Foundations of Noncommutative Geometry

Tanvir Ahamed Bhuyain

Abstract

We discuss one of the newer fields of mathematics - noncommutative geometry. Noncommutative geometry has been developed in recent years with a high degree of aspiration of physical and purely mathematical applications. We will start by describing the fundamental ideas that gave birth to this promising area of modern mathematics. We will first discuss Gelfand-Naimark theorem and it's consequences. Then we will move onto the concept of a spectral triple - the noncommutative manifold. Finally we will look at an explicit example of noncommutative space, the noncommutative torus (two dimensional). Time permitting, we will discuss an example of a result in noncommutative geometry in contrast with it's classical analog.

One of the promising modern branches of mathematics is noncommutative geometry. First formulated in the latter part of last century, it is very much work in progress. Active research is going on to understand and generalize the idea of noncommutative spaces and analyze it's correspondence with classical geometry. Primarily motivated by physics, noncommutative geometry by default has the possibility of many interesting physical applications including particle physics models, statistical dynamics model, black holes and even string theory. Beyond physics, the techniques involved with noncommutative geometry have been found useful in other areas of mathematics as well - algebraic geometry, arithmetic number theory being prominent examples. Over the years, quite a few important classical results have been shown to also hold in the noncommutative analog, at least for certain prototype noncommutative spaces. So the idea of noncommutative geometry is a very natural generalization, even without physical consideration.

A sort of 'main theorem' in noncommutative geometry is the Gelfand-Naimark theorem which tells us that the categories of locally compact topological spaces and C*-algebras are equivalent and we lose no information by passing to one from the other. With the Gelfand-Naimark theorem as the primary tool one can characterize certain topological properties as being parallel to algebraic properties. In the classical case, this theorem tells us that a topological space contain the same information as the space of continuous functionals on it. The latter is a commutative C*-algebra. Again, the theorem tells us that a commutative C*-algebra has the same information as the space of characters i.e. multiplicative linear functionals on it and that the underlying topological space and the space of characters are the same (homeomorphic). Similarly the space of continuous linear functionals on the space of characters is isomorphic to the underlying C*-algebra. This opens a new door for mathematics, where one can consider a noncommutative C*-algebra, and attempt to make sense of a corresponding topological space. This is the beginning of the study of noncommutative geometry.

However, noncommutativity comes with a price, as we immediately find out that the space of character for a noncommutative C*-algebra is almost always trivial or discrete. This implies that the classical notion of points does not work very well for noncommutative geometry. So one needs to create appropriate analogs of the notions in classical topology or geometry for noncommutative spaces, and this is not automatic.

The most interesting types of noncommutative spaces are described by spectral triples. These are analogs of manifolds in classical geometry. The minimum structure of a spectral triple consists of three objects (the triple) - an algebra (C*), a Hilbert space, and a Dirac operator. In actual studies, mathematicians put in more structure as needed to draw analogs with classical geometry and to make sure that all parts of the construction are compatible with each other and nicely behaved. An interesting class of spectral triple are even real spectral triples which are associated with a $\mathbb{Z}_2$-grading and an antilinear involution. The Dirac operator, in particular, works as an infinitesimal by which we can define the noncommutative integral.

One particularly interesting noncommutative space is the noncommutative two dimensional torus. This is the prototype noncommutative space which contains most of the essence of noncommutative geometry and at the same time has a very explicit algebraic structure which enables us to extract a lot of information about the space. It can be described as an irrational rotation algebra which is a universal C*-algebra generated by two unitaries that satisfy a particular commutation relation. It can also be described in terms of a space of Fourier series' with rapidly decreasing coefficients. The noncommutative torus has been studied extensively and many interesting results and applications have been found.

While noncommutative geometry is a relatively new branch of mathematics, it is being enriched by new generalizations, results and applications very rapidly. It is hoped that advancement in this field will not only further enrich our knowledge of geometry but also lead us to useful techniques in modeling physical systems and in other areas of mathematics. Some of these have already been achieved, while a lot more remains to be done in the future.