Thursday Apr 29

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$\mathcal{A}$ = Bost-Connes algebra σ = time evol.

φ state $\int_{\hat{\mathbb{Z}}} f(1,\rho)\, d\mu(\rho)$

$D(\mathcal{A},\varphi) = \mathrm{Coker}(\delta)$

$\hat{\mathcal{A}}_\beta \xrightarrow{\delta} C(\tilde{\Omega}_\beta)$

$HC_*(D(\mathcal{A},\varphi))$

Cokernel not an algebra but a cyclic module

$\Rightarrow$ can compute HC_* as Ext in cat. of cyclic modules

with induced scaling action

$\theta : \mathbb{R}_+^* \longrightarrow \mathrm{Aut}(HC_0(D(\mathcal{A},\varphi)))$

Note: dual $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$

is algebra of equiv. rel. of commensurability on 1-dim $\mathbb{Q}$-lattices not up to scaling

$C^*(\mathcal{G}_1)$ the map $i : \hat{\mathcal{A}} \xrightarrow{\cong} C^*(\mathcal{G}_1)$

Ψ

X

$$i(X)(k,\rho,\lambda) = \int_{\mathbb{R}} x(t)(k,\rho)\, \lambda^{it}\, dt$$

for $X = \int x(t)\, U_t\, dt$

$x(t) \in \mathcal{A}$ i.e. $f(k,\rho)$

convolution prod. compatible

$$i(X_1X_2)(k,\rho,\lambda) = \iint (x_1(t) * \sigma_t(x_2(s-t)))(k,\rho)\, \lambda^{is}\, dt\, ds$$

$$= \sum_{r,\, r\rho \in \hat{\mathbb{Z}}} \iint x_1(t)(kr^{-1}, r\rho)\, \sigma_t(x_2(s-t))(r,\rho)\, \lambda^{is}\, dt\, ds$$

$$= \sum_{r,\, r\rho \in \hat{\mathbb{Z}}} \iint x_1(t)(kr^{-1}, r\rho)\, x_2(s-t)(r,\rho)\, r^{it} \lambda^{is}\, dt\, ds = \sum_{r,\, r\rho \in \hat{\mathbb{Z}}} i(X_1)(kr^{-1}, r\rho, r\lambda)\; i(X_2)(r,\rho,\lambda)$$

then map δ on $\hat{A}$ also described as

$$\delta : \hat{A} \longrightarrow C(\tilde{\Omega}_\beta)$$

$$f = i(X) \in C^*(G_1)$$

$$\delta(X)(u,\lambda) = \sum_{n \in \mathbb{N}} f(1, nu, n\lambda)$$

$$(u,\lambda) \in \hat{\mathbb{Z}}^* \times \mathbb{R}^*_+ = C_{\mathbb{Q}} \simeq \tilde{\Omega}_\beta$$

In fact have repres. on $\ell^2(\mathbb{N})$:

$$(\pi_{(u,\lambda)}(x)\,\xi)(n) = \sum_{m \in \mathbb{N}} x(nm^{-1}, mu)\,\xi(m) \qquad (*)$$

$\pi_{(u,\lambda)}(H)$ = diagonal operator D_λ on $\ell^2(\mathbb{N})$

$$(D_\lambda \xi)(n) = (\log n + \log\lambda)\,\xi(n)$$

$$\mathrm{Sp}(D_\lambda) = \{\log n + \log\lambda\}_{n \in \mathbb{N}}$$

$\delta(x)(u,\lambda) = \mathrm{Tr}(\pi_{(u,\lambda)}(x))$ by construction of map δ

where $\pi_{(u,\lambda)}(X) = \int \pi_{(u,\lambda)}(x(t))\, e^{itD_\lambda}\, dt$

because rep given by $(*)$

Trace = sum diagonal entries of matrix

$$\sum_{n \in \mathbb{N}} \int x(t)(1, nu)\, e^{it(\log n + \log\lambda)}\, dt$$

$$= \sum_{n \in \mathbb{N}} f(1, nu, n\lambda)$$

for $f = i(X)$

Note : a function $f(\rho,\lambda)$ $(\rho,\lambda)\in\hat{\mathbb{Z}}\times\mathbb{R}_+^*$

has an extension

$\tilde{f}(x)$ $x\in\mathbb{A}_{\mathbb{Q}}$ obtained by

setting $\tilde{f}=0$ outside of $\hat{\mathbb{Z}}\times\mathbb{R}_+ \subset \mathbb{A}_{\mathbb{Q}}$

to have ext. from $\mathbb{R}_+^*$ to $\mathbb{R}_+$ need a condition on the behavior of f at $\lambda\to 0$

Control of behavior as $\lambda\to 0$ in the BC algebra case :

$$\boxed{\mathrm{Tr}\left(\pi_{\varepsilon,H+c}(x)\right) \underset{\text{as } c\to-\infty}{=} e^{-c}\,\tau(x) + O(|c|^{-N})}$$

where τ is the canonical dual trace on $\hat{\mathcal{A}}$

Let $\varphi\in KMS_\beta$ for $(\mathcal{A},\sigma_t)$

$\Rightarrow$ dual weight $\hat{\varphi}\left(\int_{\mathbb{R}} x(t)\,U_t\,dt\right) := \int_{\mathbb{R}} \varphi(\hat{x}(s))\,ds$

with $\hat{x}(s) = \int_{\mathbb{R}} x(t)\,e^{its}\,dt$

If $\beta=0 \Rightarrow \hat{\varphi}$ is a trace

If $\beta\neq 0$: $X_\beta := \int_{\mathbb{R}} x_\beta(t)\,U_t\,dt$ $\qquad x_\beta(t) = x(t+i\beta)$

for $\pi_{\varepsilon,H}(X) = \int_{\mathbb{R}} \pi_\varepsilon(x(t))\,e^{itH}\,dt$ have

$\pi_{\varepsilon,H}(X_\beta) = \pi_{\varepsilon,H}(X)\,e^{\beta H}$

then obtain a dual trace by $\tau_\varphi(X) := \hat{\varphi}(X_\beta)$ if $\varphi\in KMS_\beta$

Also this satisfies scaling property

$$\tau_\psi(\theta_\lambda(X)) = \lambda^{-\beta}\,\tau_\psi(X)$$

In fact: $\tau_\psi(X) = \int_{\mathbb{R}} \psi(\hat{x}(s))\, e^{\beta s}\, ds$

$\tau_\psi(XY) = \int_{\mathbb{R}} \psi(\widehat{x * y}(s))\, e^{\beta s}\, ds$ then apply KMS property of ψ when exchanging order

also $\tau_\psi(\theta_\lambda(X)) = \iint \lambda^{i(t+i\beta)}\, \psi(x(t+i\beta))\, e^{its}\, dt\, ds = \lambda^{-\beta}\,\tau_\psi(X)$

* Any trace on $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$ that scales like
$\tau(\theta_\lambda(X)) = \lambda^{-\beta}\,\tau(X)$ comes in this way from a KMS_β state on $(\mathcal{A}, \sigma)$

Then the behavior of $\mathrm{Tr}(\pi_{\xi, H_c}(X))$ for $c \to -\infty$ shows that if take cyclic module

$$\hat{\mathcal{A}}^\natural_{\beta,\tau} = \mathrm{Ker}(\tau_\varphi)$$ kernel of dual trace (use φ or use ε state)

then have decay condition as $\lambda \to 0$

$\Rightarrow$ $f(\rho, \lambda) = i(X)(\rho, \lambda)$ extends to $\lambda = 0$
and then to $\tilde{f}(x)$ on $\mathcal{A}_{\mathbb{Q}}$ by zero

Need it above to pass to X_β

Note: also taking here $\hat{\mathcal{A}}^\natural_\beta \supset \hat{\mathcal{A}}^\natural_{\beta,\tau}$
where $\hat{\mathcal{A}}_\beta \subset \hat{\mathcal{A}}$ linear subspace of elements $x \in \mathcal{S}(\mathbb{R}, \mathcal{A})$
s.t. $x(t)$ analytically continues to $x(z)$ $z \in I_\beta$ bounded
$x(t+i\beta)$ has rapid decay $\in \mathcal{S}(\mathbb{R}_\beta, \mathcal{A})$

Then the map $\delta(X)(\rho,\lambda) = \sum_{n\in\mathbb{N}} f(1, n\rho, n\lambda)$ (5)

$i(X) = f$ can also be written as

$$\sum_{n\in\mathbb{N}} f(1, n\rho, n\lambda) = \sum_{q\in\mathbb{Q}^*} \tilde{f}(qx)$$

with $x = (\cancel{\xi}_f, \cancel{\xi}_\infty) \in C_{\mathbb{Q}}$ with $\underline{x = (\rho,\lambda)}$ invertible $\mathbb{Q}$-lattice since pt in Ω_β (extr. KMS_β states at low temperature)

$\hat{C}_{\mathbb{Q}}$ $\cap$

Explicitly the dual trace for BC algebra

$$\tau_\varphi(X) = \int_{\hat{\mathbb{Z}}\times\mathbb{R}_+^*} f(1,\rho,\lambda)\, d\mu(\rho)\, d\lambda \qquad \varphi \in KMS_1$$

for $f = i(X)$ $\quad X = \int x(t)\, U_t\, dt$ $\quad d\mu$ = Haar measure on $\hat{\mathbb{Z}}$

and have $\sum_{n\in\mathbb{N}} f(1, nu, n\lambda) = \lambda^{-1} \int_{\hat{\mathbb{Z}}\times\mathbb{R}_+^*} f(1,\rho,v)\, d\rho\, dv + O(|\log(\lambda)|^{-N})$

Take then $\mathcal{E} = HC_0(D(\mathcal{A},\varphi))$

Algebra of holomorphic multipliers

$\mathrm{Hol}(I_\beta)$ = holomorphic functions h on strip I_β s.t. $h\, \mathcal{S}(I_\beta) \subset \mathcal{S}(I_\beta)$

then for $h \in \mathrm{Hol}(I_\beta)$

$$\theta(\hat{h}) := \int_{\mathbb{R}_+^*} \hat{h}(\lambda)\, \theta_\lambda\, d^*\lambda$$

$d^*\lambda = \frac{d\lambda}{\lambda}$

$\int_{\mathbb{R}_+^*} \hat{h}(\lambda)\, \lambda^{it}\, d^*\lambda = h(t)$

operator acting on $\hat{\mathcal{A}}_\beta$ module over algebra $\mathrm{Hol}(I_\beta)$

Gives an induced $Hol(I_\beta)$-module structure to
$\mathcal{E} = HC_0(D(A,\varphi))$

Spectral realization :

$z \in I_\beta$ $\rightsquigarrow$ character of algebra $Hol(I_\beta)$
$\rightsquigarrow$ 1-dim representation $\mathbb{C}_z$

$$\mathcal{E} \otimes_{Hol(I_\beta)} \mathbb{C}_z \neq \{0\} \quad \text{iff} \quad \xi(-iz) = 0$$

from $(\mathcal{F} h)(t) = \int_{\mathbb{R}_+^*} h(\lambda)\, \lambda^{-it}\, d^*\lambda = \int_{\mathbb{A}_{\mathbb{Q}}^*} \tilde{f}(x)\, |x|^s\, d^*x$

(or version with characters) h in range of δ

when $h(\lambda) = \int_{\hat{\mathbb{Z}}^*} \Sigma(\tilde{f})(mx) \dfrac{d^*m}{\chi_0(m)}$ $\qquad \chi_0 = \bar{\chi}$ conj of χ

$\Sigma(\tilde{f})(x) = \sum_{q \in \mathbb{Q}^*} \tilde{f}(qx)$ $\qquad \Sigma(\tilde{f})(mx) = \chi(m)\, \Sigma(\tilde{f})(x)$

then one shows that $\mathcal{F}h \in \mathcal{S}(I_\beta)$ and $\mathcal{F}h(z) = 0$
for all $z \in I_\beta$ s.t. $\xi(-iz) = 0$

for h in range of δ : set of common zeros of all $\mathcal{F}h$ is $\{z \in I_\beta \cdot \xi(-iz) = 0\}$

(Zeros of) Riemann zeta function and the map δ

$$\tilde{f} \longmapsto \sum_{q \in \mathbb{Q}^*} \tilde{f}(qx)$$

Riemann zeta function: $\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_p (1-p^{-s})^{-1}$

w/ analytic continuation

"trivial zeros" at $-2, -4, \ldots, -2n$

"non trivial zeros" location?

relation between zeros & primes (explicit formulae)

Riemann hypothesis: all of form

$\rho = \frac{1}{2} + i\alpha$ with α real

i.e. on the line $\mathrm{Re}(z) = \frac{1}{2}$

Counting of primes

$\pi(x) = \#\{p \text{ prime } p \leq x\}$

prime number theorem: asympt. behavior

$$\pi(x) \sim \frac{x}{\log x}$$

set $Li(x) = \int_0^x \frac{du}{\log u} \sim \sum_k (k-1)! \frac{x}{\log(x)^k}$

integral logarithm

~~then~~ also set $\pi'(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots + \frac{1}{n}\pi(x^{1/n}) + \cdots$

then

$$\pi'(x) = Li(x) - \sum_\rho Li(x^\rho) + \int_x^\infty \frac{dt}{t(t^2-1)\log t} + \log \xi(0)$$

non-trivial zeros of ζ

Other more refined versions of explicit formula (Weil)

$$\xi(t) = \frac{s(s-1)}{2}\Gamma(s/2)\pi^{-s/2}\zeta(s)$$

$s = \frac{1}{2} + it$ $\quad \xi(t) = \xi(0)\prod_\alpha \left(1 - \frac{t}{\alpha}\right)$

$\{\alpha \in \mathbb{C} : \xi(\alpha) = 0\}$

Note: $\pi(x)$ has a closed formula

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{2\pi i\,\Gamma(k)/k} - 1}{e^{-2\pi i/k} - 1}$$

but rapidly oscillating
not computationally useful

Another Version of the explicit formula: zeros & primes

$\Lambda(n)$ = von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & n = p^k \quad p \text{ prime}, k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

$$\psi(x) = \sum_{n \le x} \Lambda(n)$$

step function starts at 0, jumps by $\log p$ at each prime power

at discontinuities set $\psi(x) = \dfrac{\psi(x_+) + \psi(x_-)}{2}$ average of nearby values

then explicit formula also gives

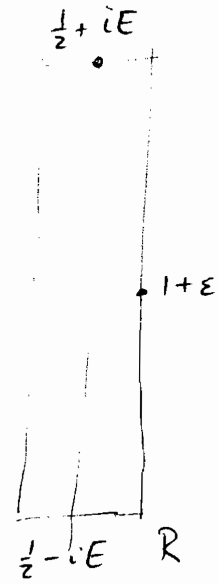
$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}) \qquad \forall x > 1$$

non trivial zeros of ξ

More general distributional explicit formula (Weil) of which these are special cases

$\rightsquigarrow$ explicitly counting primes (distribution of prime numbers among integers) related to distribution and location of non-trivial zeros of Riemann zeta function

Riemann's estimate of # of zeros growth



$$N(E) = \#\{\rho : \xi(\rho) = 0, \quad 0 < \mathrm{Im}(\rho) \le E\}$$

$$N(E) \sim \frac{E}{2\pi} \log \frac{E}{2\pi} - \frac{E}{2\pi}$$

Obtained from proving that

$$N(E) = \frac{\theta(E)}{\pi} + 1 + \frac{1}{\pi} \mathrm{Im}\left(\int_C \frac{\xi'(s)}{\xi(s)}\, ds \right)$$

$\theta(E)$ = Siegel angular function

$$= -\frac{E}{2} \log \pi + \mathrm{Im}\left(\log \Gamma\left(\frac{1}{4} + i\frac{E}{2}\right)\right)$$

C = part of ∂R from $1+\varepsilon$ to $\frac{1}{2} + iE$

$$2\left(N(E)-1\right) = \frac{1}{2\pi i}\int_{\partial R} \frac{d\xi^*(s)}{\xi^*(s)}$$

$$\xi^*(s) = \Gamma(s/2)\,\pi^{-s/2}\,\zeta(s)$$

For all this see <u>Edward "Riemann's zeta function"</u>

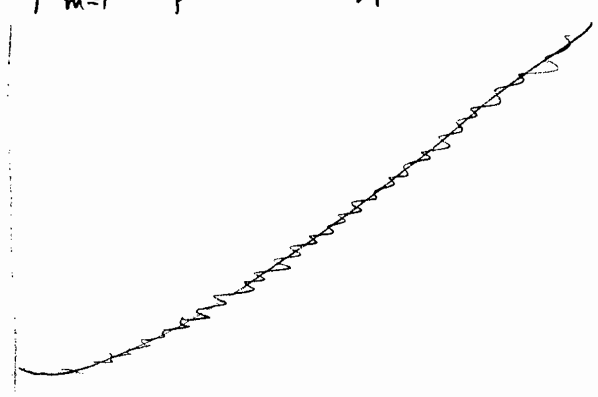
<u>In</u> fact
$$N(E) = \langle N(E)\rangle + N_{osc}(E)$$

$$\langle N(E)\rangle = \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right) + \frac{7}{8} + o(1) \qquad = 1 + \frac{\theta(E)}{\pi}$$

oscillatory part
$$N_{osc}(E) = \frac{1}{\pi}\,\mathrm{Im}\left(\int_c \frac{\zeta'(s)}{\zeta(s)}\right) = \frac{1}{\pi}\,\mathrm{Im}\log\zeta\left(\tfrac{1}{2}+iE\right)$$

$$N_{osc}(E) = O(\log E)$$

$$N_{osc}(E) \sim \frac{-1}{\pi}\sum_p \sum_{m=1}^{\infty} \frac{1}{m}\,\frac{1}{p^{m/2}}\,\sin(mE\log p)$$



H hamiltonian quantum mech
$N(E) = \#$ eigenvalues of $H \le E$
if H quantization of a classical hamiltonian $h(q,p)$ then
leading term $\langle N(E)\rangle$
= symplectic volume
$\{x \in \text{phase space} : h(x) \le E\}$

+ $N_{osc}(E)$ in terms of periodic orbits of the classical system (asympt. estimate)

$$N_{osc}(E) \sim \frac{1}{\pi}\sum_{\gamma_p}\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{1}{2\sinh\left(\frac{m\lambda_p}{2}\right)}\,\sin(mET_\gamma^{\#})$$

λ_p: Lyap. exponent; $T_\gamma^{\#}$: period

Is there a physical system that accounts for the counting of zeros?

<u>Chapter 2 of Connes, M. book</u>:

Scaling Hamiltonian

$$H(q,p) = 2\pi\, qp$$

not usual pos. energy bounded below

canonical symplectic form on phase space

$$\omega = dp \wedge dq$$

$h(q,p)$ not positive but look at symplectic volume of region $|h| \leq E$

i.e. $|pq| \leq \frac{E}{2\pi}$

with cutoff on infrared and ultraviolet

$$|q| \leq \Lambda \qquad |p| \leq \Lambda$$

and work with

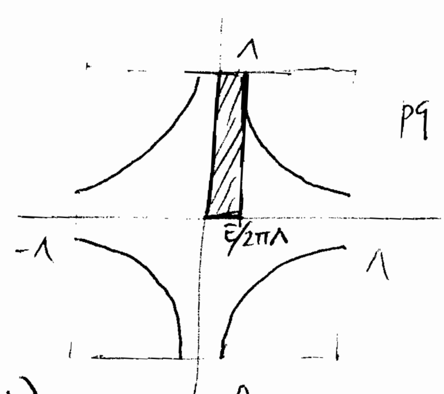
X quotient of $\mathbb{R}\times\mathbb{R}$ by symmetry $(p,q) \longmapsto (-p,-q)$

$$B = \{ (q,p) : |h| \leq E ,\ |q| \leq \Lambda ,\ |p| \leq \Lambda \}$$

Symplectic volume of B in X is

$$v(E) = 2\,\mathrm{Vol}_\omega(B_+)$$

$$\int_{B_+ \subset X} \omega = \frac{E}{2\pi} + \int_{E/2\pi\Lambda}^{\Lambda} \frac{E}{2\pi q}\, dq$$



$pq = \frac{E}{2\pi}$

$$\Rightarrow \mathrm{Vol}_\omega(B_+) = \frac{E}{2\pi}\, 2\log\Lambda - \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right)$$

divergent factor in $\Lambda \to \infty$

leading term of $\langle N(E) \rangle$ with negative sign

sympl. volume of region in X

$$W(E,\Lambda) = \{(\lambda,\theta) \in \mathbb{R}_+^* \times \mathbb{R} : \lambda \in [\Lambda^{-1},\Lambda] ,\ |\theta| \leq E\}$$ cutoff region

in prod of Haar measures $\frac{1}{2\pi}\frac{d\lambda}{\lambda} \times d\theta$

So have $v(E) = \mathrm{Vol}_\omega(W(E,\Lambda)) - 2\langle N(E)\rangle$

- Thinking of this as ~~some~~ classical limit of a quantum system can get better counting $N(E)$?

Want to quantize the scaling hamiltonian $h(q,p) = 2\pi\, pq$

Hilbert space of states

$\mathcal{H} = L^2(\mathbb{R})^{\text{even}}$ = square integrable even functions $f(q)$

(implementing $(p,q) \mapsto (-p,-q)$ symmetry)

Hamiltonian generates scaling transformations $\lambda \in \mathbb{R}_+^*$

$$(\vartheta_a(\lambda) f)(q) = f(\lambda^{-1} q)$$

not unitary: make unitary by normalizing

$$\lambda \longmapsto |\lambda|^{-1/2}\, \vartheta_a(\lambda)$$ unitary operators implementing scaling flow

Can "smear" these operators using a ~~test~~ test function

$h \in C_c^\infty(\mathbb{R}_+^*)$ $\quad$ $d^*\lambda = \frac{d\lambda}{\lambda}$ multipl. Haar meas.

$$\vartheta_a(h) = \int_{\mathbb{R}_+^*} h(\lambda)\, \vartheta_a(\lambda)\, d^*\lambda$$

Infrared cutoff $|q| \le \Lambda$ imposed by an orthogonal projector on Hilbert space $\mathcal{H}$

$$P_\Lambda = \{ f \in L^2(\mathbb{R})^{\text{even}} \mid f(q) = 0 \quad \forall q \text{ with } |q| > \Lambda \}$$

char. function of $[-\Lambda, \Lambda]$

Also want a cutoff in momentum space $|p| \leq \Lambda$ (12)

Passing from position to momentum : Fourier transform

$$\hat{P}_\Lambda = \mathcal{F} P_\Lambda \mathcal{F}^{-1}$$

<u>Note</u>: P_Λ, $\hat{P}_\Lambda$ projectors do <u>not</u> commute

($\Rightarrow$ cannot take inters. of ranges)

No function has at same time compact support <u>and</u> Fourier transform with compact support

How to simultaneously impose cutoff in position & momenta in Hilbert space setting $\rightsquigarrow$ quantum theory of laser technology ~1960's

$\exists$ a second order differential operator W_Λ on $\mathbb{R}$ that commutes with both P_Λ & $\hat{P}_\Lambda$

$$(W_\Lambda f)(q) = -\partial((\Lambda^2 - q^2)\partial)\, f(q) + (2\pi \Lambda q)^2 f(q)$$

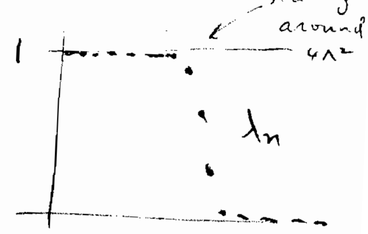
~~solutions to the diff eq~~

restriction to range P_Λ i.e. $q \in [-\Lambda, \Lambda]$

W_Λ has discrete simple spectrum

$\chi_n(\Lambda)$ $n \geq 0$ increasing seq. of positive simple eigenvalues

$\psi_n(q)$ eigenfunctions = prolate spheroidal wave functions,
even/odd fncts for even/odd n



Operator $P_\Lambda \hat{P}_\Lambda P_\Lambda$ commutes with W_Λ

$\Rightarrow \psi_n$ eigenfunctions of $P_\Lambda \hat{P}_\Lambda P_\Lambda$

$\lambda_0 > \lambda_1 > \cdots > \lambda_n > \cdots$
eigenvalues

Subspace $\mathcal{H}$ spanned by ψ_{n} $2n \leq 4\Lambda^2$ approximates projection