Endomotives

Algebraic version:

Objects $\mathcal{A}_{\mathbb{K}}$ algebras over a field $\mathbb{K}$ (number field) of the form $A \rtimes S$ where

$A = \varinjlim_{\alpha} A_\alpha$ with A_α = finite dimensional commutative algebras over $\mathbb{K}$

and S abelian discrete semigroup of endomorphisms of A

$X_\alpha = \mathrm{Spec}(A_\alpha)$ zero dimensional algebraic variety defined over the field $\mathbb{K}$

$\rho \in S$ $\quad \rho : A \xrightarrow{\sim} eAe \quad e = \rho(1)$ idempotent

Morphisms: Correspondences

$\mathcal{G}(X_\alpha, S)$ groupoid of S-action on $X = \varprojlim_{\alpha} X_\alpha$
pro-variety X

Category of Artin motives over $\mathbb{K}$

$\mathcal{AM}_{\mathbb{K}}$ or $\mathcal{AM}_{\mathbb{K},\mathbb{E}}$ w/ coefficients in $\mathbb{E}$ char 0 field $(\mathbb{E} = \mathbb{Q})$

Objects zero-dim alg. varieties over $\mathbb{K}$

$\mathrm{Hom}_{\mathcal{AM}_{\mathbb{K},\mathbb{Q}}}(X, Y) = (\mathbb{Q}^{X(\bar{\mathbb{K}}) \times Y(\bar{\mathbb{K}})})^G \qquad G = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$

fin. dim $\mathbb{Q}$-vector space

Can think of elements of $\mathrm{Hom}_{AM_{k,\mathbb{Q}}}(X, Y)$

as formal $\mathbb{Q}$-lin. combinations of

subvarieties of $X \times Y$ with coeff's in $\mathbb{Q}$

since 0-dim connected comp. / pts.

$$Z = \sum a_i \chi_{Z_i(\bar{k})}$$

$\mathbb{Q}$-valued G-invariant functions on $X(\bar{k}) \times Y(\bar{k})$

So $U \in \mathrm{Hom}(X, Y)$ $\quad U = \sum_i q_i Z_i$

$Z_i \subset X \times Y$

composition

$$\circ : \mathrm{Hom}(X_1, X_2) \times \mathrm{Hom}(X_2, X_3) \to \mathrm{Hom}(X_1, X_3)$$

$$Z \circ Z' = \pi_{13*}(\pi_{12}^* Z \cdot \pi_{23}^* Z')$$

intersection product

$$\pi_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$$

zero dim case : $U \in \mathrm{Hom}(X, Y) \rightsquigarrow U_* \in \mathrm{Hom}_G(\mathbb{Q}^{X(\bar{k})}, \mathbb{Q}^{Y(\bar{k})})$

$$(U \circ V)_* = V_* \circ U_*$$

$\rightsquigarrow$ additive ($\mathbb{Q}$-linear category)

adding Ker/Coker of projectors (and all morphisms) get a (pseudo)abelian category

in this zero-dim case abelian

fiber functor $\omega : X \longmapsto H^0(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}^{X(\bar{\mathbb{k}})}$

bimodule interpretation of morphisms of Artin motives

$X = \mathrm{Spec}(A)$ $\qquad Y = \mathrm{Spec}(B)$

A, B fin. dim. algebras over $\mathbb{k}$

$U = \sum a_i Z_i \qquad a_i \in \mathbb{Q} \qquad Z_i \subset X \times Y$

virtual bimodule $\quad \sum a_i \mathcal{E}_i$

with $\mathcal{E}_i$ bimod. associated to Z_i

bimodule structure from restriction of projection maps $X \times Y \to X$, $X \times Y \to Y$ to Z_i

$$\begin{tikzcd} & X \\ Z_i \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\ & Y \end{tikzcd}$$

$Z_i = \mathrm{Spec}(C_i)$ itself a 0-dim alg. variety over $\mathbb{K}$

$$\begin{tikzcd} & A \arrow[dl, "f"'] \\ C_i & \\ & B \arrow[ul, "g"] \end{tikzcd}$$

$A \ni a \xi b := f(a) \xi g(b)$, $\xi \in C$, $b \in B$

$\mathcal{E}_i = C_i$ with this bimod. structure

$$\mathcal{E}_Z \otimes_B \mathcal{E}_{Z'} = \mathcal{E}_{Z \circ Z'}$$

In endomotives case extend this action and resulting correspondences

$\mathcal{Z}$ = zero-dim pro-variety over $\mathbb{K}$

$$\mathcal{Z} = \varprojlim_{\alpha} \mathcal{Z}_\alpha$$

with left and right actions of groupoids

$\mathcal{G} = \mathcal{G}(X_\alpha, S) \qquad \mathcal{G}'(X'_\alpha, S') = \mathcal{G}'$

require that $\mathcal{G}'$ action satisfies an étale condition

$$g : Z \to X'$$

$$g^{-1}(F(s)) \xrightarrow{\sim} g^{-1}(E(s)) \quad \text{isomorphisms}$$

$$z \qquad\qquad z\cdot s$$

$E(s) = \rho_1^{-1}(\rho_2(1)\rho_1(1)) \qquad F(s) = \rho_2^{-1}(\rho_2(1)\rho_1(1))$

projections for $s = \rho_1/\rho_2$

i.e. $g(z\cdot s) = g(z)\cdot s$, $\quad z\cdot(ss') = (z\cdot s)\,s'$ $\quad$ on $g^{-1}(F(s)\cap s(F(s')))$

in essence: proj. limits of morphisms of Artin motives compatibly w/ semigroups S, S' actions

$\mathcal{E}_{\mathbb{K},\mathbb{Q}}$ = category of algebraic endomotives

Objects $\mathcal{G}(X_\alpha, S)$ or dually $\mathcal{A}_{\mathbb{K}} = A \rtimes S$

Morphisms $\mathrm{Hom}((X_\alpha, S), (X'_\alpha, S'))$

tensor prod.

$$(X_\alpha, S) \otimes (X'_\alpha, S') = (X_\alpha \times X'_\alpha, S \times S')$$

additive category (can always make pseudo-abelian)

Analytic endomotives

$$X(\bar{\mathbb{K}}) = \varprojlim_{\alpha} X_\alpha(\bar{\mathbb{K}})$$

$X_\alpha(\bar{\mathbb{K}}) = \mathrm{Hom}(A_\alpha, \bar{\mathbb{K}})$ Hom as commutative algebras over $\mathbb{K}$

fin. many alg. pts.

$X(\bar{\mathbb{K}})$ with topology of proj. limit : Cantor set (totally disconnected compact Hausdorff space)

$C(X(\bar{\mathbb{K}}))$ C^*-algebra of continuous functions

S semigroup acting on $A = \varinjlim_{\alpha} A_\alpha$

$\Rightarrow$ S acts on $X(\bar{\mathbb{K}}) = \mathrm{Hom}(A, \bar{\mathbb{K}})$ by pre-composition

$\chi : A \to \bar{\mathbb{K}}$, s

$\Rightarrow$ $C(X(\bar{\mathbb{K}})) \rtimes S$ C^*-algebra semidirect product

endomorphism action of $s \in S$

$$U_s^* U_s = 1 \qquad U_s U_s^* = \rho_s(1)$$

$$U_{s_1 s_2} = U_{s_1} U_{s_2} \qquad U^*_{s_2 s_1} = U^*_{s_1} U^*_{s_2}$$

$$U_s f = \rho_s(f) U_s \qquad f U_s^* = U_s^* \rho_s(f) \qquad f \in C(X(\bar{\mathbb{K}}))$$

$A_{\mathbb{C}} \hookrightarrow C(X(\bar{\mathbb{K}}))$ $\quad a \mapsto f_a$ $\quad f_a(\chi) = \chi(a)$, $\chi \in \mathrm{Hom}(A, \bar{\mathbb{K}})$

$\| $

$A \otimes_{\mathbb{K}} \mathbb{C}$

choosing an embedding $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$

$$\mathcal{A}_{\mathbb{K}} = A \rtimes S \hookrightarrow \mathcal{A} = C(X(\bar{\mathbb{K}})) \rtimes S$$

arithmetic subalgebra

Morphisms: similar completion of bimodules $\mathcal{E}_Z$ to Hilbert modules

Galois action

$$X(\bar{\mathbb{K}}) = \mathrm{Hom}(A, \bar{\mathbb{K}})$$

$\gamma \in \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$ action by composition

$$\chi : A \to \bar{\mathbb{K}} \circlearrowleft \gamma$$

commutes with action of endomorphisms S (pre-composition)

$\Rightarrow$ action of $G = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$ by automorphisms of

$$C(X(\bar{\mathbb{K}})) \rtimes S$$

State: (probability) measure on $X(\bar{\mathbb{K}})$ from finite measures on $X_\alpha(\bar{\mathbb{K}})$

$\mu \Rightarrow$ state on $C(X(\bar{\mathbb{K}})) \rtimes S$ φ

$\Rightarrow$ An endomotive gives

$$\Big(\underset{\text{algebra over } \mathbb{K}}{\mathcal{A}_{\mathbb{K}} = A \rtimes S},\ \underset{C^*\text{-algebra}}{\mathcal{A} = C(X(\bar{\mathbb{K}})) \rtimes S},\ \underset{\text{state}}{\varphi} \Big)$$

The Bost-Connes endomotive

$A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$ $\qquad A = \varinjlim_n A_n = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$

$\mathcal{A}_{\mathbb{Q}} = A \rtimes \mathbb{N} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$

$X(\bar{\mathbb{Q}}) = \mathrm{Hom}(\mathbb{Q}[\mathbb{Q}/\mathbb{Z}], \bar{\mathbb{Q}})$ $\quad$ (= roots of 1)

$= \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$

$\rho_n(A_k) \subset A_{nk}$

$\rho_n(e(r)) = \frac{1}{n} \sum_{nl=r} e(l) \qquad \rho_n(1) = \frac{1}{n} \sum_{nl=0} e(l)$

measure μ on $\hat{\mathbb{Z}}$ $\Rightarrow$ state $\varphi(f) = \int_{\hat{\mathbb{Z}}} f(1,\rho)\, d\mu(\rho)$
(is the KMS, state, unique, of the BC system)

General construction; endomotives from self-maps of algebraic varieties

(Y, y_0) alg. variety (any dim) over $\mathbb{K}$ $\qquad y_0$ base pt.

S a unital abelian semigroup of self maps $s: Y \to Y$

$s(y_0) = y_0$

and finite $\deg(s) = \# s^{-1}(y_0)$
unramified over y_0

$X_s = \{ y \in Y : s(y) = y_0 \}$ $\qquad X = \varprojlim X_s$

$\xi_{s,s'} : X_{sr} \longrightarrow X_s$ $\qquad$ pro-var. over $\mathbb{K}$
$s' = sr \qquad y \longmapsto \xi_{s,s'}(y) = r(y)$ $\qquad$ zero-dim.

$X(\bar{\mathbb{K}})$

$s \in S$: β_s acts on X by

$\xi_u \beta_s(x) = s\, \xi_u(x)$ $\qquad$ $\xi_u : X \to X_u$

maps of proj. system

$\Rightarrow$ $\beta_s : X \xrightarrow{\simeq} X^{e_s} = \xi_s^{-1}(Z_s)$

Z_s = component Y_0 in X_s

$e_s = \chi_{Z_s}$

$\xi_u \beta_s^{-1}(x) = \xi_{su}(x)$

$\Rightarrow$ action of S on X $\quad\Rightarrow$ $C(X(\bar{\mathbb{K}})) \rtimes S$ and

$A_K = A \rtimes S$

Again Bost-Connes example:

$\mathbb{Q}[t, t^{-1}] / (t^n - 1) = A_n$ $\qquad$ $A = \mathbb{Q}[t, t^{-1}]$

$s_n : t \mapsto t^n$ $\qquad$ $Y_0 = 1$

$Y = \mathbb{G}_m$ $\qquad$ $s_n : u \mapsto u^n$ $\qquad$ $S = \mathbb{N}$

A general von Neumann algebra procedure (Tomita-Takesaki)

An algebra $\mathcal{A}$ ~~...~~ C^*-algebra and a state φ

$\mathcal{H}_\varphi$ = GNS representation of $\mathcal{A}$

$\mathcal{M}$ = von Neumann algebra : Weak completion of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H}_\varphi)$

Assume: $\exists\, \xi$ cyclic separating vector: $\mathcal{M}\xi$ & $\mathcal{M}'\xi$ dense in $\mathcal{H}_\varphi$

$$S_\varphi : \mathcal{M}\xi \to \mathcal{M}\xi \qquad a\xi \xmapsto{S} a^*\xi$$

$$S_\varphi^* : \mathcal{M}'\xi \to \mathcal{M}'\xi \qquad a'\xi \xmapsto{S^*} (a')^*\xi$$

polar decomposition $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$

J_φ conj-linear involution $J_\varphi = J_\varphi^* = J_\varphi^{-1}$

$\Delta_\varphi = S_\varphi^* S_\varphi$ self-adjoint positive operator

$$J_\varphi \Delta_\varphi J_\varphi = S_\varphi S_\varphi^* = \Delta_\varphi^{-1}$$

Main results of Tomita-Takesaki theory:

$$J_\varphi \mathcal{M} J_\varphi = \mathcal{M}' \qquad \Delta_\varphi^{it}\mathcal{M}\Delta_\varphi^{-it} = \mathcal{M}$$

$$\alpha_\varphi(a) = \Delta_\varphi^{it}\, a\, \Delta_\varphi^{-it} \qquad a \in \mathcal{M}$$

φ is a $KMS_{\beta=1}$ state for $\sigma_t^\varphi = \alpha_{-t}$

Get for free a time evolution

Under good conditions (endomotives)

σ_t^φ preserves $\mathcal{A} \subset \mathcal{M}$

In general don't know if low temperature KMS states are Gibbs states

but for "good" systems

Ω_β low temperature extremal KMS states s.t.

$\exists\, \pi_\varepsilon : \mathcal{A} \to B(\mathcal{H}(\varepsilon))$ irred. repres.

$\varepsilon \in \Omega_\beta$ s.t. $\underset{\text{GNS}}{\mathcal{H}_\varepsilon} = \mathcal{H}(\varepsilon) \otimes \mathcal{H}'$ with action by $\pi_\varepsilon(a) \otimes 1$

and

$$\pi_\varepsilon(\sigma_t^\varphi(a)) = e^{itH_\varepsilon}\, \pi_\varepsilon(a)\, e^{-itH_\varepsilon} \quad \text{with } \mathrm{Tr}(e^{-\beta H_\varepsilon}) < \infty$$

$$\varepsilon(a) = \frac{\mathrm{Tr}(\pi_\varepsilon(a)\, e^{-\beta H_\varepsilon})}{\mathrm{Tr}(e^{-\beta H_\varepsilon})}$$

H_ε defined only up to a shift

$$\lambda(\varepsilon, H) = (\varepsilon, H + \log\lambda) \qquad \lambda \in \mathbb{R}_+^*$$

$\mathbb{R}_+^* \to \tilde{\Omega}_\beta \to \Omega_\beta$ fibration w/ section $\mathrm{Tr}(e^{-\beta H}) = 1$

$$\tilde{\Omega}_\beta = \Omega_\beta \times \mathbb{R}_+^*$$

injections $\Omega_\beta \to \Omega_{\beta'}$ $\beta' > \beta$ stabilize

$\Rightarrow$ Classical points of the endomotive

Ω_β ($\tilde{\Omega}_\beta$ = cl. pts + choice of Hamiltonian)

Dual system

$$(\hat{\mathcal{A}}, \theta) \qquad \hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$$

$$(x * y)(s) = \int_{\mathbb{R}} x(t)\, \sigma_t(y(s-t))\, dt$$

$$x, y \in S(\mathbb{R}, \mathcal{A})$$

Write elements of $\hat{\mathcal{A}}$ as formally

$$\int x(t)\, U_t\, dt$$

Scaling action θ

$$\theta_\lambda \left(\int x(t) U_t\, dt \right) = \int \lambda^{it}\, x(t)\, U_t\, dt$$

$(\varepsilon, H) \in \tilde{\Omega}_\beta \rightsquigarrow$ irred. repres. of $\hat{\mathcal{A}}$

$$\pi_{\varepsilon,H}\left(\int x(t)\,U_t\,dt\right) = \int \pi_\varepsilon(x(t))\,e^{itH}\,dt$$

$$\pi_{\varepsilon,H} \circ \theta_\lambda = \pi_{\lambda(\varepsilon,H)}$$

with resp. to scaling action

Trace class property

$$\pi_{\varepsilon,H}\left(\int x(t)\,U_t\,dt\right) \in \mathcal{L}^1(\mathcal{H}(\varepsilon))$$

for $x \in \hat{\mathcal{A}}_\beta$ a subalgebra of analytic elements

analytic contin. from $x(t)$ to $x(z)$ $z \in I_\beta$ rapid decay on boundary

Then can consider a "restriction map"

$$\begin{array}{ccccc} \hat{\mathcal{A}}_\beta & \xrightarrow{\pi} & C(\tilde{\Omega}_\beta, \mathcal{L}^1) & \xrightarrow{\mathrm{Tr}} & C(\tilde{\Omega}_\beta) \\ \psi & & \psi & & \\ x & & (\varepsilon, H) & & \end{array}$$

$$\pi(x)(\varepsilon, H) = \pi_{\varepsilon,H}(x)$$

Not a morphism of algebras but a morphism in an abelian category containing algebras

Category of cyclic modules

$$\hat{\mathcal{A}}_\beta^\natural \xrightarrow{(\mathrm{Tr}\circ\pi)^\natural} C(\tilde{\Omega}_\beta)^\natural$$

where bimodules and traces become morphisms