KMS states of the BC system   Thursday Apr 1

(1) for $0<\beta\le 1$ $\exists!$ KMS_β state, explicitly given by the formula

$$\varphi_\beta(e(\tfrac{a}{b})) = f_{-\beta+1}(b) \Big/ f_1(b)$$

$$f_k(b) = \sum_{d|b} \mu(d)\left(\tfrac{b}{d}\right)^k$$

$\mu(d)$ = Möbius function

$\mu(n) = 1$ square free even # prime factors
-1 sq-free odd # prime
0 n is not square-free

(f_1 Euler totient function)

$\sum_{d|n}\mu(d) = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$

$\mu(ab) = \mu(a)\mu(b)$

(2) for $1<\beta\le\infty$ extremal $\mathcal{E}_\beta \subset KMS_\beta$
parameterized by $\rho\in\hat{\mathbb{Z}}^*$ (invertible $\mathbb{Q}$-lattice)

$$\mathcal{E}_\beta \simeq \hat{\mathbb{Z}}^* \quad \text{(action free trans. of } \hat{\mathbb{Z}}^* \text{ symmetries)}$$

(positive energy repres.)

$1<\beta<\infty$: $\varphi_{\beta,\rho}(e(r)) = \dfrac{1}{\zeta(\beta)} Li_\beta(\rho(\zeta_r))$

$\mu(n) = \sum_{\substack{1\le k\le n \\ \gcd(k,n)=1}} e^{2\pi i \frac{k}{n}}$

Sum of primitive n-th roots of 1

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \quad \text{polylog. function}$$

(3) At $\beta=\infty$

$$\varphi_{\infty,\rho}(A_{\mathbb{Q}}) \subset \mathbb{Q}^{cycl}$$

extension of $\mathbb{Q}$
gen. by roots of 1

and $\gamma\in Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$

$$\gamma\,\varphi_{\infty,\rho}(a) = \varphi_{\infty,\rho}(\theta^{-1}(\gamma)a)$$

θ "class field th. isom"
realizing
$Gal(\mathbb{Q}^{cycl}/\mathbb{Q}) \underset{\theta}{\cong} \hat{\mathbb{Z}}^*$

(Will give more general proof later classifying KMS states for extensions of BC : except last point) ②

don't have good analog of $\mathcal{A}_{\mathbb{Q}}$ in more gen. cases

General problem in NT (Hilbert 12th problem)

given a number field $\mathbb{K}$ finite extension of $\mathbb{Q}$

$[\mathbb{K}:\mathbb{Q}]=n$ (sol'n of polyn. eq. over $\mathbb{Q}$)

Would like to give explicit ~~[illegible]~~ sets of generators for field extensions of $\mathbb{K}$ with abelian Galois group

(abelian extension) in partic. max abelian ext

$\mathbb{K}^{ab}$ s.t. $Gal(\mathbb{K}^{ab}/\mathbb{K}) = Gal(\bar{\mathbb{K}}/\mathbb{K})^{ab}$

abelianization of absolute Galois group

with explicit description of Galois action on these generators

Kronecker-Weber theorem :

$$\mathbb{Q}^{ab} = \mathbb{Q}^{cycl}$$

$(\mathbb{G}_m)_{tors}$ = roots of unity

(torsion points of multipl. grp $\mathbb{C}^*$)

are the generators

and Galois group action is action of

$\hat{\mathbb{Z}}^*$ = invertible homom. $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$

[We'll see another case: imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ $d \in \mathbb{N}$]

Idea: Would like to use NCG to construct

C^*alg. $(\mathcal{A}_{\mathbb{K}}, \sigma_{\mathbb{K}})$ $\quad\mathbb{K}$ number field

s.t. $\exists\, \mathcal{A}_{\mathbb{K},arithm}$ defined over $\mathbb{K}$

s.t. $\mathcal{E}_\infty \subset KMS_\infty$ states

$\varphi(\mathcal{A}_{\mathbb{K},arithm}) \subset \mathbb{K}^{ab} \subset \mathbb{C}$

some embedding

and

$$\gamma\, \varphi(a) = \varphi(\theta^{-1}(\gamma)\, a)$$

Some grp. of symmetries (isom's + endom's) of $(\mathcal{A}_{\mathbb{K}}, \sigma_{\mathbb{K}})$

If could do this would solve Hilbert 12th problem

(already for real quadratic fields $\mathbb{Q}(\sqrt{d})$ not known)

* BC system for $\mathbb{Q}$
* imaginary quadratic fields
* general systems for # fields but lack good $\mathcal{A}_{\mathbb{K},arithm}$

(Systems obtained from considering $\mathbb{K}$-lattices instead of $\mathbb{Q}$-lattices)

Before generalizations of BC system

More arithmetic properties of BC algebra: (papers: "Fun with $\mathbb{F}_1$" and "Cyclotomy and endomotives" both on arXiv)

lifting from $\mathbb{Q}$ to $\mathbb{Z}$: problem relation $\frac{1}{n}\sum_{ns=r} e(s)$ involves denominators

BC model over $\mathbb{Z}$:

generators and relations

$$\left.\begin{array}{l} \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ e(r) \quad \text{with} \quad e(0)=1 \;\&\; e(r+s)=e(r)e(s) \end{array}\right\} \text{this part}$$

and $\tilde{\mu}_n, \mu_n^*$ (no longer adjoints of one another)

$$\left.\begin{array}{l} \tilde{\mu}_n\tilde{\mu}_m = \tilde{\mu}_{nm} \\ \mu_n^*\mu_m^* = \mu_{nm}^* \end{array}\right\} \forall n,m\in\mathbb{N}$$

$\mathcal{A}_{\mathbb{Z}}$

$$\mu_n^*\tilde{\mu}_n = n$$

$$\tilde{\mu}_n\mu_m^* = \mu_m^*\tilde{\mu}_n \quad \text{if } (n,m)=1$$

no longer a crossed product by a semigroup.

Still consider endomorphisms

$$\sigma_n(e(r)) = e(nr) \qquad \sigma_n: \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

Satisfies

$$\mu_n^*\, x = \sigma_n(x)\,\mu_n^* \qquad \forall x\in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

$$x\,\tilde{\mu}_n = \tilde{\mu}_n\,\sigma_n(x)$$

but now if set

$$\tilde{\rho}_n(x) = \tilde{\mu}_n \times \mu_n^* \qquad \text{no longer a ring endom. of } \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

because $\mu_n^* \tilde{\mu}_n = n$ not 1

it replaces the relation $\rho_n(x) = \mu_n \times \mu_n^*$ which involves denominators

$$\tilde{\rho}_n(e(r)) = \sum_{ns=r} e(s)$$

$$\sigma_{nm} = \sigma_n \sigma_m \qquad \tilde{\rho}_{nm} = \tilde{\rho}_n \tilde{\rho}_m$$

$$\tilde{\rho}_m(\sigma_m(x) y) = x\, \tilde{\rho}_m(y)$$

$$\sigma_c(\tilde{\rho}_b(x)) = (b,c)\, \tilde{\rho}_{b'}(\sigma_{c'}(x)) \qquad b' = \frac{b}{(b,c)} \quad c' = \frac{c}{(b,c)}$$

g.c.d.

To check these relations observe that

if $E_n(x) = \{y \in \mathbb{Q}/\mathbb{Z} : ny = x\}$

then $E_{nm}(x) = \bigcup_{y \in E_n(x)} E_m(y)$ and $E_m(y_1) \cap E_m(y_2) = \emptyset$ if $y_1 \neq y_2$

$$\Rightarrow \tilde{\rho}_m(\tilde{\rho}_n(e(x))) = \tilde{\rho}_m\Big(\sum_{y \in E_n(x)} e(y)\Big) = \sum_{y \in E_n(x)} \sum_{z \in E_m(y)} e(z) = \tilde{\rho}_{nm}(e(x))$$

for other rel: multipl. by $c = nc'$

is n-to-1 map $E_b(s) \to E_{b'}(c's)$

$A_{\mathbb{Z}}$: generated additively by

$$\tilde{\mu}_a \times \mu_b^* \qquad (a,b)=1 \qquad x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

(with $\tilde{\mu}_1 = \mu_1^* = 1$)

The endomotive description of the BC algebra

paper: (NCG & motives: thermodyn. of endomotives)

$\mathbb{Q}[t,t^{-1}]$ ring   Spec is $\mathbb{G}_{m,\mathbb{Q}}$

$s_k : P(t,t^{-1}) \longmapsto P(t^k, t^{-k})$ ring homomorphisms

(1 is fixed by all
inverse image under $t \mapsto t^k$ is :)

$$X_k = \operatorname{Spec}\left(\mathbb{Q}[t,t^{-1}]/(t^k-1)\right) \qquad \text{alg. variety (zero dim)}$$

$$X = \varprojlim_k X_k \qquad \text{ordered by divisibility}$$

maps of proj. system :

$u(n)$ = class of t mod t^n-1 (generator of $\mathbb{Q}[t,t^{-1}]/t^n-1$)

$$\xi_{m,n}(u(n)) = u(m)^a \qquad a = \frac{m}{n}$$

for $n|m$

gives corresp direct limit of algebras $A = \varinjlim_k A_k$

$$A_k = \mathbb{Q}[t,t^{-1}]/t^k-1 \qquad A_k = \mathbb{Q}[\mathbb{Z}/k\mathbb{Z}]$$

So again $A = \varinjlim_{k} A_k = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$

identified $u(n)$ with $e(\frac{1}{n})$ in previous notation for generators

Also write $\mu^{(\infty)}$ for X and $\mu^{(n)}$ for X_n zero dim varieties

pro-variety

"Field with one element"
(no such thing exists: if it has one element it ain't no field)

but can pretend such object exists and think of interpolation of counting of pts of varieties X over finite fields $\mathbb{F}_q$ ($q = p^m$ some $p > 0$ pos. char.) when "$q \to 1$"

Formal equality $\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[t, t^{-1}]/t^n - 1$

"extensions" $\mathbb{F}_{1^n}$ "like" $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$

What this strange notation means is that "something" (variety, ..) defined over $\mathbb{F}_1$ is a var. (..) def over $\mathbb{Z}$ with additional conditions that "descend" from $\mathbb{Z}$ to $\mathbb{F}_1$

(philosophical concept, not mathematical definition)

A way to try to make this idea more rigorous (8)

Soulé's approach to defining varieties over $\mathbb{F}_1$

Affine var. over $\mathbb{F}_1$

$\mathcal{R}$ = cat. of rings (over $\mathbb{Z}$) gen. by the $A_n = \mathbb{Z}[t,t^{-1}]/t^n - 1$ and their tensor products (as $\mathbb{Z}$-modules)

X = a "gadget" over $\mathbb{F}_1$ is a triple $(\underline{X}, \mathcal{A}_X, e_X)$

(1) $\underline{X} : \mathcal{R} \to$ Sets a covariant functor

(2) $\mathcal{A}_X$ an algebra over $\mathbb{C}$

(3) A natural transformation e_X from the functor $\underline{X}$ to the functor

$R \mapsto \mathrm{Hom}(\mathcal{A}_X, R_{\mathbb{C}})$ $\qquad R_{\mathbb{C}} = R \otimes_{\mathbb{Z}} \mathbb{C}$

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"'] & \underline{X}(R) \arrow[d, "\underline{X}(f)"] \\ R' \arrow[r] & \underline{X}(R') \end{tikzcd}$$

$$\begin{tikzcd} \underline{X}(R) \arrow[r, "e_X(R)"] \arrow[d, "\phi(R)"'] & \mathrm{Hom}(\mathcal{A}_X, R_{\mathbb{C}}) \arrow[d, "\phi^*"] \\ \underline{Y}(R) \arrow[r, "e_Y(R)"'] & \mathrm{Hom}(\mathcal{A}_Y, R_{\mathbb{C}}) \end{tikzcd}$$

Morphism of gadgets

E.g. $V = \mathrm{Sp}(B)$ aff. var. $\mathbb{Z}$

$\Rightarrow X = X_V$ gadget $\qquad \underline{X}_V(R) = \mathrm{Hom}_{\mathbb{Z}}(B, R)$

$\mathcal{A}_{X_V} = B \otimes_{\mathbb{Z}} \mathbb{C} \qquad e_{X_V}(f) = f \otimes id_{\mathbb{C}} \quad \forall f \in \underline{X}_V(R)$

Call $X_V = \mathcal{F}(V)$ functor from $\mathbb{Z}$-varieties to gadgets

Affine variety over $\mathbb{F}_1$ is a gadget X s.t.

$\exists\ X_{\mathbb{Z}}$ variety over $\mathbb{Z}$ and

immersion $i: X \hookrightarrow \mathcal{F}(X_{\mathbb{Z}})$ of gadgets s.t.

$\forall\ V$ aff. var over $\mathbb{Z}$ $\qquad \forall\ \varphi: X \to \mathcal{F}(V)$

$\exists$ alg. morphism of varieties

$\varphi_{\mathbb{Z}}: X_{\mathbb{Z}} \to V \qquad$ s.t. $\varphi = \mathcal{F}(\varphi_{\mathbb{Z}}) \circ i$

<u>Note</u> : what is the gadget structure saying?

think $R = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ for some n

then $\mathrm{Hom}(A_{\mathbb{C}}, R_{\mathbb{C}}) = X_{\mathbb{C}}(R_{\mathbb{C}})$

for $X_{\mathbb{C}} = \mathrm{Sp}(A_{\mathbb{C}})$ $\quad R_{\mathbb{C}}$-points of the affine scheme $X_{\mathbb{C}}$

for $R_{\mathbb{C}} = \mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ the $R_{\mathbb{C}}$-points of $X_{\mathbb{C}}$ are the cyclotomic pts

(pts def. over a cycl. ext. of $\mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$)

~~[illegible]~~

$\underline{X}(R) \xrightarrow{e_X} \mathrm{Hom}(A_X, R_{\mathbb{C}})$

evaluation maps at cyclotomic points

~~[illegible]~~

~~[illegible]~~

~~[illegible]~~

<u>idea of Soulé's definition</u>:

$X_{\mathbb{Z}}$ determined completely by its cyclotomic points (a finite "combinatorial" information)

$\Rightarrow X_{\mathbb{Z}}$ is coming from $\mathbb{F}_1$

Example the roots of unity

$\underline{\mu}^{(k)}$ as gadget

$\underline{\mu}^{(k)} : \mathcal{R} \to Sets$ $\quad\quad \underline{\mu}^{(k)}(R) = \{ x \in R : x^k = 1 \}$

k-th roots of 1 in R

$\underline{\mu}^{(k)}(R) = \mathrm{Hom}_{\mathbb{Z}}(A_k, R)$

aff. Var. over $\mathbb{F}_1$

$$\underline{\mu}^{(\infty)}(R) := \lim_{\substack{\leftarrow \\ k}} \mathrm{Hom}(A_k, R) = \mathrm{Hom}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], R)$$

pro-variety over $\mathbb{F}_1$

Then the homomorphisms of rings

$$\xi_{m,n} : A_n \longrightarrow A_m \quad \text{when } n | m$$

$$\xi_{m,n}(u(n)) = u(m)^a \quad\quad a = \frac{m}{n}$$

determine the "extensions" $\mathbb{F}_{1^n} \subset \mathbb{F}_{1^m}$

$\Rightarrow$ (meaningful only after change of coeff formula

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[t, t^{-1}] / t^n - 1 = A_n \quad)$$

So the abelian part $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ of BC algebra meaning of tower of extensions $\mathbb{F}_{1^n}$ of $\mathbb{F}_1$

The maps of proj. system coming from σ_n endomorphisms (Frobenius action)

A different approach to $\mathbb{F}_1$-geometry

(Borger) Λ-rings

R ring (underlying abelian group torsion free)

+ action of $\mathbb{N}$ multipl. semigroup by endomorphisms $\{s_p\}_{p \text{ prime}}$

so that $s_p(x) - x^p \in pR \quad \forall x \in R$

(that is: the action of s_p is ~~the~~ a lift of the Frobenius mod p)

a morphism of Λ-rings

$f : R \to R'$ s.t. $f(s_p(x)) = s'_p(f(x)) \quad \forall x \in R$
$\forall p$ prime

i.e. $f(s_n(x)) = s'_n(f(x))$
$\forall n \in \mathbb{N} \quad \forall x \in R$

Prototype case of Λ-ring

$$R = \mathbb{Z}[t, t^{-1}]/t^n - 1 = A_n$$

$$s_k(P)(t, t^{-1}) = P(t^k, t^{-k})$$

$\Rightarrow$ The BC algebra is direct limit of Λ-rings

meaning $\lim\limits_{k} A_k \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$

with s_k's giving σ_k-endom.

and compatible integral models $\rightsquigarrow A_{\mathbb{Z}}$

Various generalizations of Bost-Connes system:

(1) $GL_1 \rightsquigarrow GL_2$ two-dimensional $\mathbb{Q}$-lattices
(elliptic curves; modular forms; Shimura variety of GL_2)

(2) Systems for general Shimura varieties + for number fields

Q: How much $(\mathcal{A}_{\mathbb{K}}, \sigma_{\mathbb{K}})$ determines $\mathbb{K}$? (anabelian geometry)

(3) Function field case: QSM in positive characteristic
Drinfeld modules

(4) "Multivariable" BC systems; Λ-rings; $\mathbb{F}_1$-geometry

(5) Discussion of real quadratic fields and noncommutative geometry

(6) An analogy: → 3-manifolds and surgery presentations
↳ Loop quantum gravity: spin networks and foams

(7) Endomotives

Bost-Connes "Hecke algebras, type III factors ..." Selecta 1995
Connes-Marcolli "QSM of $\mathbb{Q}$-lattices" 2006
Ha-Paugam "Bost-Connes-Marcolli systems for Shimura varieties" 2005
Connes-Marcolli-Ramachandran "KMS states and complex multiplication" 2005
Laca-Larsen-Neshveyev "BC for number fields" J. Number Theory 2008
Consani-Marcolli "QSM over function fields" (see also B. Jacob's paper)
Marcolli "NCG and Arithmetic" arXiv (ICM talk)
Marcolli "Cyclotomy & endomotives" p-Adic numbers, Ultrametr. analysis & Appl. 2009
⋮
etc.