{2}) - \frac{1}{4}\zeta(\sigma,\frac{3}{2})\Big)\mathcal{R}_\sigma\,\mathcal{U}_\sigma(\phi^2/\Lambda^2)\\ + \mathcal{S}^{osc}_{D,\phi}(\Lambda),\end{aligned} \tag{6.1}$$

where the last term collects the fluctuations coming from the log-oscillatory terms contributed by the poles of the zeta functions that are off the real line, described in proposition 6.2 below. The potentials $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{U}_\sigma$ respectively, are, given by

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u))\mathrm{d}u, \qquad \mathcal{W}(x) = \int_0^x h(u)\mathrm{d}u, \tag{6.2}$$

$$\mathcal{U}_\sigma(x) = \int_0^\infty u^{(\sigma-1)/2}(h(u+x) - h(u))\mathrm{d}u, \tag{6.3}$$

where $\sigma = \sigma_4(\mathcal{P})$ is the packing constant (2.5), $\mathcal{R}_\sigma$ is the residue of $\zeta_{\mathcal{L}}(s)$ at $s=\sigma$, and $\mathfrak{h}_1$, $\mathfrak{h}_3$, $\mathfrak{h}_\sigma$ are as in equation (3.11) and (3.12).

Proof. The argument follows directly from proposition 3.6, along the lines of theorem 7 of [9]. We have

$$\int_0^\infty h(\rho^2)\rho^3\mathrm{d}\rho = \frac{1}{2}\int_0^\infty uh(u)\mathrm{d}u,$$

which gives rise to the term

$$\int_0^\infty u(h(u+x) - h(u))\mathrm{d}u$$

in the $\mathcal{V}$ part of the potential; similarly, we have

$$\int_0^\infty h(\rho^2)\rho \mathrm{d}\rho = \frac{1}{2}\int_0^\infty h(u)\mathrm{d}u,$$

which gives the term

$$\int_0^x h(u)\mathrm{d}u$$

in the $\mathcal{W}$ part of the potential, and the term

$$\int_0^\infty h(\rho^2)\rho^\sigma \mathrm{d}\rho = \frac{1}{2}\int_0^\infty u^{\frac{\sigma-1}{2}} h(u)\mathrm{d}u,$$

which gives the term

$$\int_0^\infty u^{\frac{\sigma-1}{2}}(h(u+x) - h(u))\mathrm{d}u$$

in the $\mathcal{U}_\sigma$ part of the potential. □

Proposition 6.2. *Under the assumptions of definition 3.3, the fluctuation terms $\mathcal{S}^{osc}_{D,\phi}(\Lambda)$ are of the form $\mathcal{S}^{osc}_{D,\phi}(\Lambda) = \mathcal{S}_{S^1_\beta\times\mathcal{P}}(\Lambda)^{osc} + \mathbb{U}^{osc}_\sigma(\phi)$, where $\mathcal{S}_{S^1_\beta\times\mathcal{P}}(\Lambda)^{osc}$ is as in (3.13). Let $\mathbb{U}^{osc}_\sigma(\phi)_{\leqslant R}$ be the approximation to $\mathbb{U}^{osc}_\sigma(\phi)$ that only counts the contribution of non-real poles with $|\Im(s)| \leqslant R$. The oscillatory term $\mathbb{U}^{osc}_\sigma(\phi)_{\leqslant R}$ is approximated by a sequence*

$$\begin{aligned} &4\beta\sum_{j=1}^{N_n}\Lambda^{\sigma_{n,j}+1}\sum_m \Lambda^{i(\alpha_{n,j}+\frac{2\pi m}{\log b_n})}\left(\zeta\left(s_{n,j,m}-2, \frac{3}{2}\right)\right.\\ &\left.-\frac{1}{4}\zeta\left(s_{n,j,m}, \frac{3}{2}\right)\right)Res_{s=s_{n,j,m}}\zeta_{\mathcal{L}_n}(s)\,\mathcal{U}_{s_{n,j,m}}(\phi^2/\Lambda^2), \end{aligned} \tag{6.4}$$

where $n\to\infty$ and $R\to\infty$, with

$$\mathcal{U}_{s_{n,j,m}}(x) = \int_0^\infty u^{(s_{n,j,m}-1)/2}(h(u+x) - h(u))\mathrm{d}u,$$

and where $s_{n,j,m} = \sigma_{n,j} + i\left(\alpha_{n,j} + \frac{2\pi m}{\log b_n}\right)$ are the complex zeros of the series of self-similar strings $\mathcal{L}_n$ with the lattice property approximating the complex poles of $\zeta_{\mathcal{L}(\mathcal{P})}(s)$.

Proof. The result is obtained as in proposition 6.1, using the results of proposition 3.6. □

The slow-roll inflation potential $V(\phi)$ is obtained from the combination of functions $\mathcal{V}$, $\mathcal{W}$, $\mathcal{U}$ that appears in the expansion of the spectral action above.

Corollary 6.3. *The function*

$$\begin{aligned} &\pi\Lambda^4\beta\zeta_{\mathcal{L}}(3)\mathcal{V}(\phi^2/\Lambda^2) + \frac{\pi}{2}\Lambda^2\beta\zeta_{\mathcal{L}}(1)\mathcal{W}(\phi^2/\Lambda^2)\\ &+ 4\beta\Lambda^{\sigma+1}\left(\zeta(\sigma-2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2})\right)\mathcal{R}_\sigma\,\mathcal{U}_\sigma(\phi^2/\Lambda^2) + \mathbb{U}^{osc}_\sigma(\phi) \end{aligned}$$

depends explicitly on the presence of fractality, through the coefficients $\zeta_{\mathcal{L}}(3)$, $\zeta_{\mathcal{L}}(1)$, the residue $\mathcal{R}_\sigma$, the packing constant σ, and the oscillatory fluctuations.

Corollary 6.4. *In the case of the fractal arrangement $\mathcal{P}_Y$ of dodecahedral spaces considered in the previous section, the form of the fluctuations in the inflation potential is simpler and given by the Fourier series*

$$\mathbb{U}_\sigma^{osc}(\phi) = \frac{4\beta\Lambda^{\sigma+1}}{\log(2+\phi)} \sum_m \Bigg(\zeta(\sigma + \frac{2\pi i m}{\log(2+\phi)} - 2, \frac{3}{2}) \\ - \frac{1}{4}\zeta(\sigma + \frac{2\pi i m}{\log(2+\phi)}, \frac{3}{2}) \Bigg) \mathcal{U}_{s_m}\left(\frac{\phi^2}{\Lambda^2}\right),$$

with

$$\mathcal{U}_{s_m}(x) = \int_0^\infty u^{(\sigma + \frac{2\pi i m}{\log(2+\phi)} - 1)/2} (h(u+x) - h(u)) \mathrm{d}u.$$

Proof. These follows directly from the results of the previous section, by proceeding as in the proof of proposition 6.1. □

7. Conclusions and further questions

7.1. Conclusions

In this paper we considered a model of gravity based on the spectral action functional. This is known to recover, via its asymptotic expansion, the usual Einstein–Hilbert action with cosmological term, along with modified gravity terms (conformal and Gauss–Bonnet gravity). We considered simple models of (Euclidean, compactified) spacetimes of the form $S^1_\beta \times \mathcal{P}$, where β is the size of the S^1-compactification and $\mathcal{P}$ is a fractal configuration built out of 3-spheres (Apollonian packings) or of other spherical space forms (Sierpiński fractals). We evaluated the leading terms of the expansion of the spectral action, using information on the zeta function of the Dirac operator of a spectral triple, and we compared them, respectively, with the corresponding terms in the simpler case of $S^1_\beta \times S^3_a$ (spatial sections given by a single sphere of radius a) or $S^1_\beta \times Y$ where Y is a spherical space form, in particular the Poincaré homology sphere (dodecahedral space). We regard the case of $S^1_\beta \times \mathcal{P}$, where $\mathcal{P}$ is an Apollonian packing of 3-spheres or a configuration obtained from a Sierpiński fractal dodecahedron, as a model of the possible presence of fractality in spacetime geometry: a version of PSCC models. We showed that the resulting leading terms of the expansion of the spectral action for $S^1_\beta \times \mathcal{P}$ differ from those of the ordinary $S^1_\beta \times S^3$ (or $S^1_\beta \times Y$) case in the following ways:

- The term $2\Lambda^4\beta a^3\mathfrak{h}_3 - \frac{1}{2}\Lambda^2\beta a\mathfrak{h}_1$, respectively corresponding to the cosmological and the Einstein–Hilbert term, are replaced by terms of the form $2\Lambda^4\beta\zeta_{\mathcal{L}}(3)\mathfrak{h}_3 - \frac{1}{2}\Lambda^2\beta\zeta_{\mathcal{L}}(1)\mathfrak{h}_1$, which can be seen as a zeta regularization of the divergent series of the 3-sphere terms summed over the packing.
- There is an additional term in the gravity action functional of the form

$$\Lambda^{\sigma+1}\left(\zeta\left(\sigma - 2, \frac{3}{2}\right) - \frac{1}{4}\zeta\left(\sigma, \frac{3}{2}\right)\right)\mathcal{R}_\sigma\mathfrak{h}_\sigma,$$

where $\zeta(s, x)$ is the Hurwitz zeta function and σ is the packing constant of $\mathcal{P}$, with $\mathcal{R}_\sigma$ the residue at σ of the zeta function $\zeta_{\mathcal{L}}(s)$ of the fractal packing. For a fractal dodecahedron σ is the Hausdorff dimension, while for an Apollonian packing it is conjecturally the Hausdorff dimension of the residual set. In both cases this term detects modifications to the gravity action functional due to the presence of a fractal structure.

- This additional term is further corrected by a Fourier series of log-oscillatory fluctuations, coming from the presence of points of the dimension spectrum off the real line (another purely fractal phenomenon).
- The perturbation $D^2 \mapsto D^2 + \phi^2$ of the Dirac operator determines a slow-roll inflation potential $V(\phi)$ for the field ϕ. The shape of the potential detects the presence of fractality through the coefficients $\zeta_{\mathcal{L}}(3)$, $\zeta_{\mathcal{L}}(1)$, the packing constant σ, and the residue at σ of $\zeta_{\mathcal{L}}(s)$, and oscillatory fluctuations.

7.2. Further questions

There are a number of questions, both mathematical and physical, that arise in relation to improving the model of gravity, based on the spectral action, on cosmologies exhibiting fractality. On the mathematical side, as we have seen, one needs a better understanding of the properties of higher-dimensional Apollonian packings, especially with respect to characterizing the presence of exact self-similarity, extending results like [15] beyond the two-dimensional case of circle packings.

From the physics viewpoint, the logic we followed in this paper is along the lines of several other recent results, where one considers a classical model of (Euclidean, compactified) spacetime and computes what the expansion of the spectral action looks like, either with the full infinite series of the asymptotic expansion, or at least with the leading terms up to order Λ^0, see for instance [10, 9, 25, 27, 54]. Under this perspective, one can consider the classical PSCCs. These originate in a spacetime model introduced in the late 1960s [47] as a prototype model of isotropic but non-homogeneous spacetimes (as opposed to, for instance, the Bianchi IX examples of homogeneous non-isotropic spaces). In other words, the original construction of the PSCC is dictated by imposing certain kind of regularity requirements on the geometry. These classical models have then been tested as possible models of fractal and multifractal structures in spacetime. Here we investigate how the spectral action functional, seen as our choice of action functional of gravity, behaves on this specific classical geometry. A first important question in this direction, which would make the model more realistic (closer to the original construction of [47]) would be to start from a (Euclidean, compactified) Robertson–Walker spacetime and carve out balls, so as to obtain a residual Apollonian sphere packing, rather than adopting the simplified model we considered here of a product $\mathcal{P} \times S^1$. In terms of computations of the spectral action, this would mean adapting the computation for the Robertson–Walker metric of [10] to the resulting fractal packing, rather than (as we did here) adapting the computations of [9] for S^3 to the case of the Apollonian packings of 3-spheres, or the fractal dodecahedral packing of Poincaré homology spheres. A second question would be to adopt a different viewpoint and derive spacetime models (possibly with some form of fractal structures or of noncommutativity) from a *least action principle* applied to the spectral action functional. More explicitly, the question would be whether such a variational principle can be reinterpreted in terms of classical theory as a modified gravity model, with an effective stress-energy tensor (as for instance in the case of *f*(*R*)-modified gravity).

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