Modular field generators

Fricke functions f_u $\quad u = (a,b) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$

$$f_{(\frac{a}{N},\frac{b}{N})}(\tau) = -2^7 3^5 \frac{g_2 g_3}{\Delta} E_{(\frac{a}{N},\frac{b}{N})}(q) \qquad q = e^{2\pi i \tau}$$

where

for $y^2 = 4x^3 - g_2 x - g_3$ $\quad g_2(\tau), g_3(\tau)$

discriminant $\Delta = g_2^3 - 27 g_3^2$ $\quad$ and

$$E_{(\frac{a}{N},\frac{b}{N})}(q) = \frac{1}{12} + \frac{z}{(1-z)^2} + \sum_{n,\, d|n} d\,(z^d + z^{-d} - 2)\, q^n$$

when $z = e^{2\pi i \frac{a}{N}} q^{-\frac{b}{N}}$

Properties: $\quad$ ($F_N \subset F_N(\mathbb{C})$ generated over $\mathbb{Q}$ by j & f_u)

(1) $f_u \in F_N(\mathbb{C})$

(2) q- expansion has all coeff's in $\mathbb{Q}(e^{2\pi i/N})$

(3) $j(\tau)$ $\quad$ j-invariant of elliptic curve

$$j(\tau) = 256(1-f+f^2)^3 / (f^2(1-f)^2)$$

with $f = (f_{(\frac{1}{2},\frac{1}{2})} - f_{(0,\frac{1}{2})}) / (f_{(\frac{1}{2},0)} - f_{(0,\frac{1}{2})})$

(4) the f_u $\quad u \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$ $\quad u \neq 0$

generate $F_N(\mathbb{C})$ over $\mathbb{C}(j)$

(5) F_N Galois extension of $\mathbb{Q}(j)$ w/ Galois group

$$GL_2(\mathbb{Z}/N\mathbb{Z}) / \{\pm 1\}$$

(6) If $\tau \in \mathbb{H}$ s.t. $j(\tau) \notin \bar{\mathbb{Q}}$ transcendental then $j \overset{\varepsilon_\tau}{\longmapsto} j(\tau)$, $f_u \mapsto f_u(\tau)$

determines embedding $F_N \hookrightarrow \mathbb{C}$

Adeles :

$$\mathbb{A}_{\mathbb{Q}} = \prod_{v \in \Sigma_{\mathbb{Q}}}^{res} \mathbb{Q}_v$$

$\xleftarrow{\quad}$ all but fin. many in $\mathbb{Z}_p$

places of $\mathbb{Q}$: non-archim. places = primes of $\mathbb{Z}$ = p primes; archim. places = embeddings $\mathbb{Q} \hookrightarrow \mathbb{C}$, unique one for $\mathbb{Q}$

where $\mathbb{Q}_v = \mathbb{Q}_p$ - padic completion of $\mathbb{Q}$, or $\mathbb{R}$ - archimedean completion of $\mathbb{Q}$ $\leftarrow$ local fields

Equivalent description

$$\hat{\mathbb{Z}} = \lim_{\leftarrow n} \mathbb{Z}/n\mathbb{Z}$$

$$\mathbb{A}_{\mathbb{Q},f} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \qquad \mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q},f} \times \mathbb{R}$$

finite adeles $\qquad$ full adeles

$\hat{\mathbb{Z}}$ is maximal compact subring of $\mathbb{A}_{\mathbb{Q},f}$

$$\hat{\mathbb{Z}} = \prod_{p \text{ primes}} \mathbb{Z}_p$$

idèles = invertible adèles $\qquad \mathbb{A}_{\mathbb{Q}}^* = \mathbb{A}_{\mathbb{Q},f}^* \times \mathbb{R}_+^*$

$$\mathbb{A}_{\mathbb{Q}}^* = GL_1(\mathbb{A}_{\mathbb{Q}}) \qquad \mathbb{A}_{\mathbb{Q},f}^* = GL_1(\mathbb{A}_{\mathbb{Q},f})$$

Note: topology on $\mathbb{A}_{\mathbb{Q}}^*$ not induced from $\mathbb{A}_{\mathbb{Q}}$ but from $x \mapsto (x, x^{-1})$ embedding

$GL_1(\mathbb{Q}) \subset \mathbb{A}_{\mathbb{Q}}^*$ discrete subgrp quotient

$C_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^* / \mathbb{Q}^*$ idèles class group

$C_{\mathbb{Q}}/D_{\mathbb{Q}} \cong \mathbb{A}^*_{\mathbb{Q},f}/\mathbb{Q}^*_+ \cong \hat{\mathbb{Z}}^*$ (related to abelian class field theory of $\mathbb{Q}$: max abelian extension $\mathbb{Q}^{cycl}$ & Galois group) (3)

$D_{\mathbb{Q}}$: conn. comp. of identity

Shimura varieties for GL_1 & GL_2

$$Sh(G,X) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times X$$

- $Sh(GL_2, \mathbb{H}^\pm) = GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathbb{H}^\pm$

$$= GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathbb{H} = GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_{\mathbb{Q}}) / \mathbb{C}^*$$

full adeles

- Components:

$$Sh(GL_1, \{\pm 1\}) = GL_1(\mathbb{Q}) \backslash GL_1(\mathbb{A}_f) \times \{\pm 1\} = \mathbb{Q}^*_+ \backslash \mathbb{A}^*_f \ (\cong \hat{\mathbb{Z}}^*)$$

$$\pi_0(Sh(GL_2, \mathbb{H}^\pm)) = Sh(GL_1, \{\pm 1\})$$

Yet one more way to see BC algebra

$C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$ is Morita equivalent to

$$C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}^*_+$$

Morita equiv. realized through a projector onto $\hat{\mathbb{Z}} \subset \mathbb{A}_{\mathbb{Q},f}$: char. function of $\hat{\mathbb{Z}}$ compressing algebra with that

Then view

(*) $$C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}_+^*$$

as a "noncommutative quotient" version of

$$\mathbb{A}_{\mathbb{Q},f}/\mathbb{Q}_+^*$$

Notice: while quotient $\mathbb{A}_{\mathbb{Q},f}^*/\mathbb{Q}_+^*$ is a nice quotient and equal to $\hat{\mathbb{Z}}^*$ good classical space

The action of $\mathbb{Q}_+^*$ on $\mathbb{A}_{\mathbb{Q},f}$ is known to be ergodic
$\Rightarrow$ quotient collapses too much!
NC quotient (*) is good replacement.

The "good quotient part" is recovered from (*) as the extremal low temperature KMS states (classical points of the system)

Similar situation for the GL_2-case

Classification of KMS states:

interesting phenom. at $\beta = 2$

- $\beta \le 1$ No KMS state at all
- $1 < \beta < 2$ unique KMS state
- $\beta > 2$ $\quad \mathcal{E}_\beta = GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \mathbb{C}^*$ — classical Shimura variety of GL_2

extremal

$$\varphi_{\beta, L}(f) = \frac{1}{Z(\beta)} \sum_{m \in \Gamma \backslash M_2(\mathbb{Z})} f(1, m\rho, m(z)) \det(m)^{-\beta}$$

invertible

$$Z(\beta) = \zeta(\beta)\,\zeta(\beta - 1)$$