Tuesday April 6

2-dimensional $\mathbb{Q}$-lattices

(Λ, ϕ) $\qquad \Lambda \subset \mathbb{C} \cong \mathbb{R}^2$

$\{\underset{e_1}{1}, \underset{e_2}{-i}\}$ basis of $\mathbb{C}$ as $\mathbb{R}$-vector space

$GL_2^+(\mathbb{R})$ acts on $\mathbb{C}$ by $\mathbb{R}$-linear transformations

for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$

set $\Lambda_0 = \mathbb{Z}e_1 + \mathbb{Z}e_2 = \mathbb{Z} + i\mathbb{Z}$ standard lattice

$M_2(\hat{\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2)$

or equivalently $\rho \in M_2(\hat{\mathbb{Z}})$ $\quad \rho : \mathbb{Q}^2/\mathbb{Z}^2 \longrightarrow \mathbb{Q}\Lambda_0/\Lambda_0$

$\rho(a) = \rho_1(a)e_1 + \rho_2(a)e_2$

Then describe (Λ, ϕ) 2-dim $\mathbb{Q}$-lattice as

$(\Lambda, \phi) = (\alpha^{-1}\Lambda_0, \alpha^{-1}\rho)$ for some $\alpha \in GL_2^+(\mathbb{R})$, $\rho \in M_2(\hat{\mathbb{Z}})$

$(\alpha^{-1}\Lambda_0, \alpha^{-1}\rho) = (\beta^{-1}\Lambda_0, \beta^{-1}\rho')$ iff

$\beta\alpha^{-1}\Lambda_0 = \Lambda_0$ and $\beta\alpha^{-1}\rho = \rho'$

$\Rightarrow \beta\alpha^{-1} = \gamma \in SL_2(\mathbb{Z})$ and $\rho' = \gamma\rho \qquad \gamma \in SL_2(\mathbb{Z})$

$(\beta, \rho') = (\gamma\alpha, \gamma\rho)$

So 2-dim $\mathbb{Q}$-lattices $\quad \Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R}))$

$\Gamma \leftarrow$ diagonal action of $SL_2(\mathbb{Z})$

$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$

$\alpha(xe_1 + ye_2) = (ax+by)e_1 + (cx+dy)e_2$

$\mathbb{C}^* \ni \lambda = a+ib = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL_2^+(\mathbb{R})$

acts as above

$\Rightarrow$ $\lambda(\Lambda, \phi) = (\lambda\Lambda, \lambda\phi)$ scaling action of $\mathbb{C}^* \subset GL_2^+(\mathbb{R})$

i.e. $(\alpha, \rho) \longmapsto (\lambda^{-1}\alpha, \rho)$

Then 2-dim $\mathbb{Q}$-lattices up to scaling

$$\underset{\substack{\uparrow \\ \text{diag. action of } \Gamma}}{\Gamma} \backslash \left(M_2(\hat{\mathbb{Z}}) \times_{\cancel{}} GL_2^+(\mathbb{R}) / \mathbb{C}^* \right) = \Gamma \backslash \left(M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \right)$$

in fact

$GL_2^+(\mathbb{R})$ acts on upper half plane $\mathbb{H}$ by fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d}$$

stabilizer of point $z = i$ is $\mathbb{C}^*$

so map $\alpha \longmapsto \alpha(i)$

$GL_2^+(\mathbb{C}) \longrightarrow \mathbb{H}$ identifies $\mathbb{H} = GL_2^+(\mathbb{C}) / \mathbb{C}^*$

$\mathbb{Q}$-lattices up to scale $\leftrightsquigarrow$ pairs (ρ, z)

$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}z), \lambda(\rho_1 - z\rho_2))$

Commensurability relation

- Partially defined action of $GL_2^+(\mathbb{Q})$

if $(\Lambda, \phi) \leftrightsquigarrow (\rho, \alpha)$ $\quad \rho \in M_2(\hat{\mathbb{Z}})$ $\quad \alpha \in GL_2^+(\mathbb{R})$

then $\quad g \in GL_2^+(\mathbb{Q})$ s.t. $g\rho \in M_2(\hat{\mathbb{Z}})$

gives commensurable lattice $(g\rho, g\alpha)$

but notice : orbits of $\Gamma = SL_2(\mathbb{Z})$

So more precisely start with groupoid

$$\tilde{\mathcal{G}}_2 = \{ (g, \rho, \alpha) \in GL_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R}) \; : \; g\rho \in M_2(\hat{\mathbb{Z}}) \}$$

units of groupoid $\tilde{\mathcal{G}}_2^{(0)} = M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R})$

then take a quotient by an action of $\Gamma \times \Gamma$ $\qquad \Gamma = SL_2(\mathbb{Z})$

$$\Gamma\times\Gamma \backslash \tilde{\mathcal{G}}_2 \subset {}_{\Gamma}\backslash GL_2^+(\mathbb{Q}) \times_{\Gamma} (M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R}))$$

$\parallel$

$\mathcal{G}_2 \qquad (\gamma_1, \gamma_2) : (g, \rho, \alpha) \longmapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 \alpha)$

it is the groupoid of the commensurability relation:

$$(g, \rho, \alpha) \longmapsto ((\alpha^{-1} g^{-1} \Lambda_0, \alpha^{-1}\rho), (\alpha^{-1}\Lambda_0, \alpha^{-1}\rho))$$

class mod $\Gamma\times\Gamma$

- Groupoid algebra

$$f_1 * f_2 (g, \rho, \alpha) = \sum_{h \in \Gamma\backslash GL_2^+(\mathbb{Q}) \, : \, h\rho \in M_2(\hat{\mathbb{Z}})} f_1(gh^{-1}, h\rho, h\alpha)\, f_2(h, \rho, \alpha)$$

When considering 2-dim $\mathbb{Q}$-lattices mod $\mathbb{C}^*$-scaling

Would like to take $\mathcal{G}_2/\mathbb{C}^*$

but composition law of groupoid not well defined in the quotient

example : take point $z = i \in \mathbb{H}$ and $\mathbb{Q}$-lattices

$$(\Lambda, \phi) = (\mathbb{Z} + 2i\mathbb{Z}, 0) \qquad (\Lambda', \phi') = (\mathbb{Z} + i\mathbb{Z}, 0)$$

they are commensurable

Composition : $((\Lambda, \phi), (\Lambda', \phi')) \circ ((\Lambda', \phi'), (\Lambda, \phi))$
in $\mathcal{G}_2$ is $((\Lambda, \phi), (\Lambda, \phi))$ ok

in $\mathcal{G}_2$ also consider pair

$$(\, i(\Lambda, \phi),\ i(\Lambda', \phi'))$$

can still compose with $((\Lambda', \phi'), (\Lambda, \phi))$
because i is symm. of (Λ', ϕ') (autom.)

so $i(\Lambda', \phi') = (\Lambda', \phi')$

composition in $\mathcal{G}_2$ is $\underbrace{(i(\Lambda, \phi), (\Lambda, \phi))}_{\text{not a unit of } \mathcal{G}_2} \neq \underbrace{((\Lambda, \phi), (\Lambda, \phi))}_{\text{unit of } \mathcal{G}_2}$

but in $\mathcal{G}_2/\mathbb{C}^*$ $(i(\Lambda, \phi), i(\Lambda', \phi'))$ becomes equal to $((\Lambda, \phi), (\Lambda', \phi'))$

so composition <u>not</u> well defined

Can get around this problem and still obtain convolution algebra for $\mathbb{Q}$-latt mod scaling, just not ~~same~~ groupoid algebra take same convolution product

$$(f_1 * f_2)(g,\rho,\alpha) = \sum_{\substack{h \in \Gamma\backslash GL_2^+(\mathbb{Q}) \\ h\rho \in M_2(\hat{\mathbb{Z}})}} f_1(gh^{-1}, h\rho, h\alpha)\, f_2(h,\rho,\alpha)$$

but applied to functions on $\Gamma\backslash GL_2^+(\mathbb{Q}) \times_\Gamma (M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R}))$

that are not of comp. support but are invariant under $\mathbb{C}^*$-actions on $GL_2^+(\mathbb{R})$

and comp. supp in $\mathbb{H}$ in α

i.e. $f(g,\rho,\alpha\lambda) = \lambda^k f(g,\rho,\alpha)$

homogeneous of weight k

choose those that homogeneous of weight 0 (invariant)

$$\begin{cases} (f_1 * f_2)(g,\rho,z) = \displaystyle\sum_{\substack{h \in \Gamma\backslash GL_2^+(\mathbb{Q}) \\ h\rho \in M_2(\hat{\mathbb{Z}})}} f_1(gh^{-1}, h\rho, h(z))\, f_2(h,\rho,z) \\ f^*(g,\rho,z) = \overline{f(g^{-1}, g\rho, g(z))} \end{cases}$$

$\mathcal{A}_2$ resulting algebra

<u>Representations</u> : $y \in M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ $\qquad y = (\rho, z)$

$$G_y = \{ g \in GL_2^+(\mathbb{Q}) : g\rho \in M_2(\hat{\mathbb{Z}}) \}$$

$$(\pi_y(f)\,\xi)(g) := \sum_{\left.\substack{h \in \Gamma\backslash GL_2^+(\mathbb{Q}) \\ h\rho \in M_2(\hat{\mathbb{Z}})}\right\} h \in \Gamma\backslash G_y} f(gh^{-1}, h\rho, h(z))\, \xi(h)$$

Repres. of $\mathcal{A}_2$

$\|f\| = \sup_y \|\pi_y(f)\|$

f compact support $\Rightarrow \sup < \infty$

Note ~~exists~~ $\Gamma \backslash M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ not compact

so $f(g,\rho,z) = \begin{cases} 1 & g \in \Gamma \\ 0 & g \notin \Gamma \end{cases}$ not comp. support

so $\mathcal{A}_2$ not a unital algebra

Time evolution :

$$\sigma_t(f)(g,\rho,z) = \det(g)^{it} f(g,\rho,z)$$

again as in 1-dim case

$$\cancel{\det}\ \det(g) = \frac{Covol(\Lambda')}{Covol(\Lambda)}$$

Symmetries :

$$GL_2^+(\mathbb{Q}) \cdot GL_2(\hat{\mathbb{Z}}) = GL_2(\mathbb{A}_{\mathbb{Q},f})$$

note $GL_2^+(\mathbb{Q}) \cap GL_2(\hat{\mathbb{Z}}) = SL_2(\mathbb{Z})$

Action of $GL_2(\hat{\mathbb{Z}})$:

$$\gamma : (\Lambda, \phi) \longmapsto (\Lambda, \phi \circ \gamma)$$

$$\gamma : \mathbb{Q}^2/\mathbb{Z}^2 \xrightarrow{\sim} \mathbb{Q}^2/\mathbb{Z}^2$$

preserves commensurability : $(\Lambda,\phi) \sim (\Lambda',\phi') \Leftrightarrow (\Lambda,\phi\circ\gamma) \sim (\Lambda',\phi'\circ\gamma)$

on algebra automorphisms

$$\vartheta_\gamma(f)(g,\rho,z) = f(g,\rho\gamma,z)$$

compatible with time evolution

$$\vartheta_m(f)(g,\rho,z) = \begin{cases} f(g,\rho m \det(m)^{-1}, z) & \text{if } \rho m \det(m)^{-1} \in M_2(\hat{\mathbb{Z}}) \text{ and } g\rho m\det(m)^{-1} \in M_2(\hat{\mathbb{Z}}) \\ 0 & \text{otherwise} \end{cases}$$

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action by endomorphisms

Of these certain endom. are <u>inner</u>

$n \in N \subset M_2^+(\mathbb{Z})$ inner endom.

$\vartheta_n(f) = \mu_n f \mu_n^*$

$$\mu_n(g,\rho,z) = \begin{cases} 1 & g \in \Gamma n \\ 0 & \text{otherwise} \end{cases}$$

matrix

partially def. action of $\mathbb{Q}_+^\times \subset GL_2^+(\mathbb{Q})$

$\Rightarrow$ symmetries mod inner $\quad \mathbb{Q}_+^* \backslash GL_2^+(\mathbb{Q})\cdot GL_2(\hat{\mathbb{Z}}) = \underbrace{\mathbb{Q}_+^* \backslash GL_2(\mathbb{A}_{\mathbb{Q},f})}_{\text{Automorphisms of the modular field}}$

<u>Modular field</u> (field of modular functions)

Congruence groups level N:

$\Gamma(N) = \{ \gamma \in \Gamma = SL_2(\mathbb{Z}) : \gamma \equiv 1 \bmod N \}$

Notation: $\alpha \in GL_2^+(\mathbb{R})$

$$(f|_k \alpha)(z) = \det(\alpha)^{k/2} f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-k}$$

for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Modular functions of level N:

f holomorphic function on $\mathbb{H}$ s.t.

$$f|\gamma = f \qquad \forall \gamma \in \Gamma(N)$$

F_N field of mod functions of level N

$F = \bigcup_N F_N$ field of mod functions

- There are explicit generators for F_N
- F can be embedded (in many ways) as a subfield of $\mathbb{C}$
- Embeddings $F \subset \mathbb{C}$ parameterized by invertible 2-dim $\mathbb{Q}$-lattices (Λ, ϕ)
- $\mathrm{Aut}(F) = \mathbb{Q}^*_+ \backslash GL_2(\mathbb{A}_{\mathbb{Q},f})$

Analog of $\mathcal{A}_{\mathbb{Q}}$ arithmetic subalgebra of BC system will involve modular functions here